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Minimum implicational basis for $\land$-semidistributive lattices

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Abstract

For a $\land$-semidistributive lattice $L$, we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice $L$. We also provide a polynomial time algorithm to compute a minimum implicational basis for $L$.

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1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited nondeterminism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we show by using a dependence relation in [11] that the class of $\land$-semidistributive lattices is another tractable case.

Consider a finite set $U$. A subset $C \subseteq 2^U$ is said to be a closure system if $C$ is closed under set-intersection and containing the set $U$. An implication on $U$ is an ordered pair $(A,B)$ of subsets of $U$, denoted by $A \rightarrow B$. The set $A$ is called the premise and the set $B$ the conclusion of the implication $A \rightarrow B$. Let $\Sigma$ be a set of implications on $U$. A subset $D \subseteq U$ is $\Sigma$-closed if for each implication $A \rightarrow B$ in $\Sigma$, $A \subseteq D$ implies $B \subseteq D$.

The set of $\Sigma$-closed subsets of $U$, denoted by $\mathcal{C}(\Sigma)$, is a closure system on $U$. Conversely, given a closure system $C$ on $U$, a family $\Sigma$ of implications on $U$ is said an implicational basis for $C$ if $C = \mathcal{C}(\Sigma)$. An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice $L$ as the closure system $C_L$ on the set $J(L)$ of its join-irreducible elements. More precisely, put $J(a) = \{ j \in J(L): j \leq a \}$ for $a \in L$. .
Then \( C_L = \{ J(a) : a \in L \} \) is a closure system on \( J(L) \) which, as a lattice ordered by inclusion, is isomorphic to \( L \).

Fig. 1 gives an example of the closure system \( C_L \) associated to a lattice \( L \).

The closure system \( C_L \) can be defined by the set of its meet-irreducible elements \( M(C_L) = \{ J(m) : m \in M(L) \} \), where \( M(L) \) denotes the set of meet-irreducible elements of \( L \). Each element of \( C_L \) can be obtained as the intersection of some elements of \( M(C_L) \).

The problem we study is:

**Problem:** Minimum implicational basis

**Instance:** The set of meet-irreducible elements \( M(C_L) \) of the closure system \( C_L \).

**Question:** Find a minimum basis \( \Sigma \) for \( C_L \).

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal\(^1\) implicational basis for modular lattices. In the following, we study the case of \( \land \)-semidistributive lattices. For such lattices we show that the number of implications of a minimum implicational basis is at most \( |J(L)|^2\) and give a polynomial time algorithm to compute such a basis.

2. Some properties of \( \land \)-semidistributive lattices

Let \( L \) be a finite lattice. We note \( \lor \) the join operation, \( \land \) the meet operation and \( \lessdot \) the cover relation of \( L \). If \( j \) is a join-irreducible element of \( L \), we use \( j_* \) to denote the unique element covered by \( j \). Dually, we use \( m^* \) to denote the unique element covering a meet-irreducible element \( m \).

We will use the arrow relations introduced by Wille [15]: for \( x, y \in L, x \lor y \) means that \( x \) is a minimal element of \( \{ z \in L : z \not\leq x \} \), \( x \lhd y \) means that \( y \) is a maximal element of \( \{ z \in L : z \not\geq y \} \) and \( x \vdash y \) means that \( x \lor y \) and \( x \lessdot y \). Recall that \( \lhd, \vdash, \dashv \) are relations defined on \( J(L) \times M(L) \), where \( J(L) \) is the set of join-irreducible elements and \( M(L) \) the set of meet-irreducible elements of \( L \).

In the following, we deal essentially with \( \land \)-semidistributive lattices. Let us recall that a lattice \( L \) is said \( \land \)-semidistributive if for all elements \( x, y, z \in L, x \land y = x \land z \) implies \( x \land y = x \land (y \lor z) \). A \( \land \)-semidistributive lattice is said semidistributive if for all elements \( x, y, z, x \lor y = x \lor z \) implies \( x \lor y = x \lor (y \lor z) \). The following characterization of these lattices are well known (see, for example, [6]):

**Property 1.** A finite lattice \( L \) is \( \land \)-semidistributive if and only if for any \( j \in J(L) \) there exists a unique \( m \in M(L) \) such that \( j \vdash m \).

For any \( \land \)-semidistributive lattice \( L \) and \( j \in J(L) \), we denote by \( m(j) \) the unique element \( m \in M(L) \) such that \( j \vdash m \).

We define the mapping \( \gamma : J(L) \to 2^{M(L)} \) by \( \gamma(j) = \{ m \in M(L) : j \vdash m \} \). This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the \( \gamma \) mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming

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\(^1\) An implication is known as optimal if the sum of the cardinality of the premises and the conclusions of all the implications is minimal.
that the lattice $L$ is ∧-semidistributive) as follows (see, for example, [8,11]):

Let $j, j' \in J(L)$.
Then $j B j'$ if and only if $j \neq j'$ and $j' \not\in m(j), j' \leq m(j)$.

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation $B$ and some classes of lattices. For example, Nation has shown that a ∧-semidistributive lattice is semidistributive if and only if it contains no $B$-cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation $B$ using the mapping $\gamma$.

**Lemma 1.** Let $L$ be a ∧-semidistributive lattice, $j, j' \in J(L)$.

$j B j'$ if and only if $j \neq j'$ and $m(j) \in \gamma(j')$.

### 3. Minimum implicational basis for ∧-semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a ∧-semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system $C_L$. The first one is obvious since the elements of $C_L$ are order ideals of the induced poset by $J(L)$.

**Lemma 2.** Let $j, j' \in J(L)$ such that $j < j'$ and $X \in C_L$. Then $j' \in X$ implies $j \in X$.

Consider now a ∧-semidistributive lattice $L$ and $j, j' \in J(L)$ such that $j B j'$. We denote by $P_{jj'}$ the set $J(j) \cup J(j')$.

**Lemma 3.** Let $L$ be a ∧-semidistributive lattice and $j, j' \in J(L)$ such that $j B j'$ and $X \in C_L$. Then $P_{jj'} \subseteq X$ implies $j \in X$.

**Proof.** Let $x \in L$ such that $X = J(x)$ and $P_{jj'} \subseteq X$. Since $J(j) \subseteq X$ this implies that $j_{x} \lor j' \leq x$, and then it suffices to prove that $j \leq j_{x} \lor j'$.

Suppose that $j \not\in J(j)$ and let $m' \in M(L)$ be a maximal element of $\{z \in L \mid z \not\in j \mathrm{ and } z \geq j_{x} \lor j'\}$. By definition of $m'$, we have $j \downarrow m'$. Moreover $j \downarrow m'$ since $j_{x} \leq m'$.

Consider now the meet-irreducible $m(j)$ associated with $j$. Then $j' \not\subseteq m(j)$ since $j B j'$. Thus since $j' \leq m'$, $m'(j)$ and $m(j)$ are two distinct elements such that $j \not\in m'$ and $j \not\in m(j)$. This contradicts the fact that $L$ is ∧-semidistributive. □

We can now define a particular set of implications associated to a ∧-semidistributive lattice $L$. Let $\Sigma_{1} = \{j \rightarrow J(j)\}, \Sigma_{2} = \{P_{jj'} \rightarrow j \mid j \in J(L) \land j B j'\}$ and $\Sigma = \Sigma_{1} \cup \Sigma_{2}$.

For example, the sets of implications $\Sigma_{1}$ and $\Sigma_{2}$ for the lattice in Fig. 1 are $\Sigma_{1} = \{b \rightarrow ab, d \rightarrow cd, e \rightarrow cde, f \rightarrow cdef, g \rightarrow cdg\}$ and $\Sigma_{2} = \{ac \rightarrow e, abc \rightarrow d, acdef \rightarrow b, acd \rightarrow b, cdef \rightarrow f\}$.

The following theorem shows that $\Sigma$ is an implicational basis for $C_L$.

**Theorem 1.** Let $L$ be a ∧-semidistributive lattice. Then the set of implications $\Sigma$ is an implicational basis for $C_L$.

**Proof.** We need to show that $C_{\Sigma} = C_L$.

Let $X \in C_L$. By Lemma 2, $X$ is $\Sigma_{1}$-closed. By Lemma 3, $X$ is $\Sigma_{2}$-closed. Then $X$ is $\Sigma$-closed and $C_L \subseteq C_{\Sigma}$.

Now let us show that $C_{\Sigma} \subseteq C_L$. Let $X \in C_{\Sigma}$. Let $x_{0} = \bigvee X \mathrm{ i.e., the least closed set containing } X$. Clearly $X$ is an ideal since it is $\Sigma_{1}$-closed. Suppose that $X \not\in C_L$ and let $j$ be a minimal element of $J(x_{0}) \setminus X$. Since $j \leq x_{0}$, we have $x_{0} \not\in m(j)$. Moreover $X \not\subseteq J(m(j))$, otherwise one would have $\bigvee X \leq m(j)$ and then $\bigvee X \not\leq x_{0}$. Thus there exists an element $j' \in X$ such that $m(j) \in \gamma(j')$ and therefore $P_{jj'} \to j \in \Sigma$ with $P_{jj'} \subseteq X$ and $j \not\in X$. Then $X$ is not $\Sigma$-closed, which concludes the proof. □

**Corollary 1.** Let $L$ be a ∧-semidistributive lattice. Then there exists an implicational basis for $C_L$ with at most $|B| + |J(L)|$ implications, where $|B|$ is the number of arcs in the relation $B$. 

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**Fig. 2.** The arrow relations and mapping $\gamma$ of the lattice in Fig. 1.

**Fig. 3.** The graph of the $B$ relation for the lattice in Fig. 1.
Data: Let $L$ be a $\land$-semidistributive lattice and $\mathcal{M}(C_L)$ the set of meet-irreducible elements of $C_L$.

Result: A minimum basis $\Sigma$ of the closure system $C_L$.

begin
\[ \Sigma = \emptyset; \]
for $j \in J(L)$ do
\[ \Sigma = \Sigma \cup \{ j \rightarrow \varphi(j) \}; \]
for $j' \in J(L)$ do
\[ P = (\varphi(j))\setminus \{ j \} \cup \varphi(j'); \]
\[ \Sigma = \Sigma \cup \{ P \rightarrow (P) \}; \]
end
\[ \Sigma = \text{a nonredundant cover of } \Sigma; \]

Algorithm 1. Minimum-Basis($\mathcal{M}(C_L)$).

Clearly the set $\Sigma$ of implications obtained as above is in general not minimum. For instance, for the set $\Sigma$ associated to the lattice in Fig. 1, the implication \( ac \rightarrow d \) is redundant and can be removed from $\Sigma$ without changing $C(\Sigma)$.

In the following we give a polynomial time algorithm to compute a minimum basis for a $\land$-semidistributive lattice.

3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13]. Indeed, the algorithm in [13] computes a minimum basis (called there a minimum cover) from any given basis in polynomial time.

Let $\mathcal{M}(C_L)$ be the set of meet-irreducible elements. Consider the closure operator $\varphi : 2^J \rightarrow 2^J$, with for $X \subseteq J$, $\varphi(X) = \bigcap \{ M \in \mathcal{M}(C_L) \mid X \subseteq M \}$. The images of the mapping $\varphi$ are said closed sets, and they correspond to the elements of the closure system $C_L$.

Remark 1. We replaced $P \rightarrow j$ by $P \rightarrow \varphi(P)$ to guarantee the minimality after the calculation of a nonredundant cover of $\Sigma$.

Remark 2. Let us note that Algorithm 1 does not compute the same $\Sigma$ as that of Theorem 1. This to avoid the computation of the relation $B$. But like the whole of the implications calculated by Algorithm 1 contains all implications of Theorem 1 (relative with the preceding remark), this guaranteed us to have a cover of $C_L$.

Theorem 2. Let $L$ be a $\land$-semidistributive lattice. Then Algorithm 1 computes a minimum implicational basis $\Sigma$ of $C_L$ in $O(|J|^5 + |J|^3 |M(C_L)|)$ time complexity. Moreover, the size of $\Sigma$ is at most $|J(L)|^2$ implications.

Proof. Theorem 1 guarantees that $\Sigma$ is a basis for the closure system $C_L$. Since the conclusions of all implications are closed by the mapping $\varphi$, the result in [13] guarantees that a not redundant basis is minimum.

Computing the closure of a set $X \subseteq J(L)$ by $\varphi$ can be done in $O(|J(L)||M(C_L)|)$ time complexity. Thus the total time complexity for computing a basis is in $O(|J(L)|^3 |M(C_L)|)$. Now computing a not redundant basis can be done in $O(|J(L)||\Sigma|^2)$. Since $\Sigma$ has at most $|J(L)|^2$ implications, we conclude that the time complexity of Algorithm 1 is in $O(|J(L)|^5 + |J(L)|^3 |M(C_L)|)$. $\square$

Uncited references

[9]

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