Minimum Implicational Basis for -Semidistributive Lattices
Philippe Janssen, Lhouari Nourine

To cite this version:

HAL Id: lirmm-00101743
https://hal-lirmm.ccsd.cnrs.fr/lirmm-00101743
Submitted on 28 Sep 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Minimum implicational basis for ∧-semidistributive lattices

Philippe Janssen a, Lhouari Nourine b,*

a LIRMM, Université Montpellier II, 161, rue Ada, F34392 Montpellier cedex 5, France
b LIMOS, Université Blaise Pascal, Campus des cézeaux, F63173 Aubiere, France

Received 13 January 2006; received in revised form 21 February 2006; accepted 21 April 2006

Communicated by L. Boasson

Abstract

For a ∧-semidistributive lattice \( L \), we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice \( L \). We also provide a polynomial time algorithm to compute a minimum implicational basis for \( L \).

Keywords: Algorithms; Lattice; Closure system; Minimum implicational basis

1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited non-determinism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we show by using a dependence relation in [11] that the class of ∧-semidistributive lattices is another tractable case.

Consider a finite set \( U \). A subset \( C \subseteq \mathcal{P}(U) \) is said to be a closure system if \( C \) is closed under set-intersection and containing the set \( U \). An implication on \( U \) is an ordered pair \((A, B)\) of subsets of \( U \), denoted by \( A \rightarrow B \). Let \( \Sigma \) be a set of implications on \( U \). A subset \( D \subseteq U \) is \( \Sigma \)-closed if for each implication \( A \rightarrow B \) in \( \Sigma \), \( A \subseteq D \) implies \( B \subseteq D \). The set of \( \Sigma \)-closed subsets of \( U \), denoted by \( C(\Sigma) \), is a closure system on \( U \). Conversely, given a closure system \( C \) on \( U \), a family \( \Sigma \) of implications on \( U \) is said an implicational basis for \( \mathcal{C} \) if \( \mathcal{C} = C(\Sigma) \). An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice \( L \) as the closure system \( C_L \) on the set \( J(L) \) of its join-irreducible elements. More precisely, put \( J(a) = \{ j \in J(L) : j \leq a \} \) for \( a \in L \).
Then $C_L = \{ J(a): a \in L \}$ is a closure system on $J(L)$ which, as a lattice ordered by inclusion, is isomorphic to $L$.

Fig. 1 gives an example of the closure system $C_L$ associated to a lattice $L$.

The closure system $C_L$ can be defined by the set of its meet-irreducible elements $M(C_L) = \{ J(m): m \in M(L) \}$, where $M(L)$ denotes the set of meet-irreducible elements of $L$. Each element of $C_L$ can be obtained as an intersection of some elements of $M(C_L)$.

The problem we study is:

**Problem:** Minimum implicational basis

**Instance:** The set of meet-irreducible elements $M(C_L)$ of the closure system $C_L$.

**Question:** Find a minimum basis $\Sigma$ for $C_L$.

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal\(^1\) implicational basis for modular lattices. In the following, we study the case of $\wedge$-semidistributive lattices. For such lattices we show that the number of implications of a minimum implicational basis is at most $|J(L)|^2$ and give a polynomial time algorithm to compute such a basis.

**2. Some properties of $\wedge$-semidistributive lattices**

Let $L$ be a finite lattice. We note $\vee$ the join operation, $\wedge$ the meet operation and $\prec$ the cover relation of $L$. If $j$ is a join-irreducible element of $L$, we use $j^*$ to denote the unique element covered by $j$. Dually, we use $m^*$ to denote the unique element covering a meet-irreducible element $m$.

We will use the arrow relations introduced by Wille [15]: for $x, y \in L$, $x \downarrow y$ means that $x$ is a minimal element of $\{ z \in L: z \not\leq x \}$, $x \uparrow y$ means that $y$ is a maximal element of $\{ z \in L: z \not\geq y \}$ and $x \downarrow y$ means that $x \uparrow y$ and $x \downarrow y$. Recall that $\uparrow, \downarrow, \rightarrow$ are relations defined on $J(L) \times M(L)$, where $J(L)$ is the set of join-irreducible elements and $M(L)$ the set of meet-irreducible elements of $L$.

In the following, we deal essentially with $\wedge$-semidistributive lattices. Let us recall that a lattice $L$ is said $\wedge$-semidistributive if for all elements $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$. A $\wedge$-semidistributive lattice is said semidistributive if for all elements $x, y, z, x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. The following characterization of these lattices are well known (see, for example, [6]):

**Property 1.** A finite lattice $L$ is $\wedge$-semidistributive if and only if for any $j \in J(L)$ there exists a unique $m \in M(L)$ such that $j \nleq m$.

For any $\wedge$-semidistributive lattice $L$ and $j \in J(L)$, we denote by $m(j)$ the unique element $m \in M(L)$ such that $j \nleq m$.

We define the mapping $\gamma: J(L) \rightarrow 2^{M(L)}$ by $\gamma(j) = \{ m \in M(L): j \nleq m \}$. This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the $\gamma$ mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming...
that the lattice $L$ is $\wedge$-semidistributive as follows (see, for example, [8,11]):

Let $j, j' \in J(L)$.  
Then $j B j'$ iff $j \neq j'$ and $j' \leq m(j)$.

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation $B$ and some classes of lattices. For example, Nation has shown that a $\wedge$-semidistributive lattice is semidistributive if and only if it contains no $B$-cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation $B$ using the mapping $\gamma$.

**Lemma 1.** Let $L$ be a $\wedge$-semidistributive lattice, $j, j' \in J(L)$.


$$j B j' \iff j \neq j' \text{ and } m(j) \in \gamma(j').$$

### 3. Minimum implicational basis for $\wedge$-semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a $\wedge$-semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system $C_L$. The first one is obvious since the elements of $C_L$ are order ideals of the induced poset by $J(L)$.

**Lemma 2.** Let $j, j' \in J(L)$ such that $j < j'$ and $X \in C_L$. Then $j' \in X$ implies $j \in X$.

Consider now a $\wedge$-semidistributive lattice $L$ and $j, j' \in J(L)$ such that $j B j'$. We denote by $P_{jj'}$ the set $J(f_x) \cup J(j')$.

**Lemma 3.** Let $L$ be a $\wedge$-semidistributive lattice and $j, j' \in J(L)$ such that $j B j'$ and $X \in C_L$. Then $P_{jj'} \subseteq X$ implies $j \in X$.

**Proof.** Let $x \in L$ such that $X = J(x)$ and $P_{jj'} \subseteq X$. Since $J(j_x) \subseteq X$ this implies that $j_x \lor j' \leq x$, and then it suffices to prove that $j \leq j_x \lor j'$.

Suppose that $j \neq j_x \lor j'$ and let $m' \in M(L)$ be a maximal element of $\{z \in L \mid z \not\leq j$ and $z \geq j_x \lor j'\}$. By definition of $m'$, we have $j \uparrow m'$. Moreover $j \downarrow m'$ since $j_x \leq m'$. Thus $j \uparrow m'$.

Consider now the meet-irreducible $m(j)$ associated with $j$. Then $j' \neq m(j)$ since $j B j'$. Thus since $j' \leq m'$, $m(j)$ and $m(j)$ are two distinct elements such that $j \uparrow m'$ and $j \downarrow m(j)$. This contradicts the fact that $L$ is $\wedge$-semidistributive. $\square$

We can now define a particular set of implications associated to a $\wedge$-semidistributive lattice $L$. Let $\Sigma_1 = \{j \rightarrow J(j)\}$, $\Sigma_2 = \{P_{jj'} \rightarrow j \mid j' \in J(L)$ and $j B j'\}$ and $\Sigma = \Sigma_1 \cup \Sigma_2$.

For example, the sets of implications $\Sigma_1$ and $\Sigma_2$ for the lattice in Fig. 1 are $\Sigma_1 = \{b \rightarrow ab, d \rightarrow cd, e \rightarrow cde, f \rightarrow cdef, g \rightarrow cdg\}$ and $\Sigma_2 = \{ac \rightarrow e, abc \rightarrow d, acdef \rightarrow b, acd \rightarrow d, cde \rightarrow f\}$.

The following theorem shows that $\Sigma$ is an implicational basis for $C_L$.

**Theorem 1.** Let $L$ be a $\wedge$-semidistributive lattice. Then the set of implications $\Sigma$ is an implicational basis for $C_L$.

**Proof.** We need to show that $C_\Sigma = C_L$.

Let $X \in C_L$. By Lemma 2, $X$ is $\Sigma_1$-closed. By Lemma 3, $X$ is $\Sigma_2$-closed. Thus $X$ is $\Sigma$-closed and $C_L \subseteq C_\Sigma$.

Now let us show that $C_\Sigma \subseteq C_L$. Let $X \in C_\Sigma$. Let $x_0 = \bigvee x$, i.e., the least closed set containing $X$. Clearly $X$ is an ideal since it is $\Sigma_1$-closed. Suppose that $X \not\subseteq C_L$ and let $j$ be a minimal element of $J(x_0) \backslash X$. Since $j \leq x_0$, we have $x_0 \neq m(j)$. Moreover $X \not\subseteq J(m(j))$, otherwise one would have $\bigvee X \leq m(j)$ and then $\bigvee X \neq x_0$.

Thus there exists an element $j' \in X$ such that $m(j) \in \gamma(j')$ and therefore $P_{jj'} \rightarrow j \in \Sigma$ with $P_{jj'} \subseteq X$ and $j \not\in X$. Then $X$ is not $\Sigma$-closed, which concludes the proof. $\square$

**Corollary 1.** Let $L$ be a $\wedge$-semidistributive lattice. Then there exists an implicational basis for $C_L$ with at most $|B| + |J(L)|$ implications, where $|B|$ is the number of arcs in the relation $B$. 

### Fig. 2. The arrow relations and mapping $\gamma$ of the lattice in Fig. 1.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>γ</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{6,7}</td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{4,5}</td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{3}</td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{5}</td>
</tr>
<tr>
<td>e</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{7}</td>
</tr>
<tr>
<td>f</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{2,4}</td>
</tr>
<tr>
<td>g</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>{1,2,4}</td>
</tr>
</tbody>
</table>

### Fig. 3. The graph of the $B$ relation for the lattice in Fig. 1.
Data: Let \( L \) be a \( \wedge \)-semidistributive lattice and \( \mathcal{M}(C_L) \)
the set of meet-irreducible elements of \( C_L \).

Result: A minimum basis \( \Sigma \) of the closure system \( C_L \).

begin
\[
\Sigma = \emptyset; \\
\text{for } j \in J(L) \text{ do} \\
\quad \Sigma = \Sigma \cup \{ j \rightarrow \varphi(j) \}; \\
\text{for } j' \in J(L) \text{ do} \\
\quad P = \{ \varphi(j') \} \cup \varphi(j'); \\
\quad \Sigma = \Sigma \cup \{ P \rightarrow (P) \}; \\
\] \( \Sigma \) is a nonredundant cover of \( \Sigma \);
end

Algorithm 1. Minimum-Basis(\( \mathcal{M}(C_L) \)).

Clearly the set \( \Sigma \) of implications obtained as above
is in general not minimum. For instance, for the set
\( \Sigma \) associated to the lattice in Fig. 1, the implication
\( ac \rightarrow d \) is redundant\(^2\) and can be removed from \( \Sigma \)
without changing \( C(\Sigma) \).

In the following we give a polynomial time algorithm
to compute a minimum basis for a \( \wedge \)-semidistributive
lattice.

3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13].
Indeed, the algorithm in [13] computes a minimum ba-
sis (called there a minimum cover) from any given basis
in polynomial time.

Let \( \mathcal{M}(C_L) \) be the set of meet-irreducible elements.
Consider the closure operator \( \varphi : 2^J \rightarrow 2^J \), with for \( X \subseteq J \),
\( \varphi(X) = \bigcap \{ M \in \mathcal{M}(C_L) \mid X \subseteq M \} \).
The images of the mapping \( \varphi \) are said closed sets, and they correspond
to the elements of the closure system \( C_L \).

Remark 1. We replaced \( P \rightarrow j \) by \( P \rightarrow \varphi(P) \) to guar-
antee the minimality after the calculation of a nonredu-
dant cover of \( \Sigma \).

Remark 2. Let us note that Algorithm 1 does not com-
pute the same \( \Sigma \) as that of Theorem 1. This to avoid
the computation of the relation \( B \). But like the whole of
the implications calculated by Algorithm 1 contains all
implications of Theorem 1 (relative with the preceding
remark), this guaranteed to us to have a cover of \( C_L \).

Theorem 2. Let \( L \) be a \( \wedge \)-semidistributive lattice. Then
Algorithm 1 computes a minimum implicational basis \( \Sigma \)
of \( C_L \) in \( O(|J|^3 + |J|^2 |\mathcal{M}(C_L)|) \) time complexity. More-
over, the size of \( \Sigma \) is at most \( |J(L)|^2 \) implications.

Proof. Theorem 1 guarantees that \( \Sigma \) is a basis for the closure system \( C_L \).
Since the conclusions of all implications are closed by the mapping \( \varphi \), the result in [13]
guarantees that a not redundant basis is minimum.

Computing the closure of a set \( X \subseteq J(L) \) by \( \varphi \)
can be done in \( O(|J(L)|^2 |\mathcal{M}(C_L)|) \) time complexity.
Thus the total time complexity for computing a ba-
sis is in \( O(|J(L)|^2 |\mathcal{M}(C_L)|) \). Now computing a not redundant basis can be done in \( O(|J(L)|\Sigma^2) \).
Since \( \Sigma \) has at most \( |J(L)|^2 \) implications, we conclude that
the time complexity of Algorithm 1 is in \( O(|J(L)|^5 +
|J(L)|^3 |\mathcal{M}(C_L)|) \). \( \square \)

Uncited references

[9]

Acknowledgements

The authors are very grateful to the referees for very
helpful comments.

References

[1] N. Caspard, B. Monjardet, The lattices of closure systems, clo-
sure operators, and implicational systems on a finite set: A sur-
[3] T. Eiter, G. Gottlob, Identifying the minimal transversals of a
hypergraph and related problems, SIAM Journal on Comput-
[4] T. Eiter, G. Gottlob, Hypergraph transversal computation and
related problems in logic and AI, in: European Conference on
Logics in Artificial Intelligence (JELIA’02), 2002, pp. 549–564.
[5] M.L. Fredman, L. Khachiyan, On the complexity of duali-
zation of monotone disjunctive normal forms, Journal of Algo-
informatives resultant d’un tableau de donnees binaires, Mathé-
tiques et Sciences humaines 95 (1986) 5–18.
distributive lattices and interval collapsing, Universalis Alge-
[10] D. Maier, The Theory of Relational Data Bases, Computer Sci-
[12] L. Nourine, Une structuration algorithmique de la théorie des
treillis, Habilitation à diriger des recherches, Université de Mont-
pellier II, France, July 2000.
