

Characterizing path graphs by forbidden induced subgraphs

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April 17, 2008

Abstract

A path graph is the intersection graph of subpaths of a tree. In 1970, Renz asked for a characterization of path graphs by forbidden induced subgraphs. We answer this question by determining the complete list of graphs that are not path graphs and are minimal with this property.

1 Introduction

All graphs considered here are finite and have no parallel edges and no loop. A *hole* is a chordless cycle of length at least four. A graph is *chordal* (or *triangulated*) if it contains no hole as an induced subgraph. Gavril [6] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree. In this paper, whenever we talk about the intersection of subgraphs of a graph we mean that the *vertex sets* of the subgraphs intersect.

An *interval graph* is the intersection graph of a family of intervals on the real line; equivalently, it is the intersection graph of a family of subpaths of a path. An *asteroidal triple* in a graph G is a set of three non adjacent vertices such that for any two of them, there exists a path between them in G that does not intersect the neighborhood of the third. Lekkerkerker and Boland [11] proved that a graph is an interval graph if and only if it is chordal and contains no asteroidal triple. They derived from this result the list of minimal forbidden subgraphs for interval graphs.

An intermediate class is the class of path graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. Clearly, the class of path graphs is included in the class of chordal graphs and contains the class of interval graphs. Several characterizations of path graphs have been given [7, 13, 15] but no characterization by forbidden subgraphs was known, whereas such results exist for intersection graphs of subpaths of a path (interval graphs [11]), subtrees of a tree (chordal graphs [6]), and also for directed subpaths of a directed tree [14].

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In 1970, Renz [15] asked for a complete list of graphs that are chordal and not path graphs and are minimal with this property, and he gave two examples of such graphs. Reference [19] extends the list of minimal forbidden subgraphs for path graphs; but that list is incomplete. Here we answer Renz's question and obtain a characterization of path graphs by forbidden induced subgraphs. We will prove that the graphs presented in Figures 1–5 are all the minimal non-path graphs. In other words:

Theorem 1 *A graph is a path graph if and only if it does not contain any of F_0, \dots, F_{16} as an induced subgraph.*

2 Special simplicial vertices in chordal graphs

In a graph G , a *clique* is set of pairwise adjacent vertices. Let $\mathcal{Q}(G)$ be the set of all (inclusionwise) maximal cliques of G . When there is no ambiguity we will write \mathcal{Q} instead of $\mathcal{Q}(G)$.

Given two vertices u, v in a graph G , a $\{u, v\}$ -separator is a set S of vertices of G such that u and v lie in two different components of $G \setminus S$ and S is minimal with this property. A set is a *separator* if it is a $\{u, v\}$ -separator for some u, v in G . Let $\mathcal{S}(G)$ be the set of separators of G . When there is no ambiguity we will write \mathcal{S} instead of $\mathcal{S}(G)$.

The neighborhood of a vertex v is the set $N(v)$ of vertices adjacent to v . Let us say that a vertex u is *complete* to a set X of vertices if $X \subseteq N(u)$. A vertex is *simplicial* if its neighborhood is a clique. It is easy to see that a vertex is simplicial if and only if it does not belong to any separator. Given a simplicial vertex v , let $Q_v = N(v) \cup \{v\}$ and $S_v = Q_v \cap N(V \setminus Q_v)$. Since v is simplicial, we have $Q_v \in \mathcal{Q}$. Remark that S_v is not necessarily in \mathcal{S} ; for example, in the graph H with vertices a, b, c, d, e and edges ab, bc, cd, de, bd , we have $S_c = \{b, d\}$ and $\mathcal{S}(H) = \{\{b\}, \{d\}\}$.

A classical result [10, 1] (see also [8]) states that, in a chordal graph G , every separator is a clique; moreover, if S is a separator, then there are at least two components of $G \setminus S$ that contain a vertex that is complete to S , and so S is the intersection of two maximal cliques.

A *clique tree* T of a graph G is a tree whose vertices are the members of \mathcal{Q} and such that, for each vertex v of G , those members of \mathcal{Q} that contain v induce a subtree of T , which we will denote by T^v . A classical result [6] states that a graph is chordal if and only if it has a clique tree.

For a clique tree T , the *label* of an edge QQ' of T is defined as $S_{QQ'} = Q \cap Q'$. Note that every edge QQ' satisfies $S_{QQ'} \in \mathcal{S}$; indeed, there exist vertices $v \in Q \setminus Q'$ and $v' \in Q' \setminus Q$, and the set $S_{QQ'}$ is a $\{v, v'\}$ -separator. The number of times an element S of \mathcal{S} appears as a label of an edge is equal to $c - 1$, where c is the number of components of $G \setminus S$ that contain a vertex complete to S [6, 12]. Note that this number is at least one and that it depends only on S and not on T , so for a given $S \in \mathcal{S}$ it is the same in every clique tree.

Given $X \subseteq \mathcal{Q}$, let $G(X)$ denote the subgraph of G induced by all the vertices that

appear in members of X . If T is a clique tree of G , then $T[X]$ denotes the subtree of T of minimum size whose vertices contains X . Note that if $|X| = 2$, then $T[X]$ is a path.

Given a subtree T' of a clique-tree T of G , let $\mathcal{Q}(T')$ be the set of vertices of T' and $\mathcal{S}(T')$ be the set of separators of $G(\mathcal{Q}(T'))$.

Dirac [5] proved that a chordal graph that is not a clique contains two non adjacent simplicial vertices. We need to generalize this theorem to the following. Let us say that a simplicial vertex v is *special* if S_v is a member of \mathcal{S} and is (inclusionwise) maximal in \mathcal{S} .

Theorem 2 *In a chordal graph that is not a clique, there exist two non adjacent special simplicial vertices.*

Proof. We prove the theorem by induction on $|\mathcal{Q}|$. By the hypothesis, G is not a clique, so $|\mathcal{Q}| \geq 2$ and $\mathcal{S} \neq \emptyset$.

Case 1: \mathcal{S} has only one maximal element S . Let Q, Q' be two maximal cliques such that $Q \cap Q' = S$. Let $v \in Q \setminus Q'$ and $v' \in Q' \setminus Q$. The set S is the only maximal separator and it does not contain v or v' . So v and v' do not belong to any element of \mathcal{S} , and so they are simplicial and $S_v = S_{v'} = S$, so they are special.

Case 2: \mathcal{S} has two distinct maximal elements S, S' . So $|\mathcal{Q}| \geq 3$. Let T be a clique tree of G . Let Q_1, Q_2, Q'_1, Q'_2 be members of \mathcal{Q} such that $S = S_{Q_1 Q_2}$, $S' = S_{Q'_1 Q'_2}$, and Q_2, Q_1, Q'_1, Q'_2 appear in this order along the path $T[Q_2, Q_1, Q'_1, Q'_2]$ (possibly $Q_1 = Q'_1$). Let Y be the subtree of $T \setminus Q_1$ that contains Q_2 , and let Z be the tree that consists of Y plus the vertex Q_1 and the edge $Q_1 Q_2$. The subtree Z does not contain Q'_2 , so $G(\mathcal{Q}(Z))$ has strictly fewer maximal cliques than G ; and G is not a clique. By the induction hypothesis, there exist two non adjacent simplicial vertices v, w of $G(\mathcal{Q}(Z))$ such that S_v, S_w are maximal elements of $\mathcal{S}(Z)$. At most one of v, w is in Q_1 since they are not adjacent, say v is not in Q_1 . We claim that v is a simplicial vertex of G and that S_v is a maximal element of \mathcal{S} . Vertex v does not belong to any element of $\mathcal{S}(Z)$. If it belongs to an element of $\mathcal{S} \setminus \mathcal{S}(Z)$, then it must also belong to $Q_1 \cap Q_2 = S \in \mathcal{S}(Z)$, a contradiction. So v does not belong to any element of \mathcal{S} and so it is a simplicial vertex of G . The set S_v is a maximal element of $\mathcal{S}(Z)$. If it is not a maximal element of \mathcal{S} , then it is included in $S \in \mathcal{S}(Z)$, a contradiction. So v is a special simplicial vertex of G . Likewise, let Y' be the subtree of $T \setminus Q'_1$ that contains Q'_2 , and let Z' be the tree that consists of Y' plus the vertex Q'_1 and the edge $Q'_1 Q'_2$. Just like with v , we can find a simplicial vertex v' of $G(\mathcal{Q}(Z'))$ not in Q'_1 that is a simplicial vertex of G with $S_{v'}$ being a maximal element of \mathcal{S} . Vertices v and v' are not adjacent since S is a $\{v, v'\}$ -separator. So v and v' are the desired vertices. \square

Algorithms LexBFS [16] and MCS [18] are linear time algorithms that were developed to find a simplicial vertex in a chordal graph. But a simplicial vertex found by these algorithms is not necessarily special. For example, on the graph with vertices a, b, c, d, e, f and edges $ab, bc, cd, eb, ec, fb, fc$, every application of LexBFS or MCS will end on one of simplicial vertices a, d , which are not special. The proof of Theorem 2 can be turned

into a polynomial time algorithm to find a special simplicial vertex in a chordal graph. We do not know how to find such a vertex in linear time.

3 Forbidden induced subgraphs

A *clique path tree* T of G is a clique tree of G such that, for each vertex v of G , the subtree T^v induced by cliques that contain v is a path. Gavril [7] proved a graph is a path graph if and only if it has a clique path tree.

Consider graphs F_0, \dots, F_{16} presented in Figures 1–5. Let us make a few remarks about them. Each graph in Figure 2 is obtained by adding a universal vertex to some minimal forbidden subgraph for interval graphs. Clearly, in a path graph the neighborhood of every vertex is an interval graph; so F_1, \dots, F_5 are not path graphs. Graphs $F_{10}(n)_{n \geq 8}$ are also forbidden in interval graphs. Graphs F_6 and $F_{10}(8)$ are from Renz [15, Figures 1 and 5]. For $i \in \{0, 1, 3, 4, 5, 6, 7, 9, 10, 13, 15, 16\}$, Panda [14] proved that F_i is a minimal non directed path graph, so $F_i \setminus x$ is a directed path graph for every vertex x (obviously every directed path graph is a path graph). In general we have the following:

Theorem 3 F_0, \dots, F_{16} are minimal non path graphs.

Proof. Clearly, F_0 is a minimal non path graph. For the other graphs, we prove the theorem in one case and then show how the same arguments can be applied to all cases.

Consider $F = F_{11}(4k)$, $k \geq 2$; see Figure 4. Name its vertices such that u_1, \dots, u_{2k-1} are the simplicial vertices of degree 2, clockwise; v_{j-1}, v_j are the two neighbors of u_j ($j = 1, \dots, 2k-1$), with subscripts modulo $2k-1$; and a, b are the remaining vertices. Let Q_j be the maximal clique that contains u_j ($j = 1, \dots, 2k-1$), and call these $2k-1$ cliques “peripheral”. Let $R_a = \{a, v_1, \dots, v_{2k-1}\}$ and $R_b = \{b, v_1, \dots, v_{2k-1}\}$ be the maximal cliques that contain respectively a and b , and call these two cliques “central”. Thus $\mathcal{Q}(F) = \{R_a, R_b, Q_1, \dots, Q_{2k-1}\}$. Since F is chordal, it admits a clique tree. Let T be any clique tree of F . Then R_a and R_b are adjacent in T (for otherwise, there would be at least one interior vertex Q on the path $T[R_a, R_b]$, so we should have $R_a \cap R_b \subseteq Q$, but no member Q of $\mathcal{Q}(F) \setminus \{R_a, R_b\}$ satisfies this inclusion). By the same argument, each Q_j ($j = 1, \dots, 2k-1$) must be adjacent to R_a or R_b in T . Suppose that we are trying to build a clique path tree T for F . By symmetry, we may assume that Q_1 is adjacent to R_a . Then, for $j = 2, \dots, 2k-2$ successively, Q_j must be adjacent to R_b (if j is even) and to R_a (if j is odd) in T , for otherwise, for some $v \in \{v_{j-1}, v_j\}$ the subtree T^v induced by the cliques that contain v would not be a path. Note that in this fashion we obtain a clique path tree T' of $F \setminus u_{2k-1}$. Now if Q_{2k-1} is adjacent to R_a , then the subtree $T^{v_{2k-1}}$ is not a path, and if Q_{2k-1} is adjacent to R_b , then the same holds for $T^{v_{2k-2}}$. This shows that F is not a path graph.

Now consider any vertex x of F . If x is one of the u_j 's, then by symmetry we may assume that $x = u_{2k-1}$, and we have seen above that $F \setminus x$ is a path graph with clique

path tree T' . Suppose that x is one of the v_j 's, say $x = v_{2k-1}$. Then by adding vertex Q_{2k-1} and edge $Q_{2k-1}R_a$ to T' , it is easy to see that we obtain a clique path tree of $F \setminus x$. Finally, suppose that x is one of a, b , say $x = b$. Then the tree with vertices $R_a, Q_1, \dots, Q_{2k-1}$ and edges $R_aQ_1, \dots, R_aQ_{2k-1}$ is a clique path tree of $F \setminus x$. So F is a minimal non path graph.

When F is any other F_i ($i = 1, \dots, 16$), the same arguments apply as follows. For $i = 1, \dots, 10$, call peripheral the three cliques that contain a simplicial vertex. For $i = 11, \dots, 16$, call peripheral the cliques that contain a simplicial vertex of degree 2, plus, in the case of F_{12} , the clique that contain the bottom simplicial vertex (which has degree 3). Call central all other maximal cliques. Then it is easy to prove, as above, that the central cliques must form a subpath in any clique tree of F , and all the peripheral cliques except one can be appended to either end of that subpath, but whichever way this is done, when the last clique is appended, the subtree T^v is not a path for some vertex v of F . Moreover, when any vertex x is removed, it is possible to build a clique path tree for $F \setminus x$. \square

4 Co-special simplicial vertices

Let us say that a simplicial vertex v is *co-special* if S_v is a separator such that $G \setminus S_v$ has exactly two components. Note that in that case S_v is a minimal element of \mathcal{S} and it appears exactly once as a label of any path tree of G .

Lemma 1 *Let G be a minimal non path graph. Then either G is one of F_{11}, \dots, F_{15} or every simplicial vertex of G is co-special.*

Proof. Suppose on the contrary that G is a minimal non path graph, different from F_{11}, \dots, F_{15} , and there is a simplicial vertex q of G that is not co-special. All simplicial vertices of $F_0, \dots, F_{10}, F_{16}$ are co-special, so G is not any of these graphs; moreover it does not contain any of them strictly (for otherwise G would not be minimal). Therefore G contains none of F_0, \dots, F_{16} .

Let T_0 be a clique path tree of $G \setminus q$. Let $Q' \in \mathcal{Q}(G \setminus q)$ be such that $S_q \subseteq Q'$. If $Q' = S_q$, then we can add q to Q' to obtain a clique path tree of G , a contradiction. So $Q' \neq S_q$, and $S_q \in \mathcal{S}$ (as there is a vertex $q' \in Q' \setminus S_q$ and S_q is a $\{q, q'\}$ -separator).

Let T' be the maximal subtree of T_0 that contains Q' and such that no label of the edges of T_0 is included in S_q . Remark that T' plus vertex Q and edge QQ' is a clique tree of $G(\mathcal{Q}(T') \cup \{Q\})$ (but not necessarily a clique path tree), and in that tree only one label is included in S_q . Since q is not co-special, there is an edge of T_0 whose label is included in S_q , and so T' has strictly fewer vertices than T_0 . So $G(\mathcal{Q}(T') \cup \{Q\})$ is a path graph. Let T be a clique path tree of this graph.

We claim that Q is a leaf of T . If not, then there are at least two labels of T that are included in S_q , which contradicts the definition of T' (the number of times a label

appears in a clique tree is constant).

Let T_1, \dots, T_ℓ be the subtrees of $T_0 \setminus T'$ ($\ell \geq 1$). For $1 \leq i \leq \ell$, let $Q_i Q'_i$ be the edge between T_i and T' with $Q_i \in T_i$ and $Q'_i \in T'$. Note that Q_1, \dots, Q_ℓ are pairwise disjoint (but Q', Q'_1, \dots, Q'_ℓ are not necessarily pairwise disjoint). Let $S_i = Q_i \cap Q'_i$ and $v_i \in Q_i \setminus Q'_i$. Let $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$ be the intersection graph of S_1, \dots, S_ℓ , that is, $V_{\mathcal{H}} = \{S_1, \dots, S_\ell\}$ and $E_{\mathcal{H}} = \{S_i S_j \mid S_i \cap S_j \neq \emptyset\}$.

Claim 1 \mathcal{H} contains no odd cycle.

Proof. Suppose on the contrary, without loss of generality, that $S_1 \cdots S_p S_1$ is an odd cycle in \mathcal{H} , with length $p = 2r + 1$ ($r \geq 1$). Let $I_j = S_j \cap S_{j+1}$ ($j = 1, \dots, p$), with $S_{p+1} = S_1$. Suppose that for some $j \neq k$ we have $I_j \cap I_k \neq \emptyset$; then there is a common vertex in the cliques $Q_j, Q_{j+1}, Q_k, Q_{k+1}$, and the number of different cliques among these is at least three, which contradicts the fact that T_0 is a clique path tree as these three cliques do not lie on a common path of T_0 . For $1 \leq j \leq p$, let $s_j \in I_j$. By the preceding remark, the s_j 's are pairwise distinct. By the definition of T' , we have $S_j \subseteq S_q$ for each $1 \leq j \leq p$, so the s_j 's are all in Q and Q' . Let $q' \in Q' \setminus Q$. Let us consider the subgraph induced by $q, q', v_1, \dots, v_p, s_1, \dots, s_p$. Each of the non-adjacent vertices q and q' is adjacent to all of the clique formed by the s_j 's. Each vertex v_j is adjacent to s_{j-1} and s_j (with $s_0 = s_p$) and not to any other s_i or to q . Vertex q' can have at most two neighbors among the v_j 's. If q' has zero or one neighbor among them, then $q, q', v_1, \dots, v_p, s_1, \dots, s_p$ induce respectively $F_{11}(4r+4)_{r \geq 1}$ or $F_{12}(4r+4)_{r \geq 1}$. If q' has two consecutive neighbors v_j, v_{j+1} , then $q, q', v_j, v_{j+1}, s_{j-1}, s_j, s_{j+1}$ induce F_2 . If q' has two non-consecutive neighbors v_j, v_k , then we can assume that $1 \leq j < j+1 < k \leq p$ and $k - j$ is odd, $k - j = 2s + 1$ with $s \geq 1$, and then $q, q', v_j, \dots, v_k, s_j, \dots, s_{k-1}$ induce $F_{14}(4s+5)_{s \geq 1}$. In all cases we obtain a contradiction. Thus the claim holds. \diamond

By the preceding claim, \mathcal{H} is a bipartite graph.

For $1 \leq i \leq \ell$, let $\mathcal{R}_i = \{S \in \mathcal{S}(T') \mid S_i \cap S \neq \emptyset \text{ and } S_i \setminus S \neq \emptyset\}$. Let $X = \{S_i \mid \mathcal{R}_i \neq \emptyset\}$.

Claim 2 \mathcal{H} contains no odd path between two vertices in X .

Proof. Suppose on the contrary, without loss of generality, that $S_1 \cdots S_p$ is an odd path in \mathcal{H} between two vertices S_1, S_p of X (with $p = 2k$, $k \geq 1$), and assume that p is minimum with this property. By the minimality, all interior vertices S_j ($1 < j < p$) are not in X . For $1 \leq j < p$, let s_j be a vertex in $S_j \cap S_{j+1}$. As in the preceding claim, the s_j 's are pairwise distinct and lie in Q and Q' . Let P be the path $T'[Q'_1, Q'_2]$. If $p \neq 2$, then S_2 is not in X , so $Q'_3 = Q'_1$, for otherwise $T_0^{s_2}$ would not be a path; then S_3 is not in X , so $Q'_4 = Q'_2$, and so on. Thus the two extremities of P are $Q'_1 = Q'_3 = \dots = Q'_{p-1}$ and $Q'_2 = Q'_4 = \dots = Q'_p$. Since S_1 and S_p are in X , the sets $\mathcal{R}_1, \mathcal{R}_p$ are non empty.

Let L_1 be the closest vertex to Q'_1 in P such that there exists an edge incident to L_1 with label in \mathcal{R}_1 , and let $L_1 K_1$ be such an edge and R_1 be its label (such an edge exists

because $\mathcal{R}_1 \neq \emptyset$). Similarly, let L_p be the closest vertex to Q'_p in P such that there exists an edge incident to L_p with label in \mathcal{R}_p , and let $L_p K_p$ be such an edge and R_p be its label. So $S_1 \subseteq L_1$, $S_1 \not\subseteq K_1$ and $S_p \subseteq L_p$, $S_p \not\subseteq K_p$. Each of K_1, K_p may be in P or not. Since T' is a clique path tree, Q' lies between Q'_1 and L_1 and between L_p and Q'_p along P . So Q'_1, L_p, Q', L_1, Q'_p lie in this order on P , and S_1 is included in all labels between Q'_1 and L_1 in P , and S_p is included in all labels between Q'_p and L_p in P .

Let $v_0 \in K_1 \setminus L_1$ and $v_{p+1} \in K_p \setminus L_p$. Since T_0 is a clique path tree, v_0 and v_{p+1} are distinct from v_1, \dots, v_p and not adjacent to q .

Let $s_0 \in S_1 \cap R_1$ and $s_p \in S_p \cap R_p$. Then v_0 and s_0 are adjacent, and v_{p+1} and s_p are adjacent. Since T_0 is a clique path tree, if K_1 or K_p is not in P , then s_0 and s_p are different from each other, from s_1, \dots, s_{p-1} and from v_0, \dots, v_{p+1} . Furthermore, if K_1 is not in P , then v_0 is not adjacent to any of s_1, \dots, s_p ; and if K_p is not in P , then v_{p+1} is not adjacent to any of s_0, \dots, s_{p-1} .

Let $s'_0 \in S_1 \setminus R_1$ and $s'_p \in S_p \setminus R_p$. Then v_0 and s'_0 are not adjacent, and v_{p+1} and s'_p are not adjacent. Since T_0 is a clique path tree, if K_1 or K_p is in P , then s'_0 and s'_p are different from each other, from s_1, \dots, s_{p-1} and from v_0, \dots, v_{p+1} . Furthermore, if K_1 is in P , then v_0 is adjacent to s'_p and to s_1, \dots, s_p ; and if K_p is in P , then v_{p+1} is adjacent to s'_0 and to s_0, \dots, s_{p-1} .

Note that the set $\{q, s'_0, s_0, s_1, s_2, \dots, s_p, s'_p\}$ induces a clique in G . Moreover, v_1 is adjacent to s'_0 , v_p is adjacent to s'_p , for $i = 1, \dots, p$, v_i is adjacent to s_{i-1} and s_i , and there is no other edge between v_1, \dots, v_p and that clique.

Suppose that $K_1 = K_p$. Then $L_1 = L_p = Q'$ and K_1 is not in P . By the definition of T' , there exists $y \in R_1 \setminus S_q$. Vertex y is distinct from all s_i 's as it is not in S_q , and it is adjacent to all of v_0, s_0, \dots, s_p and to none of q, v_1, \dots, v_p . Then $q, y, v_0, \dots, v_p, s_0, \dots, s_p$ induce $F_{12}(4k+4)_{k \geq 1}$, a contradiction. So $K_1 \neq K_p$, and v_0 and v_{p+1} are distinct non adjacent vertices. We can choose vertices x_1, \dots, x_r ($r \geq 1$) not in S_q and on the labels of $T'[K_1, K_p]$ such that $v_0-x_1-\dots-x_r-v_{p+1}$ is a chordless path in G . Vertices x_1, \dots, x_r are distinct from and adjacent to $s'_0, s'_p, s_0, \dots, s_p$, and they are distinct from and not adjacent to any of v_1, \dots, v_p .

Suppose that $L_1 = Q'_p$ and $L_p = Q'_1$. Then K_1 and K_p are not in P . If $r = 1$, then $q, v_0, \dots, v_{p+1}, s_0, \dots, s_p, x_1$ induce $F_{14}(4k+5)_{k \geq 1}$. If $r = 2$, then $q, v_0, \dots, v_{p+1}, s_0, \dots, s_p, x_1, x_2$ induce $F_{15}(4k+6)_{k \geq 1}$. If $r \geq 3$, then $q, v_0, v_{p+1}, s_0, s_p, x_1, \dots, x_r$ induce $F_{10}(r+5)_{r \geq 3}$, a contradiction.

Suppose now that $L_1 \neq Q'_p$ and $L_p = Q'_1$. Then K_p is not in P and we may assume that K_1 is in P . If $r = 1$, then $q, v_0, \dots, v_{p+1}, s'_0, s_1, \dots, s_p, x_1$ induce $F_{13}(4k+5)_{k \geq 1}$. If $r \geq 2$, then $q, v_0, v_{p+1}, x_1, \dots, x_r, s'_0, s_p$ induce $F_5(r+5)_{r \geq 2}$, a contradiction.

Suppose finally that $L_1 \neq Q'_p$ and $L_p \neq Q'_1$. Then we may assume that K_1 and K_p are in P . If $r = 1$, then $q, v_0, v_{p+1}, s'_0, s_1, s'_p, x_1$ induce F_2 . If $r = 2$, then $q, v_0, v_{p+1}, s'_0, s_1, s'_p, x_1, x_2$ induce F_3 . If $r \geq 3$, then $q, v_0, v_{p+1}, x_1, \dots, x_r, s'_0, s'_p$ induce $F_{10}(r+5)_{r \geq 3}$, a contradiction. Thus the claim holds. \diamond

By the preceding two claims, \mathcal{H} is a bipartite graph $(A, B, E_{\mathcal{H}})$ such that $X \subseteq A$. Now all the subtrees T_i can be linked to T to get a clique path tree of G as follows. For each $S_i \in A$, we add an edge QQ_i between T and T_i . This creates a clique path tree on the corresponding subset of cliques because A is a stable set of \mathcal{H} and Q is a leaf of T . For each $S_i \in B$, let $Q_i'' \in \mathcal{Q}(T)$ be such that $Q_i'' \cap S_i \neq \emptyset$ and the length of $T[Q, Q_i'']$ is maximal. Since $S_i \in B$, we have $\mathcal{R}_i = \emptyset$, so $S_i \subseteq Q_i''$ and we can add an edge $Q_i''Q_i$ between T and T_i . This creates a clique path tree of G because B is a stable set of \mathcal{H} and by the definition of Q_i'' , a contradiction. \square

5 Characterization of path graphs

In this section we prove the main theorem, that is, path graphs are exactly the graphs that do not contain any of F_0, \dots, F_{16} . We could not find a characterization similar to the one found by Lekkerkerker and Boland [11] for interval graphs (“an interval graph is a chordal graph with no asteroidal triple”). We know that in a path graph, the neighborhood of every vertex contains no asteroidal triple; but this condition is not sufficient. So we prove directly that a graph that does not contain any of the excluded subgraphs is a path graph.

Lemma 2 *In a graph that does not contain any of F_0, \dots, F_5, F_{10} , the neighborhood of every vertex does not contain an asteroidal triple.*

Proof. It suffices to check that when a universal vertex is added to a minimal forbidden induced subgraph for interval graphs ([11]), then one obtains a graph that contains one of F_0, \dots, F_5, F_{10} . The easy details are left to the reader. \square

Given three non adjacent vertices a, b, c , we say that a is the *middle* of b, c if every path between b and c contains a vertex from $N(a)$. If a, b, c is not an asteroidal triple, then at least one of them is the middle of the others.

Lemma 3 *In a chordal graph G with clique tree T , a vertex a is the middle of two vertices b, c if and only if for all cliques Q_b and Q_c such that $b \in Q_b$ and $c \in Q_c$, there is an edge of the path $T[Q_b, Q_c]$ such that a is complete to its label.*

Proof. Suppose that a is the middle of b, c . Let Q_b and Q_c be cliques such that $b \in Q_b$ and $c \in Q_c$, and suppose there is no edge of $T[Q_b, Q_c]$ such that a is complete to its label. For each edge on $T[Q_b, Q_c]$, one can select a vertex that is not adjacent to a . Then the set of selected vertices forms a path from b to c that uses no vertex from $N(a)$, a contradiction.

Suppose now that for all cliques Q_b and Q_c with $b \in Q_b$ and $c \in Q_c$, there is an edge of the path $T[Q_b, Q_c]$ such that a is complete to its label. Suppose that there

exists a path $x_0 \cdots x_r$, with $b = x_0$ and $c = x_r$ and none of the x_i 's is in $N(a)$. We may assume that this path is chordless. For $1 \leq i \leq r$, let Q_i be a maximal clique containing x_{i-1}, x_i . Then Q_1, \dots, Q_r appear in this order along a subpath of T . On each $T[Q_i, Q_{i+1}]$ ($1 \leq i \leq r-1$), vertex a is not adjacent to x_i , so a is not complete to any label of $T[Q_1, \dots, Q_r]$, but Q_1 contains b and Q_r contains c , a contradiction. \square

Now we are ready to prove the main theorem. Part of the proof has been done in the previous section. Lemma 1 deals with the case where there exists a simplicial vertex that is the middle of two other vertices; now we have to look at the case where all simplicial vertices are not the middle of any pair of vertices.

Proof of Theorem 1 By Theorem 3, a path graph does not contain any of F_0, \dots, F_{16} . Suppose now that there exists a graph G that does not contain any of F_0, \dots, F_{16} and is a minimal non path graph. Since G contains no F_0 , it is chordal. By Theorem 2, there is a special simplicial vertex q of G . By Lemma 1, q is co-special. Let $Q = Q_q$ and $S_Q = S_q \in \mathcal{S}$. Let T_0 be a clique path tree of $G(\mathcal{Q} \setminus Q)$. Let $Q' \in \mathcal{Q} \setminus Q$ be such that $S_Q \subseteq Q'$. We add the edge QQ' to T_0 to obtain a clique tree T'_0 of G .

Claim 1 *For all non-adjacent vertices $u, w \notin Q$, there exists a path between u and w that avoids the neighbourhood of q .*

Proof. Suppose the contrary. Let $U, W \in \mathcal{Q}$ be such that $u \in U$ and $w \in W$. We have $U \neq W$ since u, w are not adjacent. By Lemma 3, there is an edge of $T_0[U, W]$ whose label is included in S_Q , contradicting that q is co-special. Thus the claim holds. \diamond

For each clique $L \in \mathcal{Q} \setminus \{Q, Q'\}$, let L' be the neighbor of L along $T_0[L, Q']$. Let $S_L = L \cap L'$. Let \mathcal{S}_L be the set of labels of edges incident to L in T_0 . Let \bar{L} be the clique in $T_0[L, Q'] \setminus \{Q'\}$ such that $S_{\bar{L}} \subseteq S_L$ and no other edge of $T_0[\bar{L}, Q']$ has a label included in S_L . (Possibly $\bar{L} = L$.)

Let \mathcal{L} be the set of cliques L of $\mathcal{Q} \setminus \{Q, Q'\}$ such that no element of $\mathcal{S}_L \setminus S_L$ contains $S_{\bar{L}}$. For each clique $L \in \mathcal{L}$, we define a subtree T_L of T'_0 , where T_L is the biggest subtree of T'_0 that contains Q' and for which no label is included in S_L . Note that \bar{L}' is in T_L and \bar{L} is not in T_L . Since q is special and co-special we have $S_Q \not\subseteq S_L$, so T_L contains Q .

Claim 2 *For each clique $L \in \mathcal{L}$ we have $L' \in T_L$.*

Proof. Suppose on the contrary that $L' \notin T_L$. Then $\bar{L} \neq L$. When we remove the edges LL' and $\bar{L}\bar{L}'$ from T'_0 , there remain three subtrees T_1, T_2, T_3 , where T_1 is the subtree that contains L , T_2 is the subtree that contains L' and \bar{L} , and T_3 is the subtree that contains \bar{L}', Q', Q . Let T_4 be the tree formed by T_1, T_3 plus the edge $L\bar{L}'$. Then, since $S_{\bar{L}} \subseteq S_L$, T_4 is a clique tree of $G(\mathcal{Q}(T_4))$. The set $\mathcal{Q}(T_4)$ contains strictly fewer maximal cliques than \mathcal{Q} , so there exists a clique path tree T_5 of $G(\mathcal{Q}(T_4))$. Label $S_{\bar{L}}$ is on the edge $L\bar{L}'$ of T_4 , so it is also a label of T_5 . Consequently there is an edge LL'' of T_5 with a label

R such that $S_{\bar{L}} \subseteq R \subseteq L$. (Possibly $L'' = \bar{L}'$). Suppose that $R \neq S_{\bar{L}}$. Then there is an edge of T_1 or T_3 with label R . But no label of T_1 can be R by the definition of \mathcal{L} ; and all the labels of T_3 that are included in L are also included in $S_{\bar{L}}$, so no label of T_3 can be R , a contradiction. So $R = S_{\bar{L}}$. Now if we remove the edge LL'' from T_5 and replace it by the subtree T_2 and edges LL' and $\bar{L}L''$, we obtain a clique path tree of G , a contradiction. Thus the claim holds. \diamond

Let \mathcal{L}^* be the set of all $L \in \mathcal{L}$ such that T_L is a strict subtree of $T'_0 \setminus L$. For every vertex x of $G(\mathcal{Q} \setminus Q)$ let T_0^x be the subtree of T_0 induced by the cliques that contain x . Recall that T_0^x is a path because T_0 is a clique path tree. Let A be the set of vertices a of Q such that Q' is a vertex of T_0^a that is not a leaf. Then A is not empty, for otherwise T'_0 would be a clique path tree of G . Moreover:

Claim 3 *For any $a \in A$, the two leaves of T_0^a are in \mathcal{L} and at least one of them is in \mathcal{L}^* .*

Proof. Let L_1, L_2 be the leaves of T_0^a , and, for $i = 1, 2$, let $\ell_i \in L_i \setminus S_{L_i}$. We have $a \in S_{L_1}$, and a is not in any member of $\mathcal{S}(L_1) \setminus S_{L_1}$. Thus $L_1 \in \mathcal{L}$. Similarly $L_2 \in \mathcal{L}$. The three vertices q, ℓ_1, ℓ_2 are adjacent to a , so they do not form an asteroidal triple by Lemma 2, and so one of them is the middle of the other two. Vertex q cannot be the middle of ℓ_1, ℓ_2 by Claim 1. So we may assume up to symmetry that ℓ_1 is the middle of q, ℓ_2 . So, by Lemma 3, there is an edge of $T'_0[Q, L_2]$ with a label included in S_{L_1} . So T_{L_1} is a strict subtree of $T'_0 \setminus L_1$ and so $L_1 \in \mathcal{L}^*$. Thus the claim holds. \diamond

The preceding claim implies that \mathcal{L}^* is not empty. We choose $L \in \mathcal{L}^*$ such that the subtree T_L is maximal. Let $S_{Q'}$ be the label of the edge of $T_0[L, Q']$ that is incident to Q' . Vertex q is special and co-special, so there exists s_Q in $S_Q \setminus S_{Q'}$, and we have $s_Q \notin S_L$. Therefore no clique of $\mathcal{Q} \setminus \mathcal{Q}(T_L)$ contains s_Q . We add the edge LL' to T_L to obtain a clique tree T'_L of $G(\mathcal{Q}(T_L) \cup \{L\})$. Since T'_L is a strict subtree of T'_0 , we can consider a clique path tree T of $G(\mathcal{Q}(T'_L))$. Note that L is a leaf of T , for otherwise there are at least two labels of T that are included in S_L , which contradicts the definition of T_L .

Claim 4 *Let $a \in A$ be such that both leaves of T_0^a are not in T_L . Let L_a be a leaf of T_0^a that belongs to \mathcal{L}^* . Then L'_a is in T_L , and every edge KK' of T_0 with $K \notin T_L, K' \in T_L$ satisfies $S_K \subseteq S_{L_a}$.*

Proof. By Claim 3, L_a exists. Since the labels of the edges of T_L are not included in S_L , they are also not included in S_{L_a} . So T_L is a subtree of T_{L_a} . By the maximality of T_L , we have $T_L = T_{L_a}$. By Claim 2, L'_a is in T_L . By the definition of T_{L_a} , every edge KK' of T_0 with $K \notin T_L, K' \in T_L$ satisfies $S_K \subseteq S_{L_a}$. Thus the claim holds. \diamond

Claim 5 *There exist $U, W \in \mathcal{Q} \setminus \mathcal{Q}(T'_L)$ such that UL is an edge of T_0 , $S_U \setminus Q' \neq \emptyset$, $U \cap W \neq \emptyset$, $W' \in \mathcal{Q}(T_L)$ and $W \cap Q \neq \emptyset$.*

Proof. We define sets \mathcal{U}, \mathcal{V} as follows:

$$\begin{aligned}\mathcal{U} &= \{U \in \mathcal{Q} \setminus \mathcal{Q}(T'_L) \mid UL \text{ is an edge of } T_0\} \\ \mathcal{V} &= \{V \in \mathcal{Q} \setminus \mathcal{Q}(T'_L) \mid V' \in \mathcal{Q}(T_L)\}.\end{aligned}$$

We observe that the members of \mathcal{V} are pairwise disjoint. For if there is a vertex v in $V_1 \cap V_2$ for some $V_1, V_2 \in \mathcal{V}$, then v is on three labels (namely S_{V_1}, S_{V_2} and S_L) of T_0 that do not lie on a common path, contradicting that T_0 is a clique path tree.

We define sets \mathcal{U}_p ($p \geq 1$) and \mathcal{V}_p ($p \geq 0$) as follows:

$$\begin{aligned}\mathcal{V}_0 &= \{W \in \mathcal{V} \mid W \cap Q \neq \emptyset\} \\ \mathcal{U}_p &= \{U \in \mathcal{U} \setminus (\mathcal{U}_1 \cup \dots \cup \mathcal{U}_{p-1}) \mid \exists V \in \mathcal{V}_{p-1} \text{ such that } U \cap V \neq \emptyset\} \quad (p \geq 1) \\ \mathcal{V}_p &= \{V \in \mathcal{V} \setminus (\mathcal{V}_1 \cup \dots \cup \mathcal{V}_{p-1}) \mid \exists U \in \mathcal{U}_p \text{ such that } V \cap U \neq \emptyset\} \quad (p \geq 1).\end{aligned}$$

Consider the smallest $k \geq 1$ such that there exists $U \in \mathcal{U}_k$ with $S_U \setminus Q' \neq \emptyset$. If no such U exists, then let $k = \infty$. The claim states that $k = 1$, so let us suppose on the contrary that $k \geq 2$. For all $1 \leq p \leq k - 1$ and all $U \in \mathcal{U}_p$, we have $S_U \subseteq Q'$; let $U'' \in \mathcal{Q}(T)$ be such that $U'' \cap S_U \neq \emptyset$ and the length of $T[L, U'']$ is maximal. Remark that S_U is included in U'' if and only if all members of $\mathcal{Q}(T)$ that intersect S_U contain S_U . Let us prove that:

$$S_U \subseteq U'' \text{ for every } U \in \mathcal{U}_p, 1 \leq p \leq k - 1. \quad (1)$$

Suppose that there exists $U_p \in \mathcal{U}_p$, $1 \leq p \leq k - 1$, such that $S_{U_p} \not\subseteq U''$, and let p be minimum with this property. Let $V_0, \dots, V_{p-1}, U_1, \dots, U_p$ be such that $V_i \in \mathcal{V}_i$, $U_i \in \mathcal{U}_i$, $V_{i-1} \cap U_i \neq \emptyset$ and $U_i \cap V_i \neq \emptyset$. Pick $u_i \in U_i \setminus S_{U_i}$ and $v_i \in V_i \setminus S_{V_i}$. Let x_1, \dots, x_r be such that $x_1 \in V_0 \cap U_1$, $x_2 \in U_1 \cap V_1$, \dots , $x_r \in V_{p-1} \cap U_p$ with $r = 2p - 1$. We claim that $V'_0 = V'_1 = \dots = V'_{p-1}$. For otherwise there exists $i \in \{1, \dots, p - 1\}$ such that $V'_{i-1} \neq V'_i$. Then one of V'_{i-1}, V'_i contains elements of S_{U_i} but not all, and so $S_{U_i} \not\subseteq U''$, which contradicts the minimality of p .

By the definition of the \mathcal{V}_i 's, none of x_2, \dots, x_r is in Q . Let $x_0 \in V_0 \cap Q$ (maybe $x_0 = x_1$). So $x_0 \in S_{V_0} \subseteq S_L \subset L$. None of U_2, \dots, U_p can contain x_0 by the definition of \mathcal{U}_1 . Note that x_r is in U_p and $V'_{p-1} = V'_0$; on the other hand we have $S_{U_p} \not\subseteq U''$. So there exists a clique Z of T_L such that $Z' \in T_0^{x_0}$, $S_{U_p} \subseteq Z'$, $S_{U_p} \cap Z \neq \emptyset$ and $S_{U_p} \setminus Z \neq \emptyset$. Vertex Q' is on $T[L, Z']$ as $S_{U_p} \subseteq Q'$. Let $z \in Z \setminus Z'$. We can find vertices y_1, \dots, y_t on the labels of $T'_0[Z, Q]$ such that none of them is in S_L and $z-y_1-\dots-y_t-q$ is a chordless path in G . Let $\ell \in L \setminus S_L$. By Claim 1, there exists a path P between z and ℓ whose vertices are not neighbors of q .

If $Z \in T_0^{x_0}$, then let $b \in S_{U_p} \setminus Z$. As q is special and co-special, we have $S_Q \not\subseteq S_Z$, so let $c \in S_Q \setminus S_Z$. Then z, ℓ, q form an asteroidal triple (because of paths P , $z-y_1-\dots-y_t-q$ and $\ell-b-c-q$), and they lie in the neighborhood of x_0 , a contradiction. So $Z \notin T_0^{x_0}$. Let $x_{r+1} \in Z \cap U_p$. If $x_{r+1} \in Q$, then z, ℓ, q form an asteroidal triple (because of paths P , $z-y_1-\dots-y_t-q$ and $\ell-x_0-q$), and they lie in the neighborhood of x_{r+1} , a contradiction. So

$x_{r+1} \notin Q$. The S_{U_i} 's are all included in Q' and so in S_L too. They are pairwise disjoint, for otherwise T_0 is not a clique path tree. Vertex ℓ is not in any of the S_{U_i} 's, and ℓ is adjacent to all of x_0, \dots, x_{r+1} and to none of $u_1, \dots, u_p, v_0, \dots, v_{p-1}, y_1, \dots, y_t, z, q$.

Suppose that $V_0 \cap U_1 \cap Q \neq \emptyset$. Then we may assume that $x_0 = x_1$, so x_0 is in A and the two leaves of $T_0^{x_0}$ are not in T_L . By Claim 4, the leaf L_{x_0} of $T_0^{x_0}$ that is in \mathcal{L}^* is such that L'_{x_0} is in T_L , so $L_{x_0} = V_0$. But x_{r+1} is in $Z \cap U_p$, so it is not in S_{V_0} ; thus $S_L \not\subseteq S_{V_0}$, which contradicts the end of Claim 4. Therefore $V_0 \cap U_1 \cap Q = \emptyset$, so $x_0 \neq x_1$, $x_0 \notin U_1$, $x_1 \notin Q$. Now, if $t = 1$, then $u_1, \dots, u_p, v_0, \dots, v_{p-1}, x_0, \dots, x_{r+1}, y_1, q, z, \ell$ induce $F_{14}(4p+5)_{p \geq 1}$. If $t = 2$, then $u_1, \dots, u_p, v_0, \dots, v_{p-1}, x_0, \dots, x_{r+1}, y_1, y_2, q, z, \ell$ induce $F_{15}(4p+6)_{p \geq 1}$. If $t \geq 3$, then $\ell, x_0, x_{r+1}, z, y_1, \dots, y_t, q$ induce $F_{10}(s+5)_{t \geq 3}$, a contradiction. Therefore (1) holds.

Suppose that k is finite. Let $V_0, \dots, V_{k-1}, U_1, \dots, U_k$ be such that $V_i \in \mathcal{V}_i$, $U_i \in \mathcal{U}_i$, $V_{i-1} \cap U_i \neq \emptyset$, and $U_i \cap V_i \neq \emptyset$. Let $u_i \in U_i \setminus S_{U_i}$ and $v_i \in V_i \setminus S_{V_i}$. Pick vertices $x_1 \in V_0 \cap U_1$, $x_2 \in U_1 \cap V_1$, \dots , $x_r \in V_{k-1} \cap U_k$ with $r = 2k - 1$. By the definition of \mathcal{V} , none of x_2, \dots, x_r is in Q . Let $x_0 \in V_0 \cap Q$. Suppose that $V_0 \cap U_1 \cap Q \neq \emptyset$. Then we can assume that $x_0 = x_1$, so x_0 is in A and the two leaves of $T_0^{x_0}$ are not in T_L . By Claim 4, the leaf L_{x_0} of $T_0^{x_0}$ that is in \mathcal{L}^* is such that L'_{x_0} is in T_L , so $L_{x_0} = V_0$. But x_2 is in S_{V_1} and not in S_{V_0} , so $S_{V_1} \not\subseteq S_{V_0}$, which contradicts the end of Claim 4. Therefore $V_0 \cap U_1 \cap Q = \emptyset$, and $x_0 \neq x_1$, $x_0 \notin U_1$, $x_1 \notin Q$. Let $s_{U_k} \in S_{U_k} \setminus Q'$. Vertex s_{U_k} is not adjacent to any of $q, s_Q, v_0, \dots, v_{k-1}$ because $s_{U_k} \notin Q'$, and by the minimality of k , vertex s_{U_k} is not adjacent to u_1, \dots, u_{k-1} . Then $u_1, \dots, u_k, v_0, \dots, v_{k-1}, x_0, \dots, x_r, s_{U_k}, s_Q, q$ induce $F_{16}(4k+3)_{k \geq 2}$, a contradiction.

Now k is infinite. Then the members of $\bigcup_{p \geq 0} \mathcal{U}_p$ are included in Q' and pairwise disjoint, for otherwise T_0 is not a clique path tree. For each member M of $\mathcal{U} \cup \mathcal{V}$, let $T'_0(M)$ be the component of $T'_0 \setminus T'_L$ that contains M . Starting from the path tree T and the trees $T'_0(M)$ ($M \in \mathcal{U} \cup \mathcal{V}$), we build a new tree as follows. For each $V \in \bigcup_{p \geq 0} \mathcal{V}_p$, we add the edge VL between $T'_0(V)$ and T . For each $U \in \bigcup_{p \geq 1} \mathcal{U}_p$, we add the edge UU'' between $T'_0(U)$ and T . For each $U \in \mathcal{U} \setminus (\bigcup_{p \geq 1} \mathcal{U}_p)$, we add the edge UL between $T'_0(U)$ and T . For each $V \in \mathcal{V} \setminus (\bigcup_{p \geq 1} \mathcal{V}_p)$, we define $V'' \in \mathcal{Q}(T)$ such that $V'' \cap S_V \neq \emptyset$ and the length of $T[L, V'']$ is maximal. By the definition of \mathcal{V}_0 , we have $S_V \cap Q = \emptyset$, so $V'' \neq Q$, so V'' is a vertex of T_L on $T_0[L, V]$ and it contains S_V as $S_V \subseteq S_L$. Then we can add the edge VV'' between $T'_0(V)$ and T . Thus we obtain a clique path tree of G , a contradiction. So $k = 1$, and there exist $U \in \mathcal{U}_1$ and $W \in \mathcal{V}_0$ such that $S_U \setminus Q' \neq \emptyset$, $U \cap W \neq \emptyset$ and $W \cap Q \neq \emptyset$. Thus the claim holds. \diamond

Let U, W be as in the preceding claim. Let $s_U \in S_U \setminus Q'$. Vertex s_U is not adjacent to s_Q . Let $u \in U \setminus S_U$ and $w \in W \setminus S_W$.

Claim 6 $S_W = S_L$.

Proof. Assume on the contrary that $S_W \neq S_L$. Then S_W is a proper subset of S_L . Suppose that there exists $a \in U \cap W \cap Q \neq \emptyset$. Then a is in A and the two leaves of

T_0^a are not in T_L . By Claim 4, the leaf L_a of T_0^a that is in \mathcal{L}^* is such that L'_a is in T_L , so $L_a = W$. But $S_L \not\subseteq S_W$, so Claim 4 is contradicted. Therefore $U \cap W \cap Q = \emptyset$. By the definition of U and W , there exists $b \in W \cap Q$ and $c \in U \cap W$. So $b \notin U$, $c \notin Q$, $b \neq c$. Since s_U is in $S_U \setminus Q'$, we have $S_U \not\subseteq S_W$. The labels of the edges of T_L are not included in S_L , so they are also not in S_W . Thus we can choose vertices x_1, \dots, x_r on the labels of $T'_0[U, Q]$ such that none of the x_i 's is in S_W , $x_1 \in U$, $x_r \in Q$, and $u-x_1-\dots-x_r-q$ is a path from u to q that avoids $N(w)$. If $r = 1$, then x_1 is different from s_U and s_Q , and $w, b, c, u, s_U, x_1, s_Q, q$ induce F_8 . If $r = 2$, then, if x_1 is adjacent to s_Q , vertices $w, b, c, u, s_U, x_1, s_Q, q$ induce F_9 , and if x_1 is not adjacent to s_Q , vertices $w, b, c, u, x_1, x_2, s_Q, q$ induce F_9 . Finally, if $r \geq 3$, then $w, b, c, u, x_1, \dots, x_r, q$ induce $F_{10}(r+5)_{r \geq 3}$. In all cases we obtain a contradiction. Thus the claim holds. \diamond

Claim 7 $W \in \mathcal{L}^*$.

Proof. If $W \in \mathcal{L}$, then, by Claim 6, we have $T_W = T_L$ and $W \in \mathcal{L}^*$, as desired. So suppose $W \notin \mathcal{L}$. By the definition of W , there is a vertex $a \in W \cap Q$, and so $a \in L$. Let $L_1, L_2 \in \mathcal{L}$ be the leaves of T_0^a such that L_1, L, Q', W, L_2 lie in this order on that path. Let K be the member of \mathcal{L} that is closest to W on $T_0[L_2, W]$. Clearly $W \neq K$. The edges of T_L are not included in S_L , so they are also not in S_W and not in S_K . So T_K contains T_L . If $K \in \mathcal{L}^*$, then $T_K = T_L$ by the maximality of T_L , so $K' \notin T_K$, which contradicts Claim 2. Thus $K \notin \mathcal{L}^*$. This means that $T_K = T'_0 \setminus K$, and so the labels of $T'_0 \setminus K$ are not included in S_K , in particular $S_W \not\subseteq S_K$. Let XX' be the edge of $T_0[K, W]$ such that X' contains S_W and X does not (maybe $X' = W$, $X = K$). The set S_X contains a but not all of $S_{X'}$, and the members of $S_{X'} \setminus \{S_{X'}, S_X\}$ do not contain a . So no element of $S_{X'} \setminus S_X$ contains $S_{X'}$, which means that $X' \in \mathcal{L}$, a contradiction to the definition of K . Thus the claim holds. \diamond

By Claim 7, we have $W \in \mathcal{L}^*$. By Claim 6, we have $T_W = T_L$, so T_W is also maximal and what we have proved for L can be done for W . Thus, by Claim 5, there exists $X \notin T_W$ such that XW is an edge of T_0 with $S_X \setminus Q' \neq \emptyset$ and $X \cap S_W \neq \emptyset$. Let $x \in X \setminus W$ and $s_X \in S_X \setminus Q'$. Vertex s_X is not in S_W , for otherwise it would also be in S_L and in Q' . Vertex s_U is not in S_L , for otherwise it would also be in S_W and in Q' . Vertex s_Q is not in $S_W (= S_L)$. So s_Q, s_X, s_U are pairwise non adjacent.

Suppose that there exists a vertex $a \in U \cap X \cap Q \neq \emptyset$. So $a \in A$, but none of the two leaves of T_0^a can satisfy Claim 4, a contradiction. Therefore $U \cap X \cap Q = \emptyset$.

Suppose that $U \cap X \neq \emptyset$, and let $a \in U \cap X$. So a is not in Q . Let $b \in S_W \cap Q (= S_L \cap Q)$. So b is not in $U \cap X$. If $b \notin X \cup U$, then $q, u, x, s_Q, s_U, s_X, a, b$ induce F_6 , a contradiction. So b is in one of U, X , say $b \in X \setminus U$ (if b is in $U \setminus X$ the argument is similar). Since W is in \mathcal{L} , there is a vertex $c \in S_W \setminus S_X$. Vertex c is adjacent to a, b, s_U, s_Q and not to x . Then $x, a, b, u, s_U, c, s_Q, q$ induce F_8, F_9 or $F_{10}(8)$, a contradiction. Therefore $U \cap X = \emptyset$.

Let $a \in U \cap W$, so $a \notin X$. Suppose $a \notin Q$. If there exists $b \in X \cap Q$, then b is also in L and $q, u, x, s_Q, s_U, s_X, a, b$ induce F_6 , a contradiction. So $X \cap Q = \emptyset$. Let $c \in W \cap Q$. Then $c \in L$ and $c \notin X$. Let $d \in X \cap S_W$; so $d \in L$, $d \notin Q$, $d \notin U$. If c is adjacent to u , then $q, u, x, s_Q, s_U, s_X, c, d$ induce F_6 , else $q, u, x, s_Q, s_U, s_X, a, c, d$ induce F_7 , a contradiction. So $a \in Q$. Let $e \in X \cap S_W$; so $e \in L$. If $e \notin Q$, then $q, u, x, s_Q, s_U, s_X, a, e$ induce F_6 , a contradiction. So $e \in Q$. Let $f \in S_W \setminus S_Q$ (f exists because q is special and co-special). Since $U \cap X = \emptyset$, f is adjacent to at most one of u, x , and then $q, u, x, s_U, s_X, a, e, f$ induce F_9 or $F_{10}(8)$, a contradiction. This completes the proof of Theorem 1. \square

6 Recognition algorithm

The proof that we give above yields a new recognition algorithm for path graphs, which takes any graph G as input and either builds a clique path tree for G or finds one of F_0, \dots, F_{16} . We have not analyzed the exact complexity of such a method but it is easy to see that it is polynomial in the size of the input graph. More efficient algorithms were already given by Gavril [7], Schäffer [17] and Chaplick [3], whose complexity is respectively $O(n^4)$, $O(nm)$ and $O(nm)$ for graphs with n vertices and m edges. Another algorithm was proposed in [4] and claimed to run in $O(n + m)$ time, but it has only appeared as an extended abstract (see comments in [3, Section 2.1.4]).

There are classical linear time recognition algorithms for triangulated graphs [16], and, following [2], there have been several linear time recognition algorithms for interval graphs, of which the most recent is [9]. We hope that the work presented here will be helpful in the search for a linear time recognition algorithm for path graphs.

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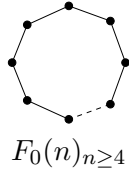


Figure 1: Forbidden subgraphs with no simplicial vertices

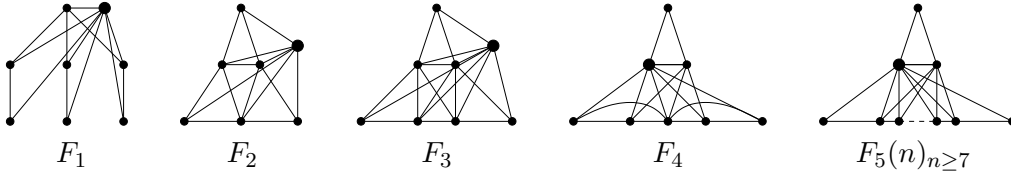


Figure 2: Forbidden subgraphs with a universal vertex

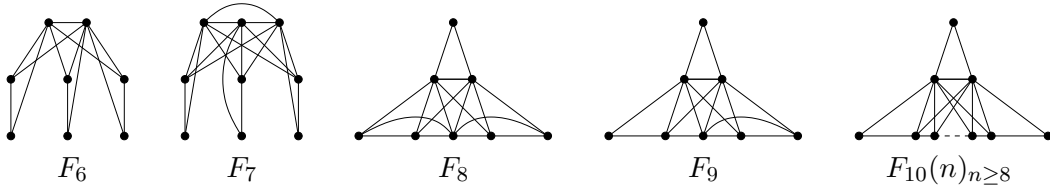


Figure 3: Forbidden subgraphs with no universal vertex and exactly three simplicial vertices

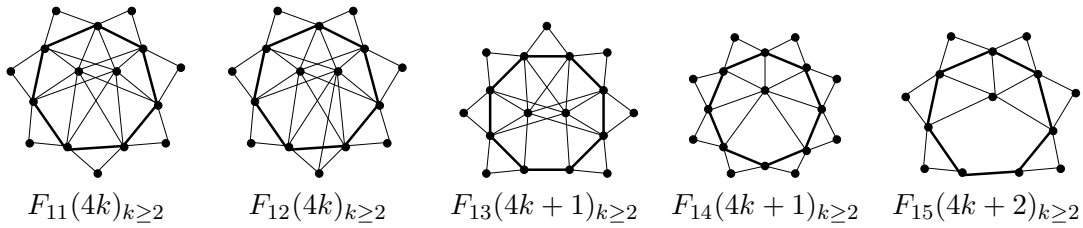


Figure 4: Forbidden subgraphs with at least one simplicial vertex that is not co-special. (bold edges form a clique)

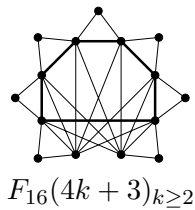


Figure 5: Forbidden subgraphs with ≥ 4 simplicial vertices that are all co-special. (bold edges form a clique)