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Minimum implicational basis for $\land$-semidistributive lattices

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Abstract

For a $\land$-semidistributive lattice $L$, we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice $L$. We also provide a polynomial time algorithm to compute a minimum implicational basis for $L$.

Keywords: Algorithms; Lattice; Closure system; Minimum implicational basis

1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited non-determinism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we show by using a dependence relation in [11] that the class of $\land$-semidistributive lattices is another tractable case.

Consider a finite set $U$. A subset $\mathcal{C} \subseteq 2^U$ is said to be a closure system if $\mathcal{C}$ is closed under set-intersection and containing the set $U$. An implication on $U$ is an ordered pair $(A,B)$ of subsets of $U$, denoted by $A \rightarrow B$. The set $A$ is called the premise and the set $B$ the conclusion of the implication $A \rightarrow B$. Let $\Sigma$ be a set of implications on $U$. A subset $\mathcal{D} \subseteq U$ is $\Sigma$-closed if for each implication $A \rightarrow B$ in $\Sigma$, $A \subseteq D$ implies $B \subseteq D$.

The set of $\Sigma$-closed subsets of $U$, denoted by $\mathcal{C}(\Sigma)$, is a closure system on $U$. Conversely, given a closure system $\mathcal{C}$ on $U$, a family $\Sigma$ of implications on $U$ is said an implicational basis for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}(\Sigma)$. An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice $L$ as the closure system $\mathcal{C}_L$ on the set $J(L)$ of its join-irreducible elements. More precisely, put $J(a) = \{ j \in J(L) : j \leq a \}$ for $a \in L$. 

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Then \( C_L = \{ J(a) : a \in L \} \) is a closure system on \( J(L) \) which, as a lattice ordered by inclusion, is isomorphic to \( L \).

Fig. 1 gives an example of the closure system \( C_L \) associated to a lattice \( L \).

The closure system \( C_L \) can be defined by the set of its meet-irreducible elements \( \mathcal{M}(C_L) = \{ J(m) : m \in M(L) \} \), where \( M(L) \) denotes the set of meet-irreducible elements of \( L \). Each element of \( C_L \) can be obtained as an intersection of some elements of \( \mathcal{M}(C_L) \).

The problem we study is:

**Problem:** Minimum implicational basis

**Instance:** The set of meet-irreducible elements \( \mathcal{M}(C_L) \) of the closure system \( C_L \).

**Question:** Find a minimum basis \( \Sigma \) for \( C_L \).

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal\(^1\) implicational basis for modular lattices. In the following, we study the case of \( \wedge \)-semidistributive lattices.

For such lattices we show that the number of implications of a minimum implicational basis is at most \(|J(L)|^2\) and give a polynomial time algorithm to compute such a basis.

### 2. Some properties of \( \wedge \)-semidistributive lattices

Let \( L \) be a finite lattice. We note \( \vee \) the join operation, \( \wedge \) the meet operation and \( \preceq \) the cover relation of \( L \). If \( j \) is a join-irreducible element of \( L \), we use \( j_* \) to denote the unique element covered by \( j \). Dually, we use \( m^* \) to denote the unique element covering a meet-irreducible element \( m \).

We will use the arrow relations introduced by Wille [15]: for \( x, y, z \in L \), \( x \downarrow y \) means that \( x \) is a minimal element of \( \{ z \in L : z \not\leq x \} \), \( x \uparrow y \) means that \( y \) is a maximal element of \( \{ z \in L : z \not\geq y \} \) and \( x \downarrow y \) means that \( x \uparrow y \) and \( x \downarrow y \). Recall that \( \uparrow, \downarrow \) are relations defined on \( J(L) \times M(L) \), where \( J(L) \) is the set of join-irreducible elements and \( M(L) \) the set of meet-irreducible elements of \( L \).

In the following, we deal essentially with \( \wedge \)-semidistributive lattices. Let us recall that a lattice \( L \) is said \( \wedge \)-semidistributive if for all elements \( x, y, z \in L \), \( x \wedge y = x \wedge z \) implies \( x \wedge y = x \wedge (y \vee z) \). A \( \wedge \)-semidistributive lattice is said semidistributive if for all elements \( x, y, z, x \vee y = x \vee z \) implies \( x \vee y = x \vee (y \wedge z) \). The following characterization of these lattices are well known (see, for example, [6]):

**Property 1.** A finite lattice \( L \) is \( \wedge \)-semidistributive if and only if for any \( j \in J(L) \) there exists a unique \( m \in M(L) \) such that \( j \downarrow m \).

For any \( \wedge \)-semidistributive lattice \( L \) and \( j \in J(L) \), we denote by \( m(j) \) the unique element \( m \in M(L) \) such that \( j \downarrow m \).

We define the mapping \( \gamma : J(L) \to 2^{M(L)} \) by \( \gamma(j) = \{ m \in M(L) : j \downarrow m \} \). This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the \( \gamma \) mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming...
that the lattice \( L \) is \( \land \)-semidistributive) as follows (see, for example, [8,11]):

Let \( j, j' \in J(L) \).
Then \( j B j' \) iff \( j \neq j', j' \neq m(j), j'_\land m(j) \).

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation \( B \) and some classes of lattices. For example, Nation has shown that a \( \land \)-semidistributive lattice is semidistributive if and only if it contains no \( B \)-cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation \( B \) using the mapping \( \gamma \).

**Lemma 1.** Let \( L \) be a \( \land \)-semidistributive lattice, \( j, j' \in J(L) \).
\[ j B j' \text{ iff } j \neq j' \text{ and } m(j) \in \gamma(j'). \]

3. Minimum implicational basis for \( \land \)-semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a \( \land \)-semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system \( C_L \). The first one is obvious since the elements of \( C_L \) are order ideals of the induced poset by \( J(L) \).

**Lemma 2.** Let \( j, j' \in J(L) \) such that \( j \neq j' \) and \( X \in C_L \). Then \( j' \in X \) implies \( j \in X \).

Consider now a \( \land \)-semidistributive lattice \( L \) and \( j, j' \in J(L) \) such that \( j B j' \). We denote by \( P_{jj'} \) the set \( J(j) \cup J(j') \).

**Lemma 3.** Let \( L \) be a \( \land \)-semidistributive lattice and \( j, j' \in J(L) \) such that \( j B j' \) and \( X \in C_L \). Then \( P_{jj'} \subseteq X \) implies \( j \in X \).

**Proof.** Let \( x \in L \) such that \( X = J(x) \) and \( P_{jj'} \subseteq X \). Since \( J(j_x) \subseteq X \) this implies that \( j_x \lor j' \leq x \), and then it suffices to prove that \( j \leq j_x \land j' \).

Suppose that \( j \neq j_x \lor j' \) and let \( m' \in M(L) \) be a maximal element of \( \{ z \in L \mid z \not\geq j \land j \geq j_x \lor j' \} \).
By definition of \( m' \), we have \( j \uparrow m' \). Moreover \( j \downarrow m' \) since \( j_x \leq m' \). Thus \( j \nmid m' \).

Consider now the meet-irreducible \( m(j) \) associated with \( j \). Then \( j' \nmid m(j) \). Thus since \( j \leq m' \) and \( m(j) \) are two distinct elements such that \( j \nmid m' \) and \( j \nmid m(j) \). This contradicts the fact that \( L \) is \( \land \)-semidistributive. \( \Box \)

We can now define a particular set of implications associated to a \( \land \)-semidistributive lattice \( L \). Let \( \Sigma_1 = \{ j \rightarrow J(j) \} \), \( \Sigma_2 = \{ P_{jj'} \rightarrow j \mid j \in J(L) \land B(j') \} \) and \( \Sigma = \Sigma_1 \cup \Sigma_2 \).

For example, the sets of implications \( \Sigma_1 \) and \( \Sigma_2 \) for the lattice in Fig. 1 are \( \Sigma_1 = \{ ab \rightarrow b, ac \rightarrow c, abc \rightarrow abc \} \) and \( \Sigma_2 = \{ acd \rightarrow e, abc \rightarrow d, acdefg \rightarrow cdefg \} \).

The following theorem shows that \( \Sigma \) is an implicational basis for \( C_L \).

**Theorem 1.** Let \( L \) be a \( \land \)-semidistributive lattice. Then the set of implications \( \Sigma \) is an implicational basis for \( C_L \).

**Proof.** We need to show that \( C_\Sigma = C_L \).
Let \( X \in C_L \). By Lemma 2, \( X \) is \( \Sigma_1 \)-closed. By Lemma 3, \( X \) is \( \Sigma_2 \)-closed. Then \( X \) is \( \Sigma \)-closed and \( C_L \subseteq C_\Sigma \).

Now let us show that \( C_\Sigma \subseteq C_L \). Let \( X \in C_\Sigma \). Let \( x_0 = \bigvee X \) i.e., the least closed set containing \( X \). Clearly \( X \) is an ideal since it is \( \Sigma_1 \)-closed. Suppose that \( X \not\in C_L \) and let \( j \) be a minimal element of \( J(x_0) \setminus X \). Since \( j \leq x_0 \), we have \( x_0 \notin m(j) \).
Moreover \( X \not\subseteq J(m(j)) \), otherwise one would have \( \bigvee X \leq m(j) \) and then \( X \neq x_0 \).
Thus there exists an element \( j' \in X \) such that \( m(j) \in \gamma(j') \) and therefore \( P_{jj'} \rightarrow j \in \Sigma \) with \( P_{jj'} \subseteq X \) and \( j \neq X \).
Then \( X \) is not \( \Sigma \)-closed, which concludes the proof. \( \Box \)

**Corollary 1.** Let \( L \) be a \( \land \)-semidistributive lattice. Then there exists an implicational basis for \( C_L \) with at most \( |B| + |J(L)| \) implications, where \( |B| \) is the number of arcs in the relation \( B \).
Data: Let $L$ be a $\wedge$-semidistributive lattice and $\mathcal{M}(C_L)$ the set of meet-irreducible elements of $C_L$.

Result: A minimum basis $\Sigma$ of the closure system $C_L$.

begin
\begin{align*}
\Sigma &= \emptyset; \\
\text{for } j \in J(L) &\text{ do} \\
\Sigma &= \Sigma \cup \{j \rightarrow \varphi(j)\}; \\
\text{for } j' \in J(L) &\text{ do} \\
P &= \{\varphi(j)\} \setminus \{j\} \cup \varphi(j'); \\
\Sigma &= \Sigma \cup \{P \rightarrow (P)\}; \\
\Sigma &= \text{a nonredundant cover of } \Sigma;
\end{align*}
\end{aligned}

Algorithm 1. Minimum-Basis($\mathcal{M}(C_L)$).

Clearly the set $\Sigma$ of implications obtained as above is in general not minimum. For instance, for the set $\Sigma$ associated to the lattice in Fig. 1, the implication $acd \rightarrow b$ is redundant\(^2\) and can be removed from $\Sigma$ without changing $C(\Sigma)$.

In the following we give a polynomial time algorithm to compute a minimum basis for a $\wedge$-semidistributive lattice.

3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13]. Indeed, the algorithm in [13] computes a minimum basis (called there a minimum cover) from any given basis in polynomial time.

Let $\mathcal{M}(C_L)$ be the set of meet-irreducible elements. Consider the closure operator $\varphi : 2^J \rightarrow 2^J$, with for $X \subseteq J$, $\varphi(X) = \bigcap \{M \in \mathcal{M}(C_L) \mid X \subseteq M\}$. The images of the mapping $\varphi$ are said closed sets, and they correspond to the elements of the closure system $C_L$.

Remark 1. We replaced $P \rightarrow j$ by $P \rightarrow \varphi(P)$ to guarantee the minimality after the calculation of a nonredundant cover of $\Sigma$.

Remark 2. Let us note that Algorithm 1 does not compute the same $\Sigma$ as that of Theorem 1. This to avoid the computation of the relation $B$. But like the whole of the implications calculated by Algorithm 1 contains all implications of Theorem 1 (relative with the preceding remark), this guaranteed to us to have a cover of $C_L$.

Theorem 2. Let $L$ be a $\wedge$-semidistributive lattice. Then Algorithm 1 computes a minimum implicational basis $\Sigma$ of $C_L$ in $O(|J|^5 + |J|^3|\mathcal{M}(C_L)|)$ time complexity. Moreover, the size of $\Sigma$ is at most $|J(L)|^2$ implications.

Proof. Theorem 1 guarantees that $\Sigma$ is a basis for the closure system $C_L$. Since the conclusions of all implications are closed by the mapping $\varphi$, the result in [13] guarantees that a not redundant basis is minimum.

Computing the closure of a set $X \subseteq J(L)$ by $\varphi$ can be done in $O(|J(L)||\mathcal{M}(C_L)|)$ time complexity. Thus the total time complexity for computing a basis is in $O(|J(L)|^3|\mathcal{M}(C_L)|)$. Now computing a not redundant basis can be done in $O(|J(L)|^3\Sigma^2)$. Since $\Sigma$ has at most $|J(L)|^2$ implications, we conclude that the time complexity of Algorithm 1 is in $O(|J(L)|^5 + |J(L)|^3|\mathcal{M}(C_L)|)$.

Uncited references

[9]

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References


