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Hoàng-Reed conjecture holds for tournaments

Frédéric Havet∗ Stéphan Thomassé† Anders Yeo‡

Abstract

Hoàng-Reed conjecture asserts that every digraph $D$ has a collection $C$ of circuits $C_1, \ldots, C_{\delta^+}$, where $\delta^+$ is the minimum outdegree of $D$, such that the circuits of $C$ have a forest-like structure. Formally, $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| \leq 1$, for all $i = 2, \ldots, \delta^+$. We verify this conjecture for the class of tournaments.

1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph $D$ on $n$ vertices and with minimum outdegree $n/k$ has a circuit of length at most $k$. Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A circuit-tree is either a singleton or consists of a set of circuits $C_1, \ldots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| = 1$ for all $i = 2, \ldots, k$, where $V(C_j)$ is the set of vertices of $C_j$. A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique $xy$-directed path for every distinct vertices $x$ and $y$. A vertex-disjoint union of circuit-trees is a circuit-forest. When all circuits have length three, we speak of a triangle-tree. For short, a $k$-circuit-forest is a circuit-forest consisting of $k$ circuits.

Conjecture 1 (Hoàng and Reed [3]) Every digraph has a $\delta^+$-circuit-forest.

This conjecture is not even known to be true for $\delta^+ = 3$. In the case $\delta^+ = 2$, C. Thomassen proved in [6] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament $T$, that is the 3-uniform hypergraph on vertex set $V$ which edges are the 3-circuits of $T$.

Indeed, if a tournament $T$ has a $\delta^+$-circuit-forest, by the fact that every circuit contains a directed triangle, $T$ also has a $\delta^+$-triangle-forest. Observe that a $\delta^+$-triangle-forest spans exactly $2\delta^++c$ vertices, where $c$ is the number of components of the triangle-forest. When $T$ is a regular tournament with outdegree $\delta^+$, hence with $2\delta^++1$ vertices, a $\delta^+$-triangle-forest of $T$ is necessarily a spanning $\delta^+$-triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

Theorem 1 Every tournament has a $\delta^+$-triangle-tree.
2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

**Lemma 1** Let \( k \geq 1 \) and let \( a_1, a_2, \ldots, a_k \) and \( b_1, b_2, \ldots, b_k \) be two sequences of positive reals. Let \( A = \sum_{i=1}^{k} a_i \) and \( B = \sum_{j=1}^{k} b_j \). If \( \sum_{i=1}^{k} a_i b_i = \frac{AB}{2} + q \), where \( q \geq 0 \), then there is an \( i \) such that \( a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q} \).

**Proof.** If \( k = 1 \), then the lemma follows immediately as \( q = \frac{AB}{2} \) and \( A + B \geq \frac{A+B}{2} + \sqrt{AB} \). So assume that \( k > 1 \). Without loss of generality, we may assume that \( (a_1, b_1) \geq (a_2, b_2) \geq \ldots \geq (a_k, b_k) \) in the lexicographical order. Let \( r \) be the minimum value such that \( b_r \geq b_i \) for all \( i = 1, 2, \ldots, k \). Note that \( a_1 \geq |A|/2 \), since otherwise \( \sum_{i=1}^{k} a_i b_i < \sum_{i=1}^{k} A b_i/2 = AB/2 \). Analogously \( b_r \geq |B|/2 \). Define \( a' \) and \( b' \) so that \( a_1 = A/2 + a' \) and \( b_r = B/2 + b' \).

If \( r \neq 1 \), then the following holds:

\[
\sum_{i=1}^{k} a_i b_i \leq a_1 b_1 + \sum_{i=2}^{k} a_i b_r \\
\leq a_1 (B - b_r) + (A - a_1) b_r \\
= \left(\frac{A}{2} + a'\right)(\frac{B}{2} - b') + \left(\frac{A}{2} - a'\right)(\frac{B}{2} + b') \\
= \frac{AB}{2} - 2a'b' \\
\leq \frac{AB}{2} + q
\]

As \( q \geq 0 \), this implies we have equality everywhere above, which means that \( b_1 = B - b_r \). As \( B = b_1 + b_r \), we must have \( k = 2 \). As there was equality everywhere above we have \( b' = 0 \) or \( a' = 0 \) which implies that \( a_1 = a_2 = A/2 \) or \( b_1 = b_2 = B/2 \). In both cases we would have \( r = 1 \), a contradiction.

Suppose now that \( r = 1 \). Then

\[
\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = \left(\frac{A}{2} + a'\right)\left(\frac{B}{2} + b'\right) + \left(\frac{A}{2} - a'\right)\left(\frac{B}{2} - b'\right)
\]

This implies that \( q \leq 2a'b' \). The minimum value of \( a' + b' \) is obtained when \( a' = b' = \sqrt{q/2} \). Therefore the minimum value of \( a_1 + b_1 \) is \( A/2 + B/2 + 2\sqrt{q/2} \). This completes the proof of the lemma. \( \blacksquare \)

**Corollary 1** Let \( G \) be a bipartite graph with partite sets \( A \) and \( B \). If \( |E(G)| = \frac{|A||B|}{2} + q \), where \( q \geq 0 \), then there is a component in \( G \) of size at least \( |V(G)|/2 + \sqrt{2q} \).

**Proof.** Let \( Q_1, Q_2, \ldots, Q_k \) be the components of \( G \). Let \( a_i = |A \cap Q_i| \) and \( b_i = |B \cap Q_i| \) for all \( i = 1, 2, \ldots, k \). We note that \( \sum_{i=1}^{k} a_i b_i \geq |A||B|/2 + q \). By Lemma 1, we have \( a_i + b_i \geq A+B + \sqrt{2q} \) for some \( i \). This completes the proof. \( \blacksquare \)

**Lemma 2** Let \( T \) be a triangle-tree in a digraph \( D \), and let \( X \subseteq V(T) \) and \( Y \subseteq V(T) \) be such that \( |X| + |Y| \geq |V(T)| + 2 \). Then there exists a triangle \( C \) in \( T \) such that the three disjoint triangle-trees in \( T - E(C) \) can be named \( T_1, T_2, T_3 \) such that \( Y \) intersects both \( T_1 \) and \( T_2 \) and \( X \) intersects both \( T_2 \) and \( T_3 \).

**Proof.** We show this by induction. As \( |X| + |Y| \geq |V(T)| + 2 \), we note that \( T \) contains at least one triangle. If \( T \) contains only one triangle then the lemma holds as either \( X \) or \( Y \) equals \( V(T) \), and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that \( T \) contains at least two triangles. Let \( T = T_1 \cup C \), where \( C \) is a triangle and \( T_1 \) is a triangle-tree. If \( |X \cap V(T_1)| + |Y \cap V(T_1)| \geq |V(T)| + 2 \), then we are done by induction. So assume that this is not the case. As \( |V(T_1)| = |V(T)| - 2 \) this implies that \( |X \cap V(T_1)| + |Y \cap V(T_1)| \geq 3 \).

Without loss of generality assume that \( |X \cap V(T_1)| \geq 2 \) and \( |Y \cap V(T_1)| \geq 1 \). Let \( T_2 \) be the singleton-tree consisting of a vertex in \( Y \cap V(T_1) \) and let \( T_3 \) be the singleton-tree \( X \cap (V(T_1) \cup V(T_2)) \). Note that
$T - E(C)$ consists of the triangle-trees $T_1, T_2$ and $T_3$. By definition, $X$ intersects both $T_2$ and $T_3$ and $Y$ intersects $T_2$. If $Y$ also intersects $T_1$, we have our conclusion. If not, since $|X| + |Y| \geq |V(T)| + 2$, we have $Y = T_2 \cup T_3$ and $X = V(T)$, and free to rename $T_1, T_2, T_3$, we have our conclusion.

3 Proof of Theorem 1.

We will need the following results:

**Theorem 2** (Tewes and Volkmann [5]) Let $D$ be a $p$-partite tournament with partite sets $V_1, V_2, \ldots V_p$. Then there exists a partition $Q_1, Q_2, \ldots, Q_k$ of $D$ such that

- each $Q_i$ induces an independent set or a strong component,
- there are no arcs from $Q_j$ to $Q_i$ for all $j > i$, and there is an arc from $Q_i$ to $Q_{i+1}$ for all $i = 1, 2, \ldots, k - 1$.

**Theorem 3** (Guo and Volkmann [2]) Let $D$ be a strong $p$-partite tournament with partite sets $V_1, V_2, \ldots V_p$. For every $1 \leq i \leq p$, there exists a vertex $x \in V_i$ which belongs to a $k$-circuit for all $3 \leq k \leq p$.

Now, we assume that $D$ is a strong tournament as otherwise we just consider the terminal strong component. Let $T$ be a maximum size triangle-tree in $D$, and assume for the sake of contradiction that $|V(T)| < 2d^+(D) + 1$. Let $D^{MT}$ be the multipartite tournament obtained from $D$ by deleting all the arcs with both endpoints in $V(T)$. Let $V_1, V_2, \ldots, V_i$ be the partite sets in $D^{MT}$ such that $V_i = V(T)$ and $|V_i| = 1$ for all $i > 1$.

Let $Q_1, Q_2, \ldots, Q_k$ be a partition of $V(D^{MT})$ given by Theorem 2.

If there is a $Q_i$ with $Q_i \cap V_1 \neq \emptyset$ and $Q_i \not\subseteq V_1$ then we obtain the following contradiction. Since $Q_i \not\subset V_1$, we observe that $Q_i$ contains at least two partite set. In addition, note that at least three partite sets intersect $Q_i$ as $D^{MT}(Q_i)$ would not be strong if there were only two partite sets since $|V_i| = 1$ for all $i > 1$. By Theorem 3, in the subgraph of $D^{MT}$ induced by $Q_i$, there is a $3$-circuit containing exactly one vertex from $V_1$. This contradicts the maximality of $T$. So every set $Q_i$ is either a subset of $V_1$ or is disjoint from $V_1$.

Note that $Q_1 \cap V_1 \neq \emptyset$ and $Q_k \cap V_1 \neq \emptyset$, as otherwise $D$ would not be strong. Applying the observation above, we obtain $Q_1 \cup Q_k \subset V_1$. Let $D' = D(V_1)$. If there is a vertex $x \in Q_k$ with $d_{D'}^+(x) \leq \frac{|V_1| - 1}{2}$, then $d_{D'}^+(x) \leq \frac{|V_1| - 1}{2}$, which implies that $|V(T)| \geq 2d^+(D) + 1$, a contradiction. So $d_{D'}^+(x) \geq \frac{|V_1| + 1}{2}$ for all $x \in Q_k$, as $|V_1|$ is odd.

Let $G_1$ denote the bipartite graph with partite sets $Q_k$ and $V_1 - Q_k$, and with $E(G_1) = \{uv \mid u \in Q_k, v \in V_1 - Q_k, uv \in E(D)\}$. Note that the following now holds by the above.

$$|Q_k| \left| \frac{|V_1| + 1}{2} \right| \leq \sum_{u \in Q_k} d_{D'}^+(u) = \left( \frac{|Q_k|}{2} \right) + |E(G_1)|$$

This implies that $|E(G_1)| \geq \frac{|Q_k|(|V_1| - |Q_k|)}{2} + |Q_k|$, which by Corollary 1 implies that there is a component in $G_1$ of size at least $|V_1|/2 + \sqrt{2|Q_k|} \geq |V_1|/2 + \sqrt{2}$. As the size of the maximum component in $G_1$ is an integer it is at least $|V_1|/2 + 3/2$. Two cases can now occur:

- If $|Q_{k-1}| > 1$ or $Q_{k-2} \not\subseteq V_1$ (or both). If $|Q_{k-1}| > 1$ then let $Z = \{z_1, z_2\}$ be any two distinct vertices in $Q_{k-1}$ otherwise let $Z$ be any two distinct vertices in $Q_{k-1} \cup Q_{k-2}$. By the definition of the $Q_i$’s we note that $Z \cap V_1 = \emptyset$ and there are all arcs from $(V_1 - Q_k)$ to $Z$ and from $Z$ to $Q_k$. We let $X = Y$ be the vertices of a component in $G_1$ of size at least $(|V_1| + 3)/2$ and use Lemma 2 to find a triangle $C$ in $T$, such that the three disjoint triangle-trees, $T_1, T_2$ and $T_3$, of $T - E(C)$ all intersect
X (as X = Y). As X are the vertices of a component in G1 there are edges, u1v1 and u2v2, from G1 such that the following holds. The edge u1v1 connects T3 and Tj, where u2v2 connects T3−j and Tj ∪ T3, generality assume that u1, u2 ∈ Qk and v1, v2 ∈ V1 − Qk. Now T − E(C) together with the vertices z1 and z2 as well as the 3-circuits v1z1u1v1 and v2z2u2v2 is a triangle-tree in D with more triangles than T, a contradiction.

- If |Qk−1| = 1 and Qk−2 ⊆ V1. Note that k > 3, as otherwise |V(D) \ V(T)| = 1 and we have a contradiction to our assumption. This implies that k > 4 as Q1 ⊆ V1, which implies that Q2 ⊆ V1. Now let Qk−1 = {z1} and let z2 ∈ Qk−3 be arbitrary. Let G2 denote the bipartite graph with partite sets A = Qk ∪ Qk−2 and B = V1 − A, and with E(G2) = {uv | u ∈ A, v ∈ B, uv ∈ E(D)}. Recall that d+y(x) ≥ |Vp|/2 for all x ∈ Qk. Analogously we get that d+y(y) ≥ |Vp|/2 − 1 for all y ∈ Qk−2 (as |Qk−1| = 1). This implies the following.

\[ |A||V1| + 1/2 − |Qk−2| ≤ \sum_{u \in A} d+y(u) = (|A|/2) + |E(G2)| \]  

This implies that |E(G2)| ≥ |A||V1| − |A| − |Qk−2|, which by Corollary 1 implies that there is a component in G2 of size at least |V1|/2 + \sqrt{2|Qk|}, as |A| − |Qk−2| = |Qk|. Note that |Qk| > 1, as otherwise the vertex in Qk−1 only has out-degree one, a contradiction. Therefore there is a component in G2 of size at least |V1|/2 + 2 and so at least |V1|/2 + 5/2 as V1 is odd.

Let X be the vertices of a component in G1 of size at least |V1|/2 + 3/2 and let Y be the vertices in a connected component of G2 of size at least |V1|/2 + 5/2. Now use Lemma 2 to find a triangle C in T, such that the three disjoint triangle-trees, T1, T2 and T3, of T − E(C) have the following property. The set Y intersects T1 and T2 and the set X intersects T2 and T3. Due to the definition of X and Y there exists edges, u1v1 ∈ E(G1) and u2v2 ∈ E(G2), such that the following holds. The edge u1v1 connects T3 and Tj, where j ∈ {1, 2} and u2v2 connects T3−j and Tj ∪ T3. Without loss of generality assume that u1, u2 ∈ Qk and v1, v2 ∈ V1 − Qk. Now T − E(C) together with the vertices z1 and z2 as well as the 3-circuits v1z1u1v1 and v2z2u2v2 is a triangle-tree in D with more triangles than T, a contradiction. This completes the proof.

\[ \blacksquare \]

References


