

The Domination Number of Grids*

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Abstract

In this paper, we conclude the calculation of the domination number of all $n \times m$ grid graphs. Indeed, we prove Chang's conjecture saying that for every $16 \leq n \leq m$, $\gamma(G_{n,m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$.

1 Introduction

A *dominating set* in a graph G is a subset of vertices S such that every vertex in $V(G) \setminus S$ is a neighbour of some vertex of S . The *domination number* of G is the minimum size of a dominating set of G . We denote it by $\gamma(G)$. This paper is devoted to the calculation of the domination number of complete grids.

The notation $[i]$ denotes the set $\{1, 2, \dots, i\}$. If w is a word on the alphabet A , $w[i]$ is the i -th letter of w , and for every a in A , $|w|_a$ denotes the number of occurrences of a in w (i.e. $|\{i \in \{1, \dots, |w|\} : w[i] = a\}|$). For a vertex v , $N[v]$ denotes the closed neighbourhood of v (i.e. the set of neighbours of v and v itself). For a subset of vertices S of a vertex set V of a graph, we denote by $N[S] = \bigcup_{v \in S} N[v]$. Note that D is a dominating set of G if and only if $N[D] = V(G)$. Let $G_{n,m}$ be the $n \times m$ complete grid, i.e. the vertex set of $G_{n,m}$ is $V_{n,m} := [n] \times [m]$, and two vertices (i, j) and (k, l) are adjacent if $|k - i| + |l - j| = 1$. The couple $(1, 1)$ denotes the bottom-left vertex of the grid and the couple (i, j) denotes the vertex of the i -th column and the j -th row. We will always assume in this paper that $n \leq m$. Let us illustrate our purpose by an example of a dominating set of the complete grid $G_{24,24}$ on Figure 1.

The first results on the domination number of grids were obtained about 30 years ago with the exact values of $\gamma(G_{2,n})$, $\gamma(G_{3,n})$, and $\gamma(G_{4,n})$ found by Jacobson and Kinch [7] in 1983. In 1993, Chang and Clark [2] found those of $\gamma(G_{5,n})$ and $\gamma(G_{6,n})$. These results were obtained analytically. Chang [1] devoted his PhD thesis to study the domination number of grids; he conjectured that this invariant behaves well provided that n is large enough. Specifically, Chang conjectured the following:

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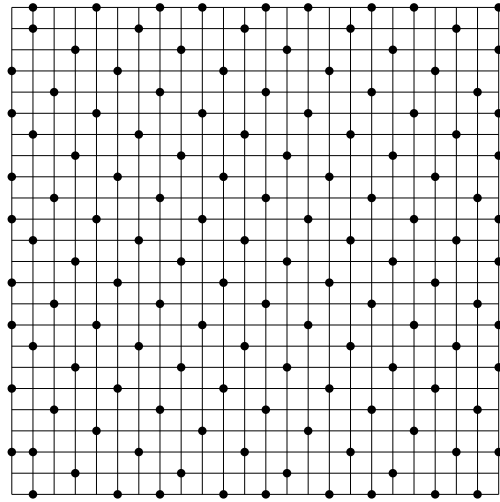


Figure 1: Example of a set of size 131 dominating the grid $G_{24,24}$

Conjecture 1 ([1]). For every $16 \leq n \leq m$,

$$\gamma(G_{n,m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$

Observe that for instance, this formula would give 131 for the domination number of the grid in Figure 1. To motivate his bound, Chang proposed some constructions of dominating sets achieving the upper bound:

Lemma 1 ([1]). For every $8 \leq n \leq m$,

$$\gamma(G_{n,m}) \leq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$$

Later, some algorithms based on dynamic programming were designed to compute a lower bound of $\gamma(G_{n,m})$. There were numerous intermediate results found for $\gamma(G_{n,m})$ for small values of n and m (see [3, 8, 9] for details). In 1995, Hare, Hedetniemi and Hare [8] gave a polynomial time algorithm to compute $\gamma(G_{n,m})$ when n is fixed. Nevertheless, this algorithm is not usable in practice when n hangs over 20. Fisher [5] developed the idea of searching for periodicity in the dynamic programming algorithms and using this technique, he found the exact values of $\gamma(G_{n,m})$ for all $n \leq 21$. We recall these values for the sake of completeness.

Theorem 2 ([5]). *For all $n \leq m$ and $n \leq 21$, we have:*

$$\gamma(G_{n,m}) = \begin{cases} \lceil \frac{m}{3} \rceil & \text{if } n = 1 \\ \lceil \frac{m+1}{2} \rceil & \text{if } n = 2 \\ \lceil \frac{3m+1}{4} \rceil & \text{if } n = 3 \\ m + 1 & \text{if } n = 4 \text{ and } m = 5, 6, 9 \\ m & \text{if } n = 4 \text{ and } m \neq 5, 6, 9 \\ \lceil \frac{6m+4}{5} \rceil - 1 & \text{if } n = 5 \text{ and } m = 7 \\ \lceil \frac{6m+4}{5} \rceil & \text{if } n = 5 \text{ and } m \neq 7 \\ \lceil \frac{10m+4}{7} \rceil & \text{if } n = 6 \\ \lceil \frac{5m+1}{3} \rceil & \text{if } n = 7 \\ \lceil \frac{15m+7}{8} \rceil & \text{if } n = 8 \\ \lceil \frac{23m+10}{11} \rceil & \text{if } n = 9 \\ \lceil \frac{30m+15}{13} \rceil - 1 & \text{if } n = 10 \text{ and } m \equiv_{13} 10 \text{ or } m = 13, 16 \\ \lceil \frac{30m+15}{13} \rceil & \text{if } n = 10 \text{ and } m \not\equiv_{13} 10 \text{ and } m \neq 13, 16 \\ \lceil \frac{38m+22}{15} \rceil - 1 & \text{if } n = 11 \text{ and } m = 11, 18, 20, 22, 33 \\ \lceil \frac{38m+22}{15} \rceil & \text{if } n = 11 \text{ and } m \neq 11, 18, 20, 22, 33 \\ \lceil \frac{80m+38}{29} \rceil & \text{if } n = 12 \\ \lceil \frac{98m+54}{33} \rceil - 1 & \text{if } n = 13 \text{ and } m \equiv_{33} 13, 16, 18, 19 \\ \lceil \frac{98m+54}{33} \rceil & \text{if } n = 13 \text{ and } m \not\equiv_{33} 13, 16, 18, 19 \\ \lceil \frac{35m+20}{11} \rceil - 1 & \text{if } n = 14 \text{ and } m \equiv_{22} 7 \\ \lceil \frac{35m+20}{11} \rceil & \text{if } n = 14 \text{ and } m \not\equiv_{22} 7 \\ \lceil \frac{44m+28}{13} \rceil - 1 & \text{if } n = 15 \text{ and } m \equiv_{26} 5 \\ \lceil \frac{44m+28}{13} \rceil & \text{if } n = 15 \text{ and } m \not\equiv_{26} 5 \\ \lfloor \frac{(n+2)(m+2)}{5} \rfloor - 4 & \text{if } n \geq 16 \end{cases}$$

Note that these values are obtained by specific formulas for every $1 \leq n \leq 15$ and by the formula of Conjecture 1 for every $16 \leq n \leq 21$. This proves Chang's conjecture for all values $16 \leq n \leq 21$.

In 2004, Conjecture 1 has been confirmed up to an additive constant:

Theorem 3 (Guichard [6]). *For every $16 \leq n \leq m$,*

$$\gamma(G_{n,m}) \geq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 9.$$

In this paper, we prove Chang's conjecture, hence finishing the computation of $\gamma(G_{n,m})$. We adapt Guichard's ideas to improve the additive constant from -9 to -4 when $24 \leq n \leq m$. Cases $n = 22$ and $n = 23$ can be proved in a couple of hours using Fisher's method (described in [5]) on a modern computer. They can be also proved by a slight improvement of the technique presented in the next section.

2 Values of $\gamma(G_{n,m})$ when $24 \leq n \leq m$

Our method follows the idea of Guichard [6]. A slight improvement is enough to give the exact bound.

A vertex of the grid $G_{n,m}$ dominates at most 5 vertices (its four neighbours and itself). It is then clear that $\gamma(G_{n,m}) \geq \frac{n \times m}{5}$. The previous inequality would become an equality if there would be a dominating set D such that every vertex of $G_{n,m}$ is dominated only once, and all vertices of D are of degree 4 (i.e. dominates exactly 5 vertices); in this case, we would have $5 \times |D| - n \times m = 0$. This is clearly impossible (e.g. to dominate the corners of the grid, we need vertices of degree at most 3). Therefore, our goal is to find a dominating set D of $G_{n,m}$ such that the difference $5 \times |D| - n \times m$ is the smallest.

Let S be a subset of $V(G_{n,m})$. The *loss* of S is $\ell(S) = 5 \times |S| - |N[S]|$.

Proposition 4. *The following properties of the loss function are straightforward:*

- (i) For every S , $\ell(S) \geq 0$ (positivity),
- (ii) If $S_1 \cap S_2 = \emptyset$, then $\ell(S_1 \cup S_2) = \ell(S_1) + \ell(S_2) + |N[S_1] \cap N[S_2]|$,
- (iii) If $S' \subseteq S$, then $\ell(S') \leq \ell(S)$ (increasing function),
- (iv) If $S_1 \cap S_2 = \emptyset$, then $\ell(S_1 \cup S_2) \geq \ell(S_1) + \ell(S_2)$ (super-additivity).

Let us denote by $\ell_{n,m}$ the minimum of $\ell(D)$ when D is a dominating set of $G_{n,m}$.

Lemma 5. $\gamma(G_{n,m}) = \left\lceil \frac{n \times m + \ell_{n,m}}{5} \right\rceil$

Proof. If D is a dominating set of $G_{n,m}$, then $\ell(D) = 5 \times |D| - |N[D]| = 5 \times |D| - n \times m$. Hence, by minimality of $\ell_{n,m}$ and $\gamma(G_{n,m})$, we have $\ell_{n,m} = 5 \times \gamma(G_{n,m}) - n \times m$. \square

Our aim is to get a lower bound for $\ell_{n,m}$. As the reader can observe in Figure 1, the loss is concentrated on the border of the grid. We now analyse more carefully the loss generated by the border of thickness 10.

We define the border $B_{n,m} \subseteq V_{n,m}$ of $G_{n,m}$ as the set of vertices (i, j) where $i \leq 10$, or $j \leq 10$, or $i > n - 10$, or $j > m - 10$ to which we add the four vertices $(11, 11)$, $(11, m - 10)$, $(n - 10, 11)$, $(n - 10, m - 10)$. Given a subset $S \subseteq V$, let $I(S)$ be the *internal vertices* of S , i.e. $I(S) = \{v \in S : N[v] \subseteq S\}$. These sets are illustrated in Figure 2. We will compute $b_{n,m} = \min_D \ell(D)$, where D is a subset of $B_{n,m}$ and dominates $I(B_{n,m})$, i.e. $D \subseteq B_{n,m}$ and $I(B_{n,m}) \subseteq N[D]$. Observe that this lower bound $b_{n,m}$ is a lower bound of $\ell_{n,m}$. Indeed, for every dominating set D of $G_{n,m}$, the set $D' := D \cap B_{n,m}$ dominates $I(B_{n,m})$, hence $b_{n,m} \leq \ell(D') \leq \ell(D)$. In the remainder, we will compute $b_{n,m}$ and we will show that $b_{n,m} = \ell_{n,m}$.

In the following, we split the border $B_{n,m}$ in four parts, $O_{m-12}, P_{n-12}, Q_{m-12}, R_{n-12}$, which are defined just below.

For $p \geq 12$, let $P_p \subset B_{n,m}$ defined as follows : $P_p = ([10] \times \{12\}) \cup ([11] \times \{11\}) \cup ([p] \times [10])$. We define the *input vertices* of P_p as $[10] \times \{12\}$ and the *output vertices* of P_p as $\{p\} \times [10]$. The set P_p , illustrated for $p = 19$ in Figure 3, corresponds to the set of black and gray vertices. The input vertices are the gray circles, and the output vertices are the gray squares. Recall that in our drawing conventions, the vertex $(1, 1)$ is the bottom-left vertex and hence the vertex (i, j) is in the i^{th} column from the left and in the j^{th} row from the bottom.

For $n, m \in \mathbb{N}^*$, let $f_{n,m} : [n] \times [m] \rightarrow [m] \times [n]$ be the bijection such that $f_{n,m}(i, j) = (j, n - i + 1)$. It is clear that the set $B_{n,m}$ is the disjoint union of the following four sets depicted in Figure 4: $P_{n-12}, Q_{m-12} = f_{n,m}(P_{m-12}), R_{n-12} = f_{n,m} \circ f_{n,m}(P_{n-12})$ and

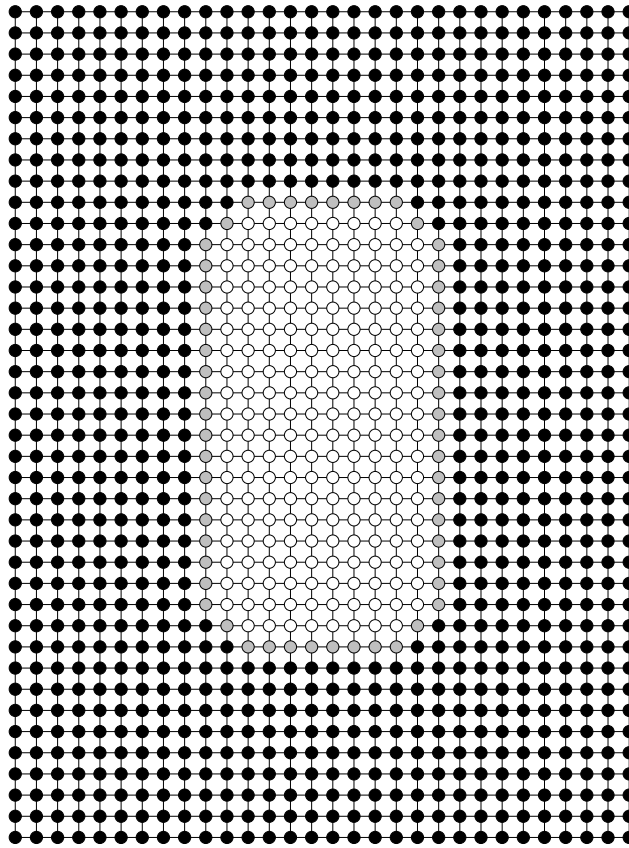


Figure 2: The graph $G_{30,40}$. The set $I(B_{30,40})$ is the set of vertices filled in black. The set $B_{30,40}$ is the set of vertices filled in black or in gray.

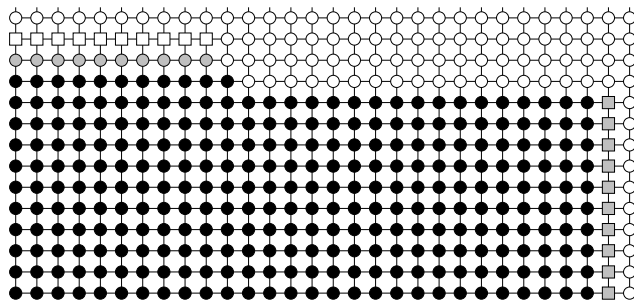


Figure 3: The set P_{19} (black and gray), the set of input vertices (gray circles) and the set of output vertices (gray squares).

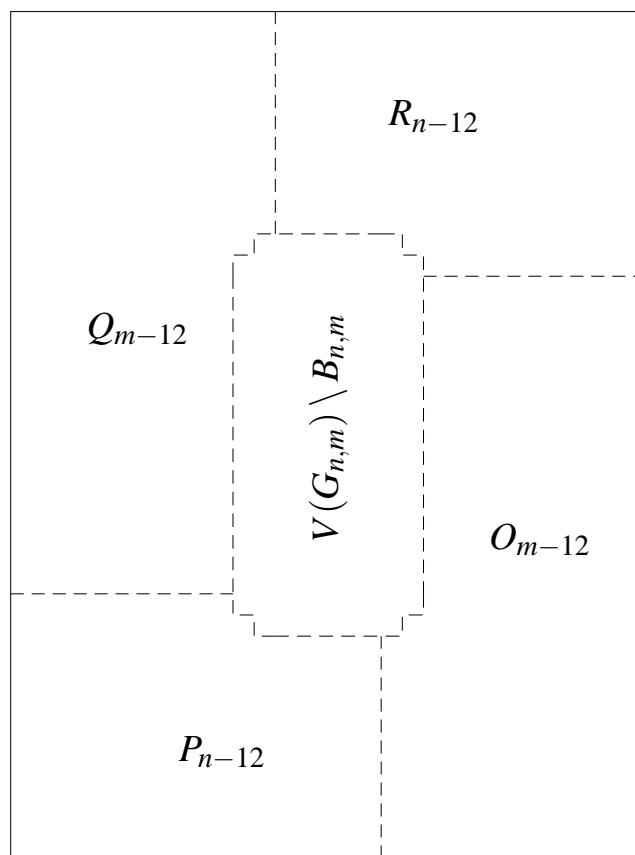


Figure 4: The sets O_{m-12} , P_{n-12} , Q_{m-12} and R_{n-12} .

$O_{m-12} = f_{n,m}^{-1}(P_{m-12})$. Similarly to P_{n-12} , the sets O_{m-12} , Q_{m-12} and R_{m-12} have input and output vertices. For instance, the output vertices of Q_{m-12} correspond in Figure 3 to the white squares. Every set playing a symmetric role, we now focus on P_{n-12} .

Given a subset S of $V_{n,m}$, let the labelling $\phi_S : V_{n,m} \rightarrow \{0, 1, 2\}$ be such that

$$\phi_S(i, j) = \begin{cases} 0 & \text{if } (i, j) \in S \\ 1 & \text{if } (i, j) \in N[S] \setminus S \\ 2 & \text{otherwise} \end{cases}$$

Note that ϕ_S is such that any two adjacent vertices in $G_{n,m}$ cannot be labelled 0 and 2.

Given $p \geq 12$ and a set $S \subseteq P_p$, the *input word* (resp. *output word*) of S for P_p , denoted by $w^{in}(S)$ (resp. $w_p^{out}(S)$), is the ten letters word on the alphabet $\{0, 1, 2\}$ obtained by reading ϕ_S on the input vertices (resp. output vertices) of P_p . More precisely, its i^{th} letter is $\phi_S(i, 12)$ (resp. $\phi_S(p, i)$). Similarly, O_p , Q_p and R_p have also input and output words. For example, the output word of $S \subseteq O_p$ for O_p is $w_p^{out}(f_{n,m}(S))$.

According to the definition of ϕ , the input and output words belong to the set \mathcal{W} of ten letters words on $\{0, 1, 2\}$ which avoid 02 and 20. The number of k -digits trinary numbers without 02 or 20 is given by the following formula [5]:

$$\frac{(1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1}}{2} \quad (1)$$

The size of \mathcal{W} is therefore $|\mathcal{W}| = 8119$.

Given two words $w, w' \in \mathcal{W}$, we define $\mathcal{D}_p^{w,w'}$ as the family of subsets D of P_p such that:

- D dominates $I(P_p)$,
- w is the input word $w^{in}(D)$,
- w' is the output word $w_p^{out}(D)$.

A relevant information for our calculation will be to know, for two given words $w, w' \in \mathcal{W}$, the minimum loss over all losses $\ell(D)$ where $D \in \mathcal{D}_p^{w,w'}$. For this purpose, we introduce the 8119×8119 square matrix C_p . For $w, w' \in \mathcal{W}$, let $C_p[w, w'] = \min_{D \in \mathcal{D}_p^{w,w'}} \ell(D)$ where the minimum of the empty set is $+\infty$.

Let $w, w' \in \mathcal{W}$ be two given words. Due to the symmetry of P_{12} with respect to the first diagonal (bottom-left to top-right) of the grid, if a vertex set D belongs to $\mathcal{D}_{12}^{w,w'}$, then $D' = \{(j, i) | (i, j) \in D\}$ belongs to $\mathcal{D}_{12}^{w',w}$. Moreover, it is clear that, always due to the symmetry, $\ell(D) = \ell(D')$. Therefore, we have $C_{12}[w, w'] = C_{12}[w', w]$ and thus C_{12} is a symmetric matrix. Despite the size of C_{12} and the size of P_{12} (141 vertices), it is possible to compute C_{12} in less than one hour by computer. Indeed, we choose a sequence of subsets $X_0 = \emptyset, X_1, \dots, X_{141} = P_{12}$ such that for every $i \in \{1, \dots, 141\}$, $X_i \subseteq X_{i+1}$ and $X_{i+1} \setminus X_i = \{x_{i+1}\}$. Moreover, we choose the sequence such that for every i , $X_i \setminus I(X_i)$ is at most 21. This can be done for example by taking $x_{i+1} = \min\{(x, y) : (x, y) \in P_{12} \setminus X_i\}$, where the order is the lexical order. For $i \in \{0, \dots, 141\}$, we compute for every labeling $f \in \mathcal{F}_i$, where \mathcal{F}_i is the set of functions $(X_i \setminus I(X_i)) \rightarrow \{0, 1, 2\}$, the minimal loss $l_{i,f}$ of a set $S \subseteq X_i$ which dominates $I(X_i)$ and such that $\phi_S(v) = f(v)$ for every $v \in X_i \setminus I(X_i)$. Note that not every labeling is possible since two adjacent vertices

cannot be labeled 0 and 2. The number of possible labellings can be computed using formula (1), and since $|X_i \setminus I(X_i)|$ can be covered by a path of at most 23 vertices, this gives, in the worst case, that this number is less than 10^9 and can be then processed by a computer. We compute inductively the sequence $(l_{i,f})_{f \in \mathcal{F}_i}$ from the sequence $(l_{i-1,f})_{f \in \mathcal{F}_{i-1}}$ by dynamical programming, and C is easily deduced from $(l_{141,f})_{f \in \mathcal{F}_{141}}$.

In the following, our aim is to glue P_{n-12} , Q_{m-12} , R_{n-12} , and O_{m-12} together. For two consecutive parts of the border, say P_{n-12} and Q_{m-12} , the output word of Q_{m-12} should be compatible with the input word of P_{n-12} . Two words w, w' of \mathcal{W} are *compatible* if the sum of their corresponding letters is at most 2, i.e. $w[i] + w'[i] \leq 2$ for all $i \in [9]$. Note that $w[10] + w'[10]$ should be greater than 2 since the corresponding vertices can be dominated by some vertices of $V_{n,m} \setminus B_{n,m}$.

Given two words $w, w' \in \mathcal{W}$, let $\ell(w, w') = |\{i \in [10] : w[i] \neq 2 \text{ and } w'[i] = 0\}| + |\{i \in [10] : w'[i] \neq 2 \text{ and } w[i] = 0\}|$.

Lemma 6. *Let D be a dominating set of $G_{n,m}$ and let us denote $D_P = D \cap P_{n-12}$ and $D_Q = D \cap Q_{m-12}$. Then $\ell(D \cap (P_{n-12} \cup Q_{m-12})) = \ell(D_P) + \ell(D_Q) + \ell(w, w')$, where $w = w^{in}(D_P)$ and $w' = w_q^{out}(f_{n,m}^{-1}(D_Q))$. Moreover, w and w' are compatible.*

Proof. By Proposition 4(ii), $\ell(D \cap (P_{n-12} \cup Q_{m-12})) = \ell(D_P) + \ell(D_Q) + |N[D_P] \cap N[D_Q]|$. It suffice then to note that $\ell(w, w') = |N[D_P] \cap N[D_Q]|$ to get $\ell(D \cap (P_{n-12} \cup Q_{m-12})) = \ell(D_P) + \ell(D_Q) + \ell(w, w')$.

In what remains, we prove that w and w' are compatible. If those two words were not compatible, there would exist an index $i \in [9]$ such that $w_{m-12}^{out}(f_{n,m}^{-1}(D_Q))[i] + w^{in}(D_P)[i] > 2$. Thus at least one of these two letters should be a 2, and the other one should not be 0.

Suppose that $w_{m-12}^{out}(f_{n,m}^{-1}(D_Q))[i] = 2$ and note that this means that the vertex $(i, 13)$ is not dominated by a vertex in D_Q . Since D is a dominating set of $G_{n,m}$, every output vertex of Q_{m-12} except $(10, 13)$ (and every input vertex of P_{n-12} except $(10, 12)$) is dominated by a vertex of D_Q or by a vertex of D_P . Thus $(i, 13)$ should be dominated by its unique neighbour in P_{n-12} , $(i, 12)$. This would imply that $(i, 12) \in D$ contradicting the fact that $w^{in}(D_P)[i] \neq 0$.

Similarly, if $w^{in}(D_P)[i] = 2$, the vertex $(i, 12)$ is not dominated by a vertex in D_P , thus $(i, 12)$ must be dominated by the vertex $(i, 13) \in D$, contradicting the fact that $w_{m-12}^{out}(f_{n,m}^{-1}(D_Q))[i] \neq 0$. \square

Lemma 6 is designed for the two consecutive parts P_{n-12} and Q_{m-12} of the border of $G_{n,m}$. Its easy to see that this extends to any pair of consecutive parts of the border, i.e. Q_{m-12} and R_{n-12} , R_{n-12} and O_{m-12} , O_{m-12} and P_{n-12} .

We define the matrix 8119×8119 square matrix L which contains, for every pair of words $w, w' \in \mathcal{W}$, the value $\ell(w, w')$:

$$L[w, w'] = \begin{cases} +\infty & \text{if } w \text{ and } w' \text{ are not compatible,} \\ \ell(w, w') & \text{otherwise.} \end{cases}$$

Note that L is symmetric since $\ell(w, w') = \ell(w', w)$.

Let \otimes be the matrix multiplication in $(\min, +)$ algebra, i.e. $C = A \otimes B$ is the matrix where for all i, j , $C[i, j] = \min_k A[i, k] + B[k, j]$.

Let $M_p = L \otimes C_p$ for $p \geq 12$.

By construction, $M_{n-12}[w, w']$ corresponds to the minimum possible loss $\ell(D \cap P_{n-12})$ of a dominating set $D \subseteq V_{n,m}$ that dominates $I(P_{n-12})$ and such that w is the output word of Q_{m-12} and w' is the output word of P_{n-12} .

Lemma 7. *For all $24 \leq n \leq m$, we have*

$$b_{n,m} \geq \min_{w_1, w_2, w_3, w_4 \in \mathcal{W}} M_{n-12}[w_1, w_2] + M_{m-12}[w_2, w_3] + M_{n-12}[w_3, w_4] + M_{m-12}[w_4, w_1].$$

Proof. Consider a set $D \subseteq B_{n,m}$ which dominates $I(B_{n,m})$ and achieving $\ell(D) = b_{n,m}$. Let $D_P = D \cap P_{n-12}$, $D_Q = D \cap Q_{m-12}$, $D_R = D \cap R_{n-12}$ and $D_O = D \cap O_{m-12}$. Let w_P (w_Q , w_R and w_O , respectively) be the input word of P_{n-12} (Q_{m-12} , R_{n-12} and O_{m-12}), and w'_P (w'_Q , w'_R and w'_O) be the output word of P_{n-12} (Q_{m-12} , R_{n-12} and O_{m-12}). By definition of C_p , the loss of D_P is at least $C_{n-12}[w_P, w'_P]$. Similarly, we have $\ell(D_Q) \geq C_{m-12}[w_Q, w'_Q]$, $\ell(D_R) \geq C_{n-12}[w_R, w'_R]$ and $\ell(D_O) \geq C_{m-12}[w_O, w'_O]$. By the definition of the loss:

$$\begin{aligned} \ell(D) &= b_{n,m} \\ &= 5 \times |D| - |N[D]| \\ &= \ell(D_O) + \ell(D_P) + \ell(D_Q) + \ell(D_R) + L[w'_O, w_P] + L[w'_P, w_Q] + L[w'_Q, w_R] + L[w'_R, w_O] \\ &\quad \text{by Lemma 6 and since } N[D_P] \cap N[D_R] = N[D_Q] \cap N[D_O] = \emptyset \\ &\geq C_{m-12}[w_O, w'_O] + C_{n-12}[w_P, w'_P] + C_{m-12}[w_Q, w'_Q] + C_{n-12}[w_R, w'_R] \\ &\quad + L[w'_O, w_P] + L[w'_P, w_Q] + L[w'_Q, w_R] + L[w'_R, w_O] \\ &\geq M_{m-12}[w_O, w_P] + M_{n-12}[w_P, w_Q] + M_{m-12}[w_Q, w_R] + M_{n-12}[w_R, w_O] \\ &\quad \text{since } w'_O \text{ and } w_P \text{ (resp. } w'_P \text{ and } w_Q, w'_Q \text{ and } w_R, w'_R \text{ and } w_O) \text{ are compatibles.} \end{aligned}$$

□

According to Lemma 7, to bound $b_{n,m}$ it would be thus interesting to know M_p for $p > 12$. It is why we introduce the following 8119×8119 square matrix, T .

Lemma 8. *There exists a matrix T such that $C_{p+1} = C_p \otimes T$ for all $p \geq 12$. This matrix is defined as follows:*

$$T[w, w'] = \begin{cases} +\infty & \text{if } \exists i \in [10] \text{ s.t. } w[i] = 0 \text{ and } w'[i] = 2 \\ +\infty & \text{if } \exists i \in [9] \text{ s.t. } w[i] = 2 \text{ and } w'[i] \neq 0 \\ +\infty & \text{if } \exists i \in \{2, \dots, 9\} \text{ s.t. } w'[i] = 1, w[i] \neq 0, w'[i-1] \neq 0 \text{ and } w'[i+1] \neq 0 \\ +\infty & \text{if } w'[1] = 1, w[1] \neq 0 \text{ and } w'[2] \neq 0 \\ +\infty & \text{if } w'[10] = 1, w[10] \neq 0 \text{ and } w'[9] \neq 0 \\ 3 \times |w'|_0 - |w|_2 - |w'|_1 + |w|_0 - 1 & \text{if } w'[10] = 0 \\ 3 \times |w'|_0 - |w|_2 - |w'|_1 + |w|_0 & \text{otherwise.} \end{cases}$$

Proof. Consider a set $S' \subseteq P_{p+1}$ dominating $I(P_{p+1})$ and let $S = S' \cap P_p$. Let $w = w_p^{\text{out}}(S)$ and $w' = w_{p+1}^{\text{out}}(S')$. Let $\Delta(S, S') = \ell(S') - \ell(S)$. By definition of the loss, $\Delta(S, S') = 5 \times |S' \setminus S| - |N[S'] \setminus N[S]|$. Let us compute $\Delta(S, S')$ in term of the number of occurrences of 0's, 1's and 2's in the words w and w' . The set $S' \setminus S$ corresponds to the vertices $\{(p+1, i) \mid i \in [10], w'[i] = 0\}$. The set $N[S'] \setminus N[S]$ corresponds to the vertices dominated by S' but not dominated by S ; these vertices clearly belong to the columns p , $p+1$ and $p+2$. Since S' dominates $I(P_{p+1})$, those in the column p are the vertices

$\{(p, i) \mid i \in [10], w[i] = 2\}$. Those in the column $p + 1$ are the vertices $\{(p + 1, i) \mid i \in [10], w'[i] \neq 2, w[i] \neq 0\}$ and possibly the vertex $(p + 1, 11)$ when $w'[10] = 0$. Finally, those in the column $p + 2$ are the vertices $\{(p + 2, i) \mid i \in [10], w'[i] = 0\}$. We then get:

$$\Delta(S, S') = \begin{cases} 3 \times |w'|_0 - |w|_2 - |w'|_1 + |w|_0 - 1 & \text{if } w'[10] = 0 \\ 3 \times |w'|_0 - |w|_2 - |w'|_1 + |w|_0 & \text{otherwise} \end{cases}$$

where $|w|_n$ denotes the number of occurrences of the letter n in the word w .

Thus $\Delta(S, S')$ only depends on the output words of S and S' , and we can denote this value by $\Delta(w, w')$. Note however that there exist pairs of words (w, w') that could not be the output words of S and S' ; there are three cases:

- Case 1. $w[i] = 0$ and $w'[i] = 2$ since the vertex $(p + 1, i)$ would be dominated by (p, i) contradicting its label 2;
- Case 2. $w[i] = 2$ and $w'[i] \neq 0$ for $i \in [9]$ since (p, i) would not be dominated contradict the fact that S' dominates $I(P_{p+1})$;
- Case 3. $w'[i] = 1$ and $w'[i - 1] \neq 0$, $w'[i + 1] \neq 0$, $w[i] \neq 0$ since $(p + 1, i)$ would be dominated according to its label but none of its neighbors belong to S' .

For these forbidden cases, we set $\Delta(w, w') = +\infty$.

By definition, $C_{p+1}[w_i, w']$ is the minimum loss $\ell(S')$ of a set $S' \subseteq P_{p+1}$ that dominates $I(P_{p+1})$, with w_i as input word and w' as output word. It is clear that $S = S' \cap P_p$ dominates $I(P_p)$ and has w_i as input word. Let w be its output word and note that $C_{p+1}[w_i, w'] = \ell(S') = \ell(S) + \Delta(w_i, w')$. The minimality of $\ell(S')$ implies the minimality of $\ell(S)$ over the sets $X \in \mathcal{D}_p^{w_i, w'}$. Indeed, any set $X \in \mathcal{D}_p^{w_i, w'}$ could be turned in a set of $X' \in \mathcal{D}_{p+1}^{w_i, w'}$ by adding vertices of the $p + 1^{\text{th}}$ column accordingly to w' . Thus

$$C_{p+1}[w_i, w'] = C_p[w_i, w] + \Delta(w, w')$$

which implies that

$$C_{p+1}[w_i, w'] \geq \min_w C_p[w_i, w] + \Delta(w, w').$$

On the other hand, for every word $w_o \in \mathcal{W}$ such that $C_p[w_i, w_o] \neq +\infty$ and $\Delta(w_o, w') \neq +\infty$, there is a set $S \in \mathcal{D}_p^{w_i, w_o}$, with $\ell(S) = C_p[w_i, w_o]$, that can be turned in a set $S' \in \mathcal{D}_{p+1}^{w_i, w'}$ with $\ell(S') = C_p[w_i, w_o] + \Delta(w_o, w')$. Thus

$$C_{p+1}[w_i, w'] \leq \min_{w_o} C_p[w_i, w_o] + \Delta(w_o, w').$$

This concludes the proof of the lemma. \square

By the definition of M_p , we have also $M_{p+1} = M_p \otimes T$. Note that T is a sparse matrix: about 95.5% of its 8119^2 entries are $+\infty$. Thus the multiplication by T in the $(\min, +)$ algebra can be done in a reasonable amount of time by a trivial algorithm.

Fact 9. *The computations give us that $M_{126} = M_{125} + 1$. Thus, since $(A + c) \otimes B = (A \otimes B) + c$ for any matrices A, B and any integer c , we have that $M_{125+k} = M_{125} + k$ for every $k \in \mathbb{N}$.*

Let us define $M'_p = \min_{k \in \mathbb{N}} (M_{p+k} - k)$. Then, for all $q \geq p$, $M_q \geq M'_p + (q - p)$. By Fact 9, $M'_p = \min_{k \in \{0, \dots, 125-p\}} (M_{p+k} - k)$

Fact 10. By computing M'_{12} , and $A' = M'_{12} \otimes M'_{12}$, we obtain that $\min_{w_1, w_3} (A' + A'^T)[w_1, w_3] = 76$ (where A^T is the transpose of A).

This implies that

$$\min_{w_1, w_3} (\min_{w_2} M'_{12}[w_1, w_2] + M'_{12}[w_2, w_3]) + (\min_{w_4} M'_{12}[w_3, w_4] + M'_{12}[w_4, w_1]) = 76$$

$$\min_{w_1, w_2, w_3, w_4} M'_{12}[w_1, w_2] + M'_{12}[w_2, w_3] + M'_{12}[w_3, w_4] + M'_{12}[w_4, w_1] = 76.$$

Theorem 11. If $24 \leq n \leq m$, then

$$\gamma(G_{n,m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$

Proof. By Chang's construction [4], $\gamma(G_{n,m}) \leq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4$. Let us now compute a lower bound for the loss of a dominating set of $G_{n,m}$.

$$\begin{aligned} \ell_{n,m} &\geq b_{n,m} \\ &\geq \min_{w_1, w_2, w_3, w_4} M_{n-12}[w_1, w_2] + M_{m-12}[w_2, w_3] + M_{n-12}[w_3, w_4] + M_{m-12}[w_4, w_1] \\ &\quad \text{by Lemma 7} \\ &\geq \min_{w_1, w_2, w_3, w_4} M'_{12}[w_1, w_2] + (n-12-12) + M'_{12}[w_2, w_3] + (m-12-12) + M'_{12}[w_3, w_4] \\ &\quad + (n-12-12) + M'_{12}[w_4, w_1] + (m-12-12) \\ &\geq 2 \times (n+m-48) + \min_{w_1, w_2, w_3, w_4} M'_{12}[w_1, w_2] + M'_{12}[w_2, w_3] + M'_{12}[w_3, w_4] + M'_{12}[w_4, w_1] \\ &\geq 2 \times (n+m-48) + 76 \\ &\geq 2 \times (n+m) - 20. \end{aligned}$$

Thus by Lemma 5, we have:

$$\begin{aligned} \gamma(G_{n,m}) &\geq \left\lfloor \frac{n \times m + 2 \times (n+m) - 20}{5} \right\rfloor \\ &\geq \left\lfloor \frac{(n+2)(m+2) - 4}{5} \right\rfloor - 4 \\ &\geq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4. \end{aligned}$$

□

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