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# Minimum implicational basis for $\wedge$ -semidistributive lattices

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## Abstract

For a  $\wedge$ -semidistributive lattice  $L$ , we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice  $L$ . We also provide a polynomial time algorithm to compute a minimum implicational basis for  $L$ .

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*Keywords:* Algorithms; Lattice; Closure system; Minimum implicational basis

## 1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited non-determinism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we

show by using a dependence relation in [11] that the class of  $\wedge$ -semidistributive lattices is another tractable case.

Consider a finite set  $U$ . A subset  $C \subseteq 2^U$  is said to be a closure system if  $C$  is closed under set-intersection and containing the set  $U$ . An implication on  $U$  is an ordered pair  $(A, B)$  of subsets of  $U$ , denoted by  $A \rightarrow B$ . The set  $A$  is called the premise and the set  $B$  the conclusion of the implication  $A \rightarrow B$ . Let  $\Sigma$  be a set of implications on  $U$ . A subset  $D \subseteq U$  is  $\Sigma$ -closed if for each implication  $A \rightarrow B$  in  $\Sigma$ ,  $A \subseteq D$  implies  $B \subseteq D$ . The set of  $\Sigma$ -closed subsets of  $U$ , denoted by  $\mathcal{C}(\Sigma)$ , is a closure system on  $U$ . Conversely, given a closure system  $C$  on  $U$ , a family  $\Sigma$  of implications on  $U$  is said an implicational basis for  $C$  if  $C = \mathcal{C}(\Sigma)$ . An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice  $L$  as the closure system  $\mathcal{C}_L$  on the set  $J(L)$  of its join-irreducible elements. More precisely, put  $J(a) = \{j \in J(L) : j \leq a\}$  for  $a \in L$ .

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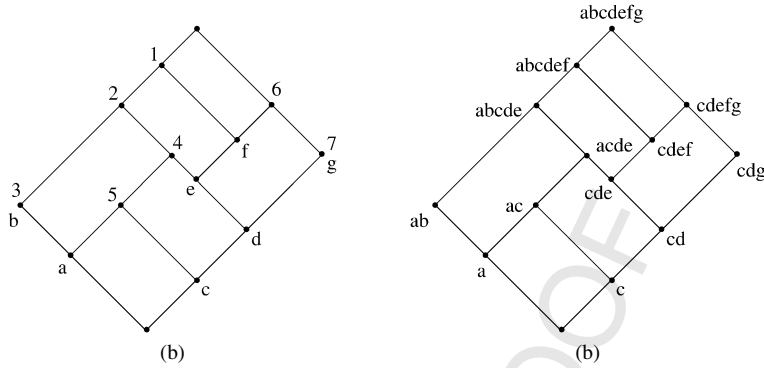


Fig. 1. (a) A lattice  $L$  where join-irreducible (resp. meet-irreducible) elements are labeled by letters (resp. numbers); (b) The closure system  $cl$  associated to  $L$ .

Then  $\mathcal{C}_L = \{J(a) : a \in L\}$  is a closure system on  $J(L)$  which, as a lattice ordered by inclusion, is isomorphic to  $L$ .

Fig. 1 gives an example of the closure system  $\mathcal{C}_L$  associated to a lattice  $L$ .

The closure system  $\mathcal{C}_L$  can be defined by the set of its meet-irreducible elements  $\mathcal{M}(\mathcal{C}_L) = \{J(m) : m \in M(L)\}$ , where  $M(L)$  denotes the set of meet-irreducible elements of  $L$ . Each element of  $\mathcal{C}_L$  can be obtained as intersection of some elements of  $\mathcal{M}(\mathcal{C}_L)$ .

The problem we study is:

**Problem:** Minimum implicational basis

**Instance:** The set of meet-irreducible elements  $\mathcal{M}(\mathcal{C}_L)$  of the closure system  $\mathcal{C}_L$ .

**Question:** Find a minimum basis  $\Sigma$  for  $\mathcal{C}_L$ .

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal<sup>1</sup> implicational basis for modular lattices. In the following, we study the case of  $\wedge$ -semidistributive lattices. For such lattices we show that the number of implications of a minimum implicational basis is at most  $|J(L)|^2$  and give a polynomial time algorithm to compute such a basis.

## 2. Some properties of $\wedge$ -semidistributive lattices

Let  $L$  be a finite lattice. We note  $\vee$  the join operation,  $\wedge$  the meet operation and  $\prec$  the cover relation of  $L$ . If  $j$  is a join-irreducible element of  $L$ , we use  $j_*$  to denote

the unique element covered by  $j$ . Dually, we use  $m^*$  to denote the unique element covering a meet-irreducible element  $m$ .

We will use the arrow relations introduced by Wille [15]: for  $x, y \in L$ ,  $x \downarrow y$  means that  $x$  is a minimal element of  $\{z \in L : z \not\leq x\}$ ,  $x \uparrow y$  means that  $y$  is a maximal element of  $\{z \in L : z \not\geq y\}$  and  $x \downarrow\uparrow y$  means that  $x \uparrow y$  and  $x \downarrow y$ . Recall that  $\uparrow, \downarrow, \downarrow\uparrow$  are relations defined on  $J(L) \times M(L)$ , where  $J(L)$  is the set of join-irreducible elements and  $M(L)$  the set of meet-irreducible elements of  $L$ .

In the following, we deal essentially with  $\wedge$ -semidistributive lattices. Let us recall that a lattice  $L$  is said  $\wedge$ -semidistributive if for all elements  $x, y, z \in L$ ,  $x \wedge y = x \wedge z$  implies  $x \wedge y = x \wedge (y \vee z)$ . A  $\wedge$ -semidistributive lattice is said semidistributive if for all elements  $x, y, z$ ,  $x \vee y = x \vee z$  implies  $x \vee y = x \vee (y \wedge z)$ . The following characterization of these lattices are well known (see, for example, [6]):

**Property 1.** A finite lattice  $L$  is  $\wedge$ -semidistributive if and only if for any  $j \in J(L)$  there exists a unique  $m \in M(L)$  such that  $j \downarrow\uparrow m$ .

For any  $\wedge$ -semidistributive lattice  $L$  and  $j \in J(L)$ , we denote by  $m(j)$  the unique element  $m \in M(L)$  such that  $j \downarrow\uparrow m$ .

We define the mapping  $\gamma : J(L) \rightarrow 2^{M(L)}$  by  $\gamma(j) = \{m \in M(L) : j \downarrow\uparrow m\}$ . This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the  $\gamma$  mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming

<sup>1</sup> An implication is known as optimal if the sum of the cardinality of the premises and the conclusions of all the implications is minimal.

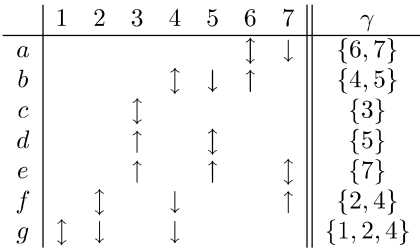


Fig. 2. The arrow relations and mapping  $\gamma$  of the lattice in Fig. 1.

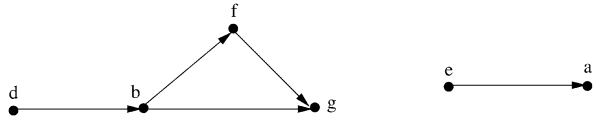


Fig. 3. The graph of the  $B$  relation for the lattice in Fig. 1.

that the lattice  $L$  is  $\wedge$ -semidistributive) as follows (see, for example, [8,11]):

Let  $j, j' \in J(L)$ .  
Then  $j B j'$  iff  $j \neq j', j' \not\leq m(j), j'_* \leq m(j)$ .

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation  $B$  and some classes of lattices. For example, Nation has shown that a  $\wedge$ -semidistributive lattice is semidistributive if and only if it contains no  $B$ -cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation  $B$  using the mapping  $\gamma$ .

**Lemma 1.** Let  $L$  be a  $\wedge$ -semidistributive lattice,  $j, j' \in J(L)$ .

$j B j'$  iff  $j \neq j'$  and  $m(j) \in \gamma(j')$ .

### 3. Minimum implicational basis for $\wedge$ -semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a  $\wedge$ -semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system  $\mathcal{C}_L$ . The first one is obvious since the elements of  $\mathcal{C}_L$  are order ideals of the induced poset by  $J(L)$ .

**Lemma 2.** Let  $j, j' \in J(L)$  such that  $j < j'$  and  $X \in \mathcal{C}_L$ . Then  $j' \in X$  implies  $j \in X$ .

Consider now a  $\wedge$ -semidistributive lattice  $L$  and  $j, j' \in J(L)$  such that  $j B j'$ . We denote by  $P_{jj'}$  the set  $J(j_*) \cup J(j')$ .

**Lemma 3.** Let  $L$  be a  $\wedge$ -semidistributive lattice and  $j, j' \in J(L)$  such that  $j B j'$  and  $X \in \mathcal{C}_L$ . Then  $P_{jj'} \subseteq X$  implies  $j \in X$ .

**Proof.** Let  $x \in L$  such that  $X = J(x)$  and  $P_{jj'} \subseteq X$ . Since  $J(j_*) \subset X$  this implies that  $j_* \vee j' \leq x$ , and then it suffices to prove that  $j \leq j_* \vee j'$ .

Suppose that  $j \not\leq j_* \vee j'$  and let  $m' \in M(L)$  be a maximal element of  $\{z \in L \mid z \not\leq j \text{ and } z \geq j_* \vee j'\}$ . By definition of  $m'$ , we have  $j \uparrow m'$ . Moreover  $j \downarrow m'$  since  $j_* \leq m'$ . Thus  $j \downarrow m'$ .

Consider now the meet-irreducible  $m(j)$  associated with  $j$ . Then  $j' \not\leq m(j)$  since  $j B j'$ . Thus since  $j' \leq m'$ ,  $m'$  and  $m(j)$  are two distinct elements such that  $j \downarrow m'$  and  $j \downarrow m(j)$ . This contradicts the fact that  $L$  is  $\wedge$ -semidistributive.  $\square$

We can now define a particular set of implications associated to a  $\wedge$ -semidistributive lattice  $L$ . Let  $\Sigma_1 = \{j \rightarrow J(j)\}$ ,  $\Sigma_2 = \{P_{jj'} \rightarrow j \mid j' \in J(L) \text{ and } j B j'\}$  and  $\Sigma = \Sigma_1 \cup \Sigma_2$ .

For example, the sets of implications  $\Sigma_1$  and  $\Sigma_2$  for the lattice in Fig. 1 are  $\Sigma_1 = \{b \rightarrow ab, d \rightarrow cd, e \rightarrow cde, f \rightarrow cdef, g \rightarrow cdg\}$  and  $\Sigma_2 = \{acd \rightarrow e, abc \rightarrow d, acdef \rightarrow b, acdg \rightarrow b, cdeg \rightarrow f\}$ .

The following theorem shows that  $\Sigma$  is an implicational basis for  $\mathcal{C}_L$ .

**Theorem 1.** Let  $L$  be a  $\wedge$ -semidistributive lattice. Then the set of implications  $\Sigma$  is an implicational basis for  $\mathcal{C}_L$ .

**Proof.** We need to show that  $\mathcal{C}_\Sigma = \mathcal{C}_L$ .

Let  $X \in \mathcal{C}_L$ . By Lemma 2,  $X$  is  $\Sigma_1$ -closed. By Lemma 3,  $X$  is  $\Sigma_2$ -closed. Then  $X$  is  $\Sigma$ -closed and  $\mathcal{C}_L \subseteq \mathcal{C}_\Sigma$ .

Now let us show that  $\mathcal{C}_\Sigma \subseteq \mathcal{C}_L$ . Let  $X \in \mathcal{C}_\Sigma$ . Let  $x_0 = \bigvee X$ , i.e., the least closed set containing  $X$ . Clearly  $X$  is an ideal since it is  $\Sigma_1$ -closed. Suppose that  $X \notin \mathcal{C}_L$  and let  $j$  be a minimal element of  $J(x_0) \setminus X$ . Since  $j \leq x_0$ , we have  $x_0 \not\leq m(j)$ . Moreover  $X \not\subseteq J(m(j))$ , otherwise one would have  $\bigvee X \leq m(j)$  and then  $\bigvee X \neq x_0$ . Thus there exists an element  $j' \in X$  such that  $m(j) \in \gamma(j')$  and therefore  $P_{jj'} \rightarrow j \in \Sigma$  with  $P_{jj'} \subseteq X$  and  $j \notin X$ . Then  $X$  is not  $\Sigma$ -closed, which concludes the proof.  $\square$

**Corollary 1.** Let  $L$  be a  $\wedge$ -semidistributive lattice. Then there exists an implicational basis for  $\mathcal{C}_L$  with at most  $|B| + |J(L)|$  implications, where  $|B|$  is the number of arcs in the relation  $B$ .

**Data:** Let  $L$  be a  $\wedge$ -semidistributive lattice and  $\mathcal{M}(\mathcal{C}_L)$  the set of meet-irreducible elements of  $\mathcal{C}_L$ .

**Result:** A minimum basis  $\Sigma$  of the closure system  $\mathcal{C}_L$ .

**begin**

```

 $\Sigma = \emptyset;$ 
for  $j \in J(L)$  do
   $\Sigma = \Sigma \cup \{j \rightarrow \varphi(j)\};$ 
  for  $j' \in J(L)$  do
     $P = (\varphi(j)) \setminus \{j\} \cup \varphi(j');$ 
     $\Sigma = \Sigma \cup \{P \rightarrow (P)\};$ 
 $\Sigma =$  a nonredundant cover of  $\Sigma;$ 
end

```

Algorithm 1. Minimum-Basis( $\mathcal{M}(\mathcal{C}_L)$ ).

Clearly the set  $\Sigma$  of implications obtained as above is in general not minimum. For instance, for the set  $\Sigma$  associated to the lattice in Fig. 1, the implication  $acd g \rightarrow b$  is redundant<sup>2</sup> and can be removed from  $\Sigma$  without changing  $\mathcal{C}(\Sigma)$ .

In the following we give a polynomial time algorithm to compute a minimum basis for a  $\wedge$ -semidistributive lattice.

### 3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13]. Indeed, the algorithm in [13] computes a minimum basis (called there a minimum cover) from any given basis in polynomial time.

Let  $\mathcal{M}(\mathcal{C}_L)$  be the set of meet-irreducible elements. Consider the closure operator  $\varphi: 2^J \rightarrow 2^J$ , with for  $X \subseteq J$ ,  $\varphi(X) = \bigcap \{M \in \mathcal{M}(\mathcal{C}_L) \mid X \subseteq M\}$ . The images of the mapping  $\varphi$  are said closed sets, and they correspond to the elements of the closure system  $\mathcal{C}_L$ .

**Remark 1.** We replaced  $P \rightarrow j$  by  $P \rightarrow \varphi(P)$  to guarantee the minimality after the calculation of a nonredundant cover of  $\Sigma$ .

**Remark 2.** Let us note that Algorithm 1 does not compute the same  $\Sigma$  as that of Theorem 1. This to avoid the computation of the relation  $B$ . But like the whole of the implications calculated by Algorithm 1 contains all implications of Theorem 1 (relative with the preceding remark), this guaranteed to us to have a cover of  $\mathcal{C}_L$ .

**Theorem 2.** Let  $L$  be a  $\wedge$ -semidistributive lattice. Then Algorithm 1 computes a minimum implicational basis  $\Sigma$

<sup>2</sup> An implication  $A \rightarrow B$  in  $\Sigma$  is said redundant in  $\Sigma$  if it can be derived using Armstrong rules from  $\Sigma \setminus \{A \rightarrow B\}$ .

of  $\mathcal{C}_L$  in  $O(|J|^5 + |J|^3 |\mathcal{M}(\mathcal{C}_L)|)$  time complexity. Moreover, the size of  $\Sigma$  is at most  $|J(L)|^2$  implications.

**Proof.** Theorem 1 guarantees that  $\Sigma$  is a basis for the closure system  $\mathcal{C}_L$ . Since the conclusions of all implications are closed by the mapping  $\varphi$ , the result in [13] guarantees that a not redundant basis is minimum.

Computing the closure of a set  $X \subseteq J(L)$  by  $\varphi$  can be done in  $O(|J(L)| |\mathcal{M}(\mathcal{C}_L)|)$  time complexity. Thus the total time complexity for computing a basis is in  $O(|J(L)|^3 |\mathcal{M}(\mathcal{C}_L)|)$ . Now computing a not redundant basis can be done in  $O(|J(L)| |\Sigma|^2)$ . Since  $\Sigma$  has at most  $|J(L)|^2$  implications, we conclude that the time complexity of Algorithm 1 is in  $O(|J(L)|^5 + |J(L)|^3 |\mathcal{M}(\mathcal{C}_L)|)$ .  $\square$

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### Acknowledgements

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