

# A Characterization of Flip-accessibility for Rhombus Tilings of the Whole Plane

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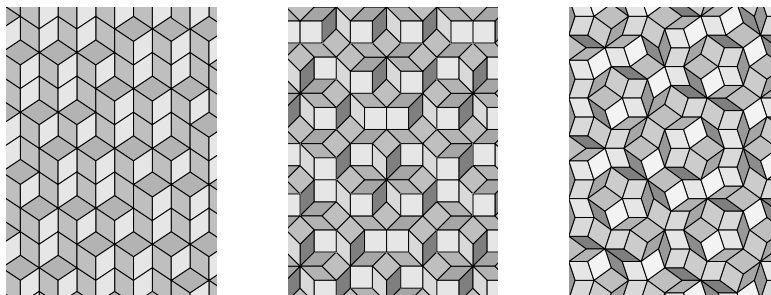
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**Abstract.** It is known that any two rhombus tilings of a polygon are flip-accessible, *i.e.* linked by a finite sequence of local transformations called flips. This paper consider flip-accessibility for rhombus tilings of the *whole plane*, asking whether any two of them are linked by a *possibly infinite* sequence of flips. The answer turning out to depend on tilings, a *characterization* of flip-accessibility is provided. This yields, for example, that any tiling by Penrose tiles is flip-accessible from a Penrose tiling.

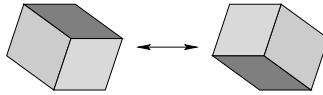
## Introduction

A *rhombus tiling* of  $D \subset \mathbb{R}^2$  is a set of rhombus-shaped compact sets, namely *rhombus tiles*, whose interiors are disjoint, which meet edge-to-edge and whose union is  $D$ . Fig. 1 depicts celebrated rhombus tilings of  $D = \mathbb{R}^2$  (see also [6]).



**Fig. 1.** Rauzy-dual, Ammann-Beenker and Penrose rhombus tilings (from left to right).

Then, the *flip* is a well-known local transformation over rhombus tilings which just exchanges three rhombus tiles sharing a vertex (see *e.g.* [1, 2, 5, 9, 11, 15], and also Fig. 2). Flips rise the question of *flip-accessibility*: can a given rhombus tiling be transformed into another one by performing a sequence of flips?



**Fig. 2.** A flip is an exchange of three rhombus tiles sharing a vertex.

A motivation for studying flip-accessibility for rhombus tilings comes from statistical physics. Indeed, rhombus tilings appeared to be a suitable model for the structure of recently discovered quasicrystalline alloys (see [14]). Moreover, elementary transformations of real quasicrystal, called *phasons*, seem being efficiently modeled by flips (see [10]). This led to study flip dynamics, thus the preliminary question of flip-accessibility.

In the case of rhombus tilings of a polygon, it is proven in [9] that any two rhombus tilings are linked by a finite sequence of flips. In other words, rhombus tilings of a polygon are all mutually flip-accessible. Many results concerning flip dynamics, in particular random sampling, have been obtained (see *e.g.* [5, 11]). The case of rhombus tilings of the whole plane is more complicated. First, note that it is natural to consider flip-accessibility in terms of *possibly infinite* sequences of flips. Then, even with this definition, tilings turn out to be not always flip-accessible. Thus, answering the question of flip-accessibility amounts to *characterize* flip-accessibility between pairs of tilings.

The paper is organized as follows. In Section 1, we more formally define rhombus tilings of the whole plane and the corresponding notion of flip-accessibility. We also show that rhombus tilings are naturally associated with a useful higher-dimensional notion, namely *stepped surfaces*. Section 2 then states the main result of this paper, that is, a characterization of flip-accessibility in terms of *shadows* (Theorem 1). As a corollary, we show that there is a large class of rhombus tilings, namely the *canonical projection tilings*, from which any other rhombus tiling over the same set of rhombus tiles is flip-accessible. The last section is devoted to the proof of this characterization. In particular, we rely on the de Bruijn lines of [3] to introduce *de Bruijn cones*, a tool which could be used for achieving efficient algorithms in the finite case.

## 1 General settings

Let us first define rhombus tilings of the whole plane. Let  $\mathbf{v}_1, \dots, \mathbf{v}_d$  be  $d \geq 3$  non-colinear unit vectors of  $\mathbb{R}^2$ . *Rhombus tiles* are the  $\binom{d}{2}$  compact sets of non-

empty interior defined for  $1 \leq i < j \leq d$  by:

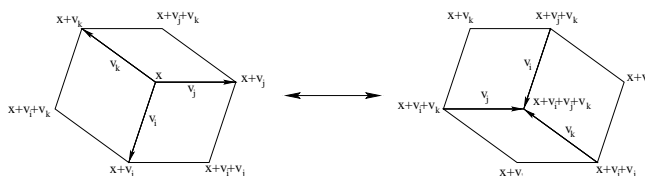
$$T_{ij} = \{\lambda \mathbf{v}_i + \mu \mathbf{v}_j, 0 \leq \lambda, \mu \leq 1\}.$$

Then, for  $\mathbf{x} \in \oplus_i \mathbb{Z} \mathbf{v}_i$ , we denote by  $\mathbf{x} + T_{ij}$  the rhombus tile obtained by translating  $T_{ij}$  by  $\mathbf{x}$ . Note that there is no loss of generality by considering rhombus tiles translated in  $\oplus_i \mathbb{Z} \mathbf{v}_i$  (instead of the whole  $\mathbb{R}^2$ ) because we are here interested in flip-accessibility; this restriction will be useful in Prop. 1, below. Let us now define rhombus tilings of the whole plane:

**Definition 1.** A  $d \rightarrow 2$  rhombus tiling is a set  $\mathcal{T}$  of translated rhombus tiles of disjoint interiors, meeting edge-to-edge<sup>4</sup> and whose union is the whole plane  $\mathbb{R}^2$ .

For example, Fig. 1 depicts  $d \rightarrow 2$  rhombus tilings for, respectively,  $d = 3, 4, 5$ .

Let us now define *flip-accessibility* for  $d \rightarrow 2$  rhombus tilings. Introduced in [15] for finite domino or lozenge tilings, flips are similarly defined for rhombus tilings (see Fig. 3).



**Fig. 3.** A flip is a local exchange of three rhombus tiles sharing a vertex.

Clearly, performing a flip on a rhombus tiling yields a (new) rhombus tiling. This also holds for a finite sequence of flips, but we need to be more precise in the case of an *infinite* sequence of flips. Let us define the distance  $d(\mathcal{T}, \mathcal{T}')$  between two tilings  $\mathcal{T}$  and  $\mathcal{T}'$  by:

$$d(\mathcal{T}, \mathcal{T}') = \inf\{2^{-r} \mid \mathcal{T}|_{B(\mathbf{0}, r)} = \mathcal{T}'|_{B(\mathbf{0}, r)}\},$$

where  $\mathcal{T}|_{B(\mathbf{0}, r)}$  denotes the set of rhombus tiles in  $\mathcal{T}$  which belong to the 2-dimensional ball of center  $\mathbf{0}$  and radius  $r$ . This allows us to indiscriminately consider finite or infinite sequences of flips for defining flip-accessibility:

**Definition 2.** Let  $\mathcal{T}$  and  $\mathcal{T}'$  be two rhombus tilings of the whole plane. If there is a sequence  $(\mathcal{T}_n)_{n \geq 0}$  of rhombus tilings such that  $\mathcal{T}_0 = \mathcal{T}$ ,  $\mathcal{T}_{n+1}$  is obtained by performing a flip on  $\mathcal{T}_n$  and  $d(\mathcal{T}_n, \mathcal{T}')$  tends towards 0, then one says that  $\mathcal{T}'$  is flip-accessible from  $\mathcal{T}$ , and one writes:

$$\mathcal{T} \xrightarrow{\text{flips}} \mathcal{T}'$$

<sup>4</sup> that is, two intersecting tiles share either a point  $\mathbf{x}$  or an edge  $\{\mathbf{x} + \lambda \mathbf{v}_i, 0 \leq \lambda \leq 1\}$

Last, let us show how rhombus tilings and flips can be seen from a higher-dimensional viewpoint. This will be very useful in the following sections.

Let  $(e_1, \dots, e_d)$  be the canonical basis of  $\mathbb{R}^d$ . For  $1 \leq i < j \leq d$  and  $\mathbf{x} \in \mathbb{Z}^d$ , the *unit face* of *type*  $t_{ij}$  located at  $\mathbf{x}$  is the subset of  $\mathbb{R}^d$  defined by:

$$(\mathbf{x}, t_{ij}) = \{\mathbf{x} + \lambda e_i + \mu e_j, 0 \leq \lambda, \mu \leq 1\}.$$

Let then  $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}^2$  be the linear map defined by:

$$\Psi(x_1, \dots, x_d) = \sum_{i=1}^d x_i \mathbf{v}_i.$$

We are now in a position to introduce so-called *stepped surfaces*:

**Definition 3.** A  $d \rightarrow 2$  stepped surface is a set  $\mathcal{S}$  of unit faces of  $\mathbb{R}^d$  such that  $\Psi$  is a homeomorphism from the union of these unit faces onto  $\mathbb{R}^2$ .

A stepped surface is thus a sort of fairly rugged subset of  $\mathbb{R}^d$  homeomorphic to a plane. Rhombus tilings and stepped surfaces turn out to be naturally connected:

**Proposition 1.** If  $\mathcal{S}$  is a  $d \rightarrow 2$  stepped surface, then  $\Psi(\mathcal{S})$  is a  $d \rightarrow 2$  rhombus tiling. Conversely, if  $\mathcal{T}$  is a  $d \rightarrow 2$  rhombus tiling, then there is a  $d \rightarrow 2$  stepped surface  $\mathcal{S}$  such that  $\Psi(\mathcal{S}) = \mathcal{T}$ , and  $\mathcal{S}$  is unique up to a translation in  $\ker(\Psi) \cap \mathbb{Z}^d$ .

*Proof.* Let  $\mathcal{S}$  be a stepped surface. First,  $\Psi$  clearly maps unit faces onto rhombus tiles whose vertices belong to  $\oplus_i \mathbb{Z} \mathbf{v}_i$ . Then, note that unit faces are of disjoint interiors and meet edge-to-edge: this still holds by applying the homeomorphism  $\Psi$ . Last,  $\Psi$  is onto  $\mathbb{R}^2$ . This shows that  $\Psi(\mathcal{S})$  is a rhombus tiling of  $\mathbb{R}^2$ .

Conversely, let  $\mathcal{T}$  be a rhombus tiling of  $\mathbb{R}^2$ . Let  $\mathbf{x}_0$  be a vertex of  $\mathcal{T}$ . Since  $\mathbf{x}_0 \in \oplus_i \mathbb{Z} \mathbf{v}_i$  (by definition), there is some  $\mathbf{y}_0 \in \mathbb{Z}^d$  such that  $\Psi(\mathbf{y}_0) = \mathbf{x}_0$ , and  $\mathbf{y}_0$  is unique up to a translation in  $\ker(\Psi) \cap \mathbb{Z}^d$ . One then define a function  $h$  from the vertices of  $\mathcal{T}$  to  $\mathbb{Z}^d$  as follows:

$$h(\mathbf{x}_0) = \mathbf{y}_0 \quad \text{and} \quad \mathbf{x}' = \mathbf{x} + \mathbf{v}_i \Rightarrow h(\mathbf{x}') = h(\mathbf{x}) + e_i.$$

Actually,  $h$  is nothing but a *height function*, and is thus consistent (see *e.g.* [4]). Here, note that  $\Psi(h(\mathbf{x})) = \mathbf{x}$  for any vertex  $\mathbf{x}$  of  $\mathcal{T}$ , and let us define the following set of unit faces:

$$\mathcal{S} = \{(h(\mathbf{x}), t_{ij}) \mid \mathbf{x} + T_{ij} \in \mathcal{T}\}.$$

It follows from the construction of  $\mathcal{S}$  that the restriction of  $\Psi$  to the union of unit faces of  $\mathcal{S}$ , denoted by  $\Psi|_{\mathcal{S}}$ , is a bijection onto  $\mathbb{R}^2$ . It is continuous as  $\Psi$  does, and its inverse is also continuous since  $\Psi|_{\mathcal{S}}$  is closed. Thus,  $\Psi$  is a homeomorphism from  $\mathcal{S}$  onto  $\mathbb{R}^2$ , that is,  $\mathcal{S}$  is a stepped surface. Last,  $\mathcal{S}$  is unique up to the initial choice of  $\mathbf{y}_0$ , that is, up to a translation in  $\ker(\Psi) \cap \mathbb{Z}^d$ .  $\square$

In other words, stepped surfaces are nothing but rhombus tilings seen from a higher-dimensional viewpoint. Actually, this is just a generalization of ideas

introduced in [15] for finite domino or lozenge tilings. Note also that the case  $d = 3$  corresponds to the notion introduced in [8], where the 3-dimensional viewpoint is very natural (see, for example, the leftmost tiling of Fig. 1).

The notion of flip is then defined over stepped surfaces so that if a stepped surface  $\mathcal{S}'$  is obtained by performing a flip on a stepped surface  $\mathcal{S}$ , then the rhombus tiling  $\Psi(\mathcal{S}')$  is obtained by performing a flip on the rhombus tiling  $\Psi(\mathcal{S})$  (it suffices to replace  $v_i$  by  $e_i$  on Fig. 3). If, moreover, one says that two stepped surfaces  $\mathcal{S}$  and  $\mathcal{S}'$  are at distance less than  $2^{-r}$  if they share the same set of unit faces within the  $d$ -dimensional ball  $B(\mathbf{0}, r)$ , then this leads to a notion of flip-accessibility for stepped surfaces which satisfies:

**Proposition 2.** *For two stepped surfaces  $\mathcal{S}$  and  $\mathcal{S}'$ , one has:*

$$\Psi(\mathcal{S}) \xrightarrow{\text{flips}} \Psi(\mathcal{S}') \Leftrightarrow \exists \mathbf{a} \in \ker(\Psi) \cap \mathbb{Z}^d \text{ s.t. } \mathcal{S} \xrightarrow{\text{flips}} \mathbf{a} + \mathcal{S}',$$

where  $\mathbf{a} + \mathcal{S}'$  denotes the stepped surface obtained by translating  $\mathcal{S}'$  by  $\mathbf{a}$ .

Fig. 4 illustrates the notion of flip-accessibility. Note that, contrarily to the case of rhombus tilings of a polygon, flip-accessibility does not always holds, and is moreover even not symmetric.

## 2 Characterization by shadows

The aim of this section is to provide a characterization of flip-accessibility for stepped surfaces (which can be then restated in terms of rhombus tilings according to Prop. 1 and 2). Let us first define the following maps, for  $1 \leq i < j \leq d$ :

$$\pi_{ij} : \begin{array}{ccc} \mathbb{R}^d & \rightarrow & \mathbb{R}^2 \\ (z_1, \dots, z_d) & \mapsto & (z_i, z_j) \end{array}$$

In particular,  $\pi_{ij}$  maps the unit face  $(\mathbf{x}, t_{kl})$  onto a unit square if  $i = k$  and  $j = l$ , onto a unit segment if  $i = k$  or  $j = l$  and onto a point otherwise. We then use these maps to define the *shadows* of a stepped surface (see *e.g.* Fig. 4):

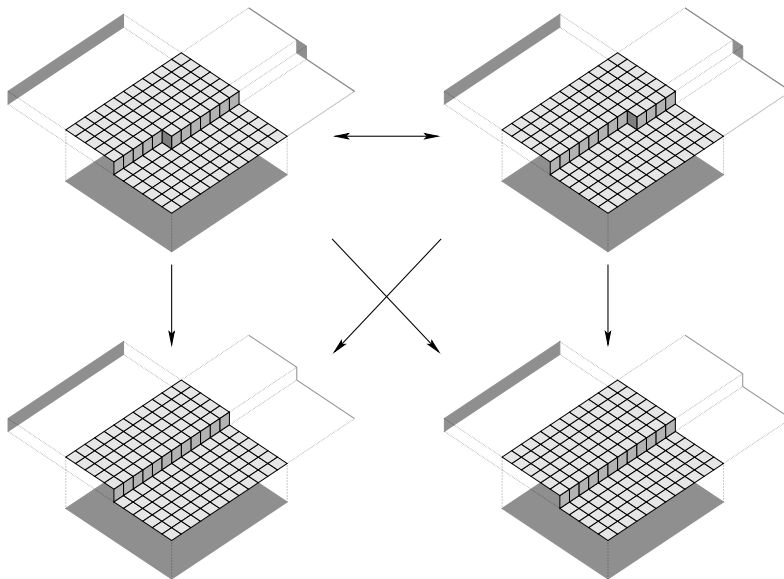
**Definition 4.** *The shadows of a  $d \rightarrow 2$  stepped surface  $\mathcal{S}$  are the  $\binom{d}{2}$  subsets of  $\mathbb{R}^2$  defined, for  $1 \leq i < j \leq d$ , by:*

$$\pi_{ij}(\mathcal{S}) = \bigcup_{(\mathbf{x}, t) \in \mathcal{S}} \pi_{ij}(\mathbf{x}, t).$$

A simple but fundamental property of shadows is that they are invariant by performing a flip (this can be easily checked on Fig. 3). This also holds for finite sequences of flips, but we have only a weaker property for infinite sequences:

**Proposition 3.** *If a stepped surface  $\mathcal{S}'$  is flip-accessible from a stepped surface  $\mathcal{S}$ , then the shadows of  $\mathcal{S}'$  are included in the shadows of  $\mathcal{S}$ :*

$$\mathcal{S} \xrightarrow{\text{flips}} \mathcal{S}' \Rightarrow \forall i, \forall j, \pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S}).$$



**Fig. 4.** Four patches of  $3 \rightarrow 2$  stepped surfaces and their shadows (see Def. 4, below). Flip-accessibility is represented by arrows: the top two stepped surfaces are mutually flip-accessible (by a finite sequence of flips), and the bottom two stepped surfaces are flip-accessible from them (by an infinite sequence of flips rejecting the “corner” to infinity in one of the two possible directions). The bottom two stepped surfaces are sort of *dead ends*: no flip can be performed on them. It is worth noticing that a stepped surface is flip-accessible from another one if and only if the shadows of the latter are included in the shadows of the former (this illustrates Th. 1, below).

*Proof.* Let  $\mathcal{S}_n$  be a sequence of stepped surfaces, obtained by performing flips on  $\mathcal{S}$ , which tends towards  $\mathcal{S}'$ . Let  $z \in \pi_{ij}(\mathcal{S}')$ :  $z$  belongs to the projection of a face  $(\mathbf{x}, t) \in \mathcal{S}'$ . Let  $r \in \mathbb{R}$  such that  $(\mathbf{x}, t) \subset B(0, r)$  and  $N \in \mathbb{N}$  such that  $d(\mathcal{S}_N, \mathcal{S}') \leq 2^{-r}$ . In particular,  $(\mathbf{x}, t) \in \mathcal{S}_N$ . Since  $\mathcal{S}_N$  is obtained from  $\mathcal{S}$  by performing a finite number of flips, both have the same shadows. Thus,  $z \in \pi_{ij}(\mathbf{x}, t) \subset \pi_{ij}(\mathcal{S}_N)$  yields  $z \in \pi_{ij}(\mathcal{S})$ . This proves  $\pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S})$ .  $\square$

In the previous proposition, inclusions of shadows can be strict (see, for example, Fig. 4). Actually, the main result of this paper is that the converse of this proposition also holds:

**Theorem 1.** *A stepped surface  $\mathcal{S}'$  is flip-accessible from a stepped surface  $\mathcal{S}$  iff the shadows of  $\mathcal{S}'$  are included in the shadows of  $\mathcal{S}$ :*

$$\mathcal{S} \xrightarrow{\text{flips}} \mathcal{S}' \Leftrightarrow \forall i, \forall j, \pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S}).$$

Th. 1 is proven in the following section. Before this, let us provide an interesting corollary. We need the following definition:

**Definition 5.** Let  $\mathbf{u}$  and  $\mathbf{v}$  be two vectors of  $\mathbb{R}^d$  with non-zero entries. The  $d \rightarrow 2$  stepped plane  $\mathcal{P}_{\mathbf{u},\mathbf{v}}$  is defined as the set of all unit faces which lie (entirely) in the following “slice” of  $\mathbb{R}^d$ :

$$\mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{v} + [0, 1]^d.$$

Roughly speaking, the stepped plane  $\mathcal{P}_{\mathbf{u},\mathbf{v}}$  is an approximation by unit faces of the real plane  $\mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{v}$  (this corresponds to a viewpoint developed in discrete geometry, see *e.g.* [12]). Actually, stepped planes are nothing but the stepped surfaces which are associated by Prop. 1 with so-called *canonical projection tilings*. These are rhombus tilings obtained by the *cut and project* method (see [7, 13]). For example, the Rauzy-dual, Ammann-Beenker and Penrose tilings depicted on Fig. 1 are canonical projection tilings associated with  $d \rightarrow 2$  stepped planes for, respectively,  $d = 3, 4, 5$  (see [6]).

Now, let us note that  $\pi_{ij}(\mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{v}) = \mathbb{R}^2$ . This easily yields that  $\pi_{ij}(\mathcal{P}_{\mathbf{u},\mathbf{v}}) = \mathbb{R}^2$ . In particular, the shadows of the stepped plane  $\mathcal{P}_{\mathbf{u},\mathbf{v}}$  contain the shadows of any other stepped surface. We thus obtain as an immediate corollary of Th. 1:

**Corollary 1.** *Any stepped surface is flip-accessible from a stepped plane.*

In terms of rhombus tilings, this means that any rhombus tiling is flip-accessible from a canonical projection tiling over the same set of rhombus tiles.

### 3 Proof of the characterization

This section provides a proof of the characterization stated in Theorem 1. The necessary condition is proven by Prop. 3. Let thus  $\mathcal{S}$  and  $\mathcal{S}'$  be two stepped surfaces such that the shadows of  $\mathcal{S}'$  are included in the shadows of  $\mathcal{S}$ , and let us prove that  $\mathcal{S}'$  is flip-accessible from  $\mathcal{S}$ .

Since the proof is not so short, it is worth giving a brief outline. The general idea is to transform  $\mathcal{S}$  into  $\mathcal{S}'$  by moving one by one unit faces. More precisely, for  $(\mathbf{x}', t_{ij}) \in \mathcal{S}'$ , inclusion of shadows ensure that there is a unit face  $(\mathbf{x}, t_{ij}) \in \mathcal{S}$  such that  $\pi_{ij}(\mathbf{x}', t_{ij}) = \pi_{ij}(\mathbf{x}, t_{ij})$ . We would like to move  $(\mathbf{x}, t_{ij})$  to  $(\mathbf{x}', t_{ij})$ . We proceed as follows. While there is  $k$  such that  $x_k < x'_k$ , we choose such a  $k$  and we define a set  $F_k^*(\mathbf{x}, t_{ij})$  such that, by performing a finite number of flips over this set, we can translate  $(\mathbf{x}, t_{ij})$  by  $\mathbf{e}_k$  (Lem. 1, 2 and 3). Similarly, we can translate  $(\mathbf{x}, t_{ij})$  by  $-\mathbf{e}_k$  for  $k$  such that  $x_k > x'_k$ . Hence, we can move  $(\mathbf{x}, t_{ij}) \in \mathcal{S}$  to  $(\mathbf{x}', t_{ij}) \in \mathcal{S}'$  by performing a finite number of flips. The last step will be to show that we can, in this way, obtain unit faces of  $\mathcal{S}'$  over growing balls centered in  $\mathbf{0}$  (Lem. 4), that is, that  $\mathcal{S}'$  is flip-accessible from  $\mathcal{S}$  (see Def. 2).

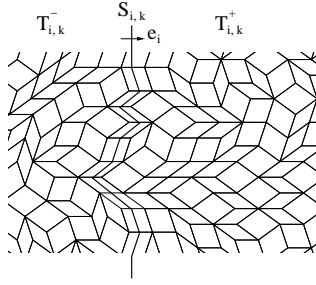
Let us now start the proof. We first define a useful tool:

**Definition 6.** Let  $\mathcal{S}$  be a stepped surface,  $k \in \mathbb{Z}$  and  $1 \leq i \leq d$ . If not empty, the following set of unit faces defines the  $k$ -th de Bruijn section of type  $i$  of  $\mathcal{S}$ :

$$S_{i,k} = \{((x_1, \dots, x_d), t_{ij}) \in \mathcal{S} \mid x_i = k\}.$$

It is easily seen that  $S_{i,k}$  is an infinite stripe of unit faces two by two adjacent along vectors  $e_i$ . Then, removing  $S_{i,k}$  naturally splits  $\mathcal{S}$  into the two following connected sets of unit faces (see Fig. 5):

$$T_{i,k}^+ = \{((x_1, \dots, x_d), t) \in \mathcal{S} \mid x_i > k\} \quad \text{and} \quad T_{i,k}^- = \mathcal{S} \setminus (S_{i,k} \cup T_{i,k}^+).$$



**Fig. 5.** A de Bruijn section  $S_{i,k}$ , here represented by a broken line crossing its unit faces, splits a stepped surface into two connected sets of unit faces,  $T_{i,k}^-$  and  $T_{i,k}^+$ .

Actually, de Bruijn sections turn out to be the set of unit faces associated by Prop. 1 with the well-known de Bruijn lines introduced in [3]. In other words,  $S_{i,k}$  is a de Bruijn section of  $\mathcal{S}$  iff  $\Psi(S_{i,k})$  is a de Bruijn line of the rhombus tiling  $\Psi(\mathcal{S})$ . In particular, two de Bruijn sections share at most one face, as well as de Bruijn lines. In such a case, they are said to *intersect*. Note that, if  $(\mathbf{x}, t_{kl}) = S_{i,n} \cap S_{j,m}$ , then  $k = i$ ,  $l = j$ ,  $x_i = n$  and  $x_j = m$ . In particular, only sections of different types can intersect, although they can also not intersect.

We use de Bruijn sections to define so-called *de Bruijn triangles*:

**Definition 7.** For  $(\mathbf{x} = (x_1, \dots, x_d), t_{ij}) \in \mathcal{S}$  and  $1 \leq k \leq d$ ,  $k \neq i$ ,  $k \neq j$ , the de Bruijn triangle  $F_k(\mathbf{x}, t_{ij})$  is the set of unit faces of  $\mathcal{S}$  defined by:

$$F_k(\mathbf{x}, t_{ij}) = (S_{i,x_i} \cup T_{i,x_i}^{\varepsilon_i}) \cap (S_{j,x_j} \cup T_{j,x_j}^{\varepsilon_j}) \cap (S_{k,x_k} \cup T_{k,x_k}^-),$$

where  $\varepsilon_i$  and  $\varepsilon_j$  respectively denote the signs of entries of  $\mathbf{v}_k$  in the basis  $(\mathbf{v}_i, \mathbf{v}_j)$ .

Roughly speaking,  $F_k(\mathbf{x}, t_{ij})$  is the triangle defined by the three “lines”  $S_{i,x_i}$ ,  $S_{j,x_j}$  and  $S_{k,x_k}$  (see Fig. 6, left). Note that it could be infinite, since the de Bruijn sections  $S_{i,x_i}$  or  $S_{j,x_j}$  do not necessarily intersect  $S_{k,x_k}$ . We will later

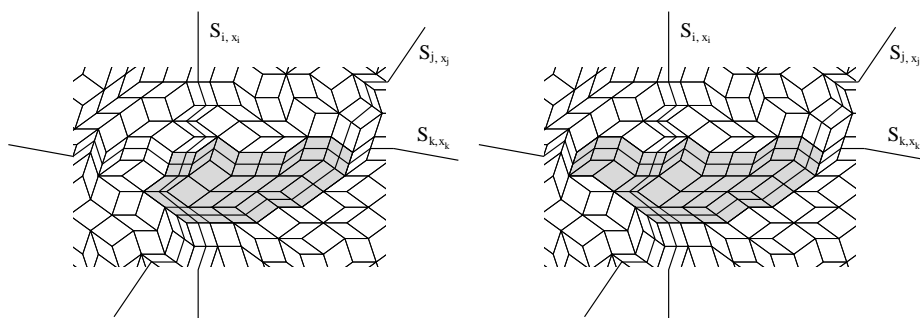
avoid this case (Lem. 3). Intuitively, for translating  $(\mathbf{x}, t_{ij})$  by  $e_k$ , we first need to translate by  $e_k$  the unit faces in  $F_k(\mathbf{x}, t_{ij})$ . However, moving a unit face of  $F_k(\mathbf{x}, t_{ij})$  requires, in turn, to move some others unit faces before. Therefore, we extend de Bruijn triangles by so-called de Bruijn cones (see also Fig. 6, right):

**Definition 8.** *With the convention  $F_k(A \cup B) = F_k(A) \cup F_k(B)$ , we define:*

$$F_k^0(\mathbf{x}, t_{ij}) = (\mathbf{x}, t_{ij}) \quad \text{and} \quad F_k^{n+1}(\mathbf{x}, t_{ij}) = F_k(F_k^n(\mathbf{x}, t_{ij})).$$

*Then, the de Bruijn cone  $F_k^*(\mathbf{x}, t_{ij})$  is defined by:*

$$F_k^*(\mathbf{x}, t_{ij}) = \bigcup_{n \geq 0} F_k^n(\mathbf{x}, t_{ij}).$$



**Fig. 6.** A de Bruijn triangle  $F_k(\mathbf{x}, t_{ij})$  (the shaded unit faces, left) and its closure, the de Bruijn cone  $F_k^*(\mathbf{x}, t_{ij})$  (right). Recall that one has always  $(\mathbf{x}, t_{ij}) = S_{i,x_i} \cap S_{j,x_j}$ .

Let us now show that  $(\mathbf{x}, t_{ij})$  can be translated by performing flips over  $F_k^*(\mathbf{x}, t_{ij})$ :

**Lemma 1.** *If  $F_k^*(\mathbf{x}, t_{ij})$  is finite, then one can translate  $(\mathbf{x}, t_{ij})$  by  $e_k$  by performing  $\text{card}(F_k^*(\mathbf{x}, t_{ij}) \setminus S_{k,x_k})$  flips over  $F_k^*(\mathbf{x}, t_{ij})$ .*

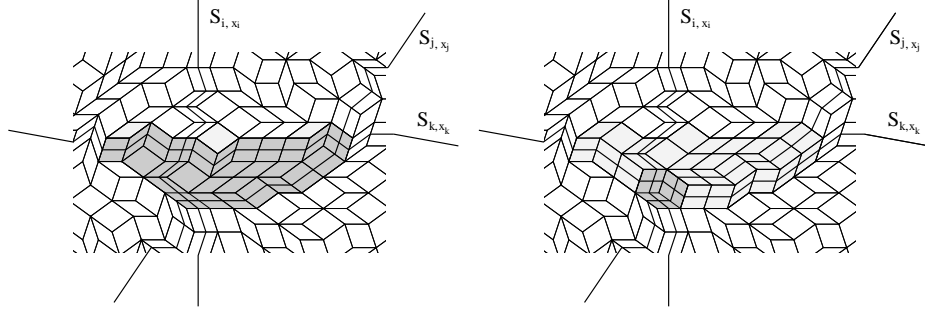
*Proof.* Def. 8 yields, for any unit faces  $(\mathbf{y}, t)$  and  $(\mathbf{y}', t')$ :

$$(\mathbf{y}, t) \in F_k^*(\mathbf{y}', t') \Rightarrow F_k^*(\mathbf{y}, t) \subset F_k^*(\mathbf{y}', t').$$

This naturally leads to define the following partial order over  $F_k^*(\mathbf{x}, t_{ij})$ :

$$\forall (\mathbf{y}, t), (\mathbf{y}', t') \in F_k^*(\mathbf{x}, t_{ij}), \quad (\mathbf{y}, t) \preceq (\mathbf{y}', t') \Leftrightarrow F_k^*(\mathbf{y}, t) \subset F_k^*(\mathbf{y}', t').$$

Let us now consider a unit face  $(\mathbf{y}, t) \in F_k^*(\mathbf{x}, t_{ij}) \setminus S_{k,x_k}$  which is minimal for this order. It is not hard to check that  $F_k^*(\mathbf{y}, t)$  is a set of three unit faces on which a flip can be performed (see, for example, Fig. 6, right). By performing this flip,  $(\mathbf{y}, t)$  is translated by  $e_k$ , so that the obtained face does no more belongs to  $F_k^*(\mathbf{x}, t_{ij})$ , which thus decreased (Fig. 7, left). This can be inductively repeated, up to translate by  $e_k$  the unit face which was originally maximal in  $F_k^*(\mathbf{x}, t_{ij})$ , that is,  $(\mathbf{x}, t_{ij})$  itself (Fig. 7, right). Since there is one flip performed for each translated unit face, there is a total of  $\text{card}(F_k^*(\mathbf{x}, t_{ij}) \setminus S_{k,x_k})$  flips performed.  $\square$



**Fig. 7.** Three flips have been performed on the minimal elements of the de Bruijn cone of Fig. 6 (left). This can be repeated, reducing the de Bruijn cone up to only three unit faces (right), on which performing a flip will translate the unit face  $(\mathbf{x}, t_{ij})$  by  $e_k$ .

Although the definition of de Bruijn cones by transitive closure suffices to prove the previous lemma, the following stronger property actually holds:

**Lemma 2.** *One has  $F_k^*(\mathbf{x}, t_{ij}) = F_k^2(\mathbf{x}, t_{ij})$ .*

*Proof.* Let  $(\mathbf{y}, t) \in F_k^2(\mathbf{x}, t_{ij})$ . If  $F_k(\mathbf{y}, t)$  is not included in  $F_k^2(\mathbf{x}, t_{ij})$ , then a case study (relying on the fact that two de Bruijn sections intersect at most once) shows that one of the two de Bruijn sections containing  $(\mathbf{y}, t)$ , say  $S_{k', y_{k'}}$ , necessarily intersects  $F_k(\mathbf{x}, t_{ij})$ . Let thus  $(\mathbf{y}', t') \in S_{k', y_{k'}} \cap F_k(\mathbf{x}, t_{ij})$ . One has  $F_k(\mathbf{y}, t) \subset F_k(\mathbf{y}', t')$ , and  $(\mathbf{y}', t') \in F_k(\mathbf{x}, t_{ij})$  yields  $F_k(\mathbf{y}', t') \subset F_k^2(\mathbf{x}, t_{ij})$ . Hence,  $F_k(\mathbf{y}, t) \subset F_k^2(\mathbf{x}, t_{ij})$ . Since this holds for any  $(\mathbf{y}, t) \in F_k^2(\mathbf{x}, t_{ij})$ , this proves  $F_k^3(\mathbf{x}, t_{ij}) \subset F_k^2(\mathbf{x}, t_{ij})$ . The result follows.  $\square$

We are now in a position to prove that one can choose  $k_0$  such that  $F_{k_0}^*(\mathbf{x}, t_{ij})$  is finite and  $(\mathbf{x}, t_{ij})$  should be translated by  $e_{k_0}$  (the condition  $k_0 \in D$  below). Lem. 1 then yields that  $(\mathbf{x}, t_{ij})$  can be effectively translated by  $e_{k_0}$ .

**Lemma 3.** *Let  $(\mathbf{x}', t_{ij}) \in \mathcal{S}'$  and  $(\mathbf{x}, t_{ij}) \in \mathcal{S}$  such that  $\pi_{ij}(\mathbf{x}', t_{ij}) = \pi_{ij}(\mathbf{x}, t_{ij})$ . If  $D = \{k \mid x'_k > x_k\} \neq \emptyset$ , then there is  $k_0 \in D$  such that  $F_{k_0}^*(\mathbf{x}, t_{ij})$  is finite.*

*Proof.* We first prove that  $F_k(\mathbf{x}, t_{ij})$  is finite for any  $k \in D$ , and then that there is  $k_0 \in D$  such that  $F_{k_0}^*(\mathbf{x}, t_{ij}) = F_{k_0}^2(\mathbf{x}, t_{ij})$  is finite.

Let  $k \in D$ . Note that  $F_k(\mathbf{x}, t_{ij})$  is finite iff both  $S_{i, x_i}$  and  $S_{j, x_j}$  intersect  $S_{k, x_k}$ . Suppose that  $S_{i, x_i}$  does not intersect  $S_{k, x_k}$ . Thus,  $S_{i, x_i} \subset T_{k, x_k}^-$ . Then, since the shadows of  $\mathcal{S}'$  are included in the shadows of  $\mathcal{S}$ , there is  $(\mathbf{z}, t) \in \mathcal{S}$  such that  $\pi_{ik}(\mathbf{x}') \in \pi_{ik}(\mathbf{z}, t)$ . This yields  $z_i = x'_i = x_i$  and  $z_k = x'_k > x_k$ . In particular,  $\mathbf{z} \in S_{i, x_i} \cap T_{k, x_k}^+$ . Since this contradicts  $S_{i, x_i} \subset T_{k, x_k}^-$ , we deduce that  $S_{i, x_i}$  intersects  $S_{k, x_k}$ . Similarly,  $S_{j, x_j}$  intersects  $S_{k, x_k}$ . The first result is proven.

Let us now choose  $k_0 \in D$  being minimal in  $D$  for the following partial order:

$$n \preceq m \Leftrightarrow T_{m, x_m}^+ \subset T_{n, x_n}^+.$$

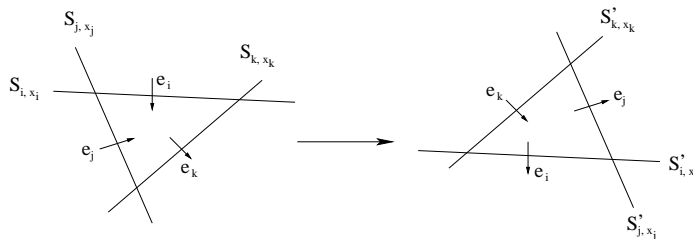
In other words,  $k_0$  is chosen such that there is no section  $S_{k,x_k}$  separating  $(\mathbf{x}, t_{ij})$  from  $S_{k_0,x_{k_0}}$ , that is, such that  $(\mathbf{x}, t_{ij}) \in T_{k,x_k}^-$  and  $S_{k_0,x_{k_0}} \subset T_{k,x_k}^+$ . This yields that a unit face  $(\mathbf{y}, t)$  of  $F_{k_0}(\mathbf{x}, t_{ij})$  belongs to two de Bruijn sections which both intersect  $S_{k_0,x_{k_0}}$ . Thus,  $F_k(\mathbf{y}, t)$  is finite. The second result follows.  $\square$

Note that the previous lemma only proves that *there is*  $k_0 \in D$  such that one can (and should) translate  $(\mathbf{x}, t_{ij})$  by  $e_{k_0}$ . Actually, one can easily check that, for  $d = 3$ , *any*  $k \in D$  is convenient, whereas this is no more true for  $d > 3$ . Without going into details, let us just say that it is strongly connected with the fact that the set of  $d \rightarrow 2$  rhombus tilings of a polygon forms a *distributive* lattice for  $d = 3$ , whereas not for  $d > 3$  (see [5, 11]).

So, following the outline given at the beginning of this section, we can now, by performing flips, translate  $(\mathbf{x}, t_{ij})$  by some  $e_{k_0}$  such that  $x'_{k_0} > x_{k_0}$ . We can repeat this up to have  $x'_k \leq x_k$  for any  $k$ . The way we can translate by  $-e_{k_0}$  a unit face  $(\mathbf{x}, t_{ij})$  such that  $x'_{k_0} < x_{k_0}$  is similar. So, we are able to move  $(\mathbf{x}, t_{ij})$  to  $(\mathbf{x}', t_{ij})$ . The end of the proof relies on the following lemma:

**Lemma 4.** *Let  $(\mathbf{x}', t_{ij}) \in \mathcal{S}'$  and  $(\mathbf{x}, t_{ij}) \in \mathcal{S}$  such that  $\pi_{ij}(\mathbf{x}', t_{ij}) = \pi_{ij}(\mathbf{x}, t_{ij})$ . If  $x'_k > x_k$ , then  $F_k^*(\mathbf{x}, t_{ij}) \cap \mathcal{S}' = \emptyset$ .*

*Proof.* (sketch) Writing down a detailed proof is rather technical and obfuscating, but the underlying geometrical idea is quite easy. Indeed,  $x'_k > x_k$  yields  $(\mathbf{x}, t_{ij}) \in T_{k,x_k}^-$  and  $(\mathbf{x}', t_{ij}) \in T_{k,x_k}^+$ , as depicted on Fig. 8. So, suppose that there is a unit face  $(\mathbf{y}, t) \in F_k(\mathbf{x}, t_{ij}) \cap \mathcal{S}'$ . Such a face thus should have the same position, in  $\mathcal{S}$  and  $\mathcal{S}'$ , relatively to any de Bruijn section. For example, if  $(\mathbf{y}, t)$  belongs to  $T_{i,x_i}^+ \cap T_{j,x_j}^+ \cap T_{k,x_k}^-$  in  $\mathcal{S}$  (as in the case of Fig. 8, left), then it should belong to  $T_{i,x_i}^+ \cap T_{j,x_j}^+ \cap T_{k,x_k}^-$  in  $\mathcal{S}'$ . However, this last set turns out to be empty (see Fig. 8, right). Thus,  $F_k(\mathbf{x}, t_{ij}) \cap \mathcal{S}' = \emptyset$ . Suppose now that  $(\mathbf{y}, t) \in F_k^2(\mathbf{x}, t_{ij}) \cap \mathcal{S}'$ . There is  $(\mathbf{z}, t_z) \in F_k(\mathbf{x}, t_{ij})$  such that  $(\mathbf{y}, t) \in F_k(\mathbf{z}, t_z)$ . We prove  $F_k(\mathbf{z}, t_z) \cap \mathcal{S}' = \emptyset$  as above, with  $(\mathbf{z}, t_z)$  instead of  $(\mathbf{x}, t_{ij})$ .  $\square$



**Fig. 8.** If  $(\mathbf{x}, t_{ij})$  must cross the section  $S_{k,x_k}$  to be transformed to  $(\mathbf{x}', t_{ij})$ , then any unit face inside the triangle  $T_{i,x_i}^+ \cap T_{j,x_j}^+ \cap T_{k,x_k}^-$  must also cross one of the sections  $S_{i,x_i}$ ,  $S_{j,x_j}$  or  $S_{k,x_k}$ , hence is moved.

This lemma ensures that, once a unit face of  $\mathcal{S}'$  is obtained, it is no more moved. We thus can get unit faces of  $\mathcal{S}'$  over growing balls, and Th. 1 follows. We end the paper by summing up the whole proof by the following pseudo-algorithm:

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for r=0 to  $\infty$ 
  while  $\mathcal{S}_{B(\mathbf{0},r)} \neq \mathcal{S}'_{B(\mathbf{0},r)}$ 
    choose  $(\mathbf{x}, t_{ij})$  in  $\mathcal{S}_{B(\mathbf{0},r)} \setminus \mathcal{S}'_{B(\mathbf{0},r)}$ 
     $(\mathbf{x}', t_{ij}) \leftarrow S'_{i,x_i} \cap S'_{j,x_j}$  ( $\pi_{ij}(\mathcal{S}') \subset \pi_{ij}(\mathcal{S})$ )
    while  $\mathbf{x} \neq \mathbf{x}'$ 
      choose  $k$  s.t.  $x_k \neq x'_k$  and  $F_k^*(\mathbf{x}, t_{ij})$  is finite (Lem. 3)
       $x_k \leftarrow x_k \pm 1$  by performing flips over  $F_k^*(\mathbf{x}, t_{ij})$  (Lem. 1)
    endwhile
  endwhile (Lem. 4)
endfor

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