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# Minimum implicational basis for $\wedge$-semidistributive lattices

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**Abstract**

For a $\wedge$-semidistributive lattice $L$, we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice $L$. We also provide a polynomial time algorithm to compute a minimum implicational basis for $L$.


*Keywords:* Algorithms; Lattice; Closure system; Minimum implicational basis

## 1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited nondeterminism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we show by using a dependence relation in [11] that the class of $\wedge$-semidistributive lattices is another tractable case.

Consider a finite set $U$. A subset $\mathcal{C} \subseteq 2^U$ is said to be a closure system if $\mathcal{C}$ is closed under set-intersection and containing the set $U$. An implication on $U$ is an ordered pair $(A, B)$ of subsets of $U$, denoted by $A \to B$. The set $A$ is called the premise and the set $B$ the conclusion of the implication $A \to B$. Let $\Sigma$ be a set of implications on $U$. A subset $D \subseteq U$ is $\Sigma$-closed if for each implication $A \to B$ in $\Sigma$, $A \subseteq D$ implies $B \subseteq D$. The set of $\Sigma$-closed subsets of $U$, denoted by $\mathcal{C}(\Sigma)$, is a closure system on $U$. Conversely, given a closure system $\mathcal{C}$ on $U$, a family $\Sigma$ of implications on $U$ is said an implicational basis for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}(\Sigma)$. An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice $L$ as the closure system $\mathcal{C}_L$ on the set $J(L)$ of its join-irreducible elements. More precisely, put $J(a) = \{j \in J(L)\colon\ j \leqslant a\}$ for $a \in L$.

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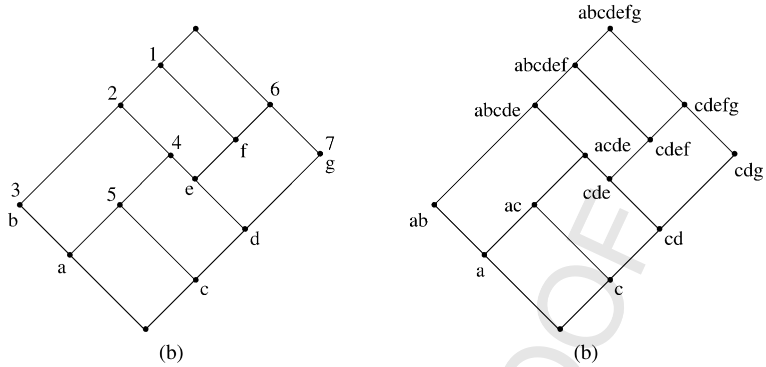


Fig. 1. (a) A lattice $L$ where join-irreducible (resp. meet-irreducible) elements are labeled by letters (resp. numbers); (b) The closure system $cl$ associated to $L$.

Then $\mathcal{C}_L = \{J(a)\colon\ a \in L\}$ is a closure system on $J(L)$ which, as a lattice ordered by inclusion, is isomorphic to $L$.

Fig. 1 gives an example of the closure system $\mathcal{C}_L$ associated to a lattice $L$.

The closure system $\mathcal{C}_L$ can be defined by the set of its meet-irreducible elements $\mathcal{M}(\mathcal{C}_L) = \{J(m)\colon\ m \in M(L)\}$, where $M(L)$ denotes the set of meet-irreducible elements of $L$. Each element of $\mathcal{C}_L$ can be obtained as intersection of some elements of $\mathcal{M}(\mathcal{C}_L)$.

The problem we study is:

**Problem**: Minimum implicational basis
**Instance**: The set of meet-irreducible elements $\mathcal{M}(\mathcal{C}_L)$ of the closure system $\mathcal{C}_L$.
**Question**: Find a minimum basis $\Sigma$ for $\mathcal{C}_L$.

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal[1] implicational basis for modular lattices. In the following, we study the case of $\wedge$-semidistributive lattices. For such lattices we show that the number of implications of a minimum implicational basis is at most $|J(L)|^2$ and give a polynomial time algorithm to compute such a basis.

## 2. Some properties of ∧-semidistributive lattices

Let $L$ be a finite lattice. We note $\vee$ the join operation, $\wedge$ the meet operation and $\prec$ the cover relation of $L$. If $j$ is a join-irreducible element of $L$, we use $j_*$ to denote the unique element covered by $j$. Dually, we use $m^*$ to denote the unique element covering a meet-irreducible element $m$.

We will use the arrow relations introduced by Wille [15]: for $x, y \in L$, $x \downarrow y$ means that $x$ is a minimal element of $\{z \in L\colon\ z \nleq x\}$, $x \uparrow y$ means that $y$ is a maximal element of $\{z \in L\colon\ z \ngeq y\}$ and $x \updownarrow y$ means that $x \uparrow y$ and $x \downarrow y$. Recall that $\uparrow, \downarrow, \updownarrow$ are relations defined on $J(L) \times M(L)$, where $J(L)$ is the set of join-irreducible elements and $M(L)$ the set of meet-irreducible elements of $L$.

In the following, we deal essentially with $\wedge$-semidistributive lattices. Let us recall that a lattice $L$ is said $\wedge$-semidistributive if for all elements $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$. A $\wedge$-semidistributive lattice is said semidistributive if for all elements $x, y, z$, $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. The following characterization of these lattices are well known (see, for example, [6]):

**Property 1.** *A finite lattice $L$ is $\wedge$-semidistributive if and only if for any $j \in J(L)$ there exists a unique $m \in M(L)$ such that $j \updownarrow m$.*

For any $\wedge$-semidistributive lattice $L$ and $j \in J(L)$, we denote by $m(j)$ the unique element $m \in M(L)$ such that $j \updownarrow m$.

We define the mapping $\gamma : J(L) \to 2^{M(L)}$ by $\gamma(j) = \{m \in M(L)\colon\ j \downarrow m\}$. This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the $\gamma$ mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming

[1] An implication is known as optimal if the sum of the cardinality of the premises and the conclusions of all the implications is minimal.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\gamma$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | | | | | | $\updownarrow$ | $\downarrow$ | $\{6,7\}$ |
| $b$ | | | | $\updownarrow$ | $\downarrow$ | $\uparrow$ | | $\{4,5\}$ |
| $c$ | | | $\updownarrow$ | | | | | $\{3\}$ |
| $d$ | | | $\uparrow$ | | $\updownarrow$ | | | $\{5\}$ |
| $e$ | | | $\uparrow$ | | $\uparrow$ | | $\updownarrow$ | $\{7\}$ |
| $f$ | | $\updownarrow$ | | $\downarrow$ | | | $\uparrow$ | $\{2,4\}$ |
| $g$ | $\updownarrow$ | $\downarrow$ | | $\downarrow$ | | | | $\{1,2,4\}$ |

Fig. 2. The arrow relations and mapping $\gamma$ of the lattice in Fig. 1.

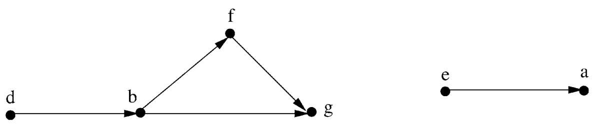


Fig. 3. The graph of the $B$ relation for the lattice in Fig. 1.

that the lattice $L$ is $\wedge$-semidistributive) as follows (see, for example, [8,11]):

Let $j, j' \in J(L)$.
Then $j\,B\,j'$ iff $j \neq j'$, $j' \nleqslant m(j)$, $j'_* \leqslant m(j)$.

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation $B$ and some classes of lattices. For example, Nation has shown that a $\wedge$-semidistributive lattice is semidistributive if and only if it contains no $B$-cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation $B$ using the mapping $\gamma$.

**Lemma 1.** *Let $L$ be a $\wedge$-semidistributive lattice, $j, j' \in J(L)$.*

$j\,B\,j'$ *iff* $j \neq j'$ *and* $m(j) \in \gamma(j')$.

## 3. Minimum implicational basis for ∧-semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a $\wedge$-semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system $\mathcal{C}_L$. The first one is obvious since the elements of $\mathcal{C}_L$ are order ideals of the induced poset by $J(L)$.

**Lemma 2.** *Let $j, j' \in J(L)$ such that $j < j'$ and $X \in \mathcal{C}_L$. Then $j' \in X$ implies $j \in X$.*

Consider now a $\wedge$-semidistributive lattice $L$ and $j, j' \in J(L)$ such that $jBj'$. We denote by $P_{jj'}$ the set $J(j_*) \cup J(j')$.

**Lemma 3.** *Let $L$ be a $\wedge$-semidistributive lattice and $j, j' \in J(L)$ such that $jBj'$ and $X \in \mathcal{C}_L$. Then $P_{jj'} \subseteq X$ implies $j \in X$.*

**Proof.** Let $x \in L$ such that $X = J(x)$ and $P_{jj'} \subseteq X$. Since $J(j_*) \subset X$ this implies that $j_* \vee j' \leqslant x$, and then it suffices to prove that $j \leqslant j_* \vee j'$.

Suppose that $j \nleqslant j_* \vee j'$ and let $m' \in M(L)$ be a maximal element of $\{z \in L \mid z \ngeqslant j \text{ and } z \geqslant j_* \vee j'\}$. By definition of $m'$, we have $j \uparrow m'$. Moreover $j \downarrow m'$ since $j_* \leqslant m'$. Thus $j \updownarrow m'$.

Consider now the meet-irreducible $m(j)$ associated with $j$. Then $j' \nleqslant m(j)$ since $j\,B\,j'$. Thus since $j' \leqslant m'$, $m'$ and $m(j)$ are two distinct elements such that $j \updownarrow m'$ and $j \updownarrow m(j)$. This contradicts the fact that $L$ is $\wedge$-semidistributive. □

We can now define a particular set of implications associated to a $\wedge$-semidistributive lattice $L$. Let $\Sigma_1 = \{j \to J(j)\}$, $\Sigma_2 = \{P_{jj'} \to j \mid j' \in J(L) \text{ and } j\,B\,j'\}$ and $\Sigma = \Sigma_1 \cup \Sigma_2$.

For example, the sets of implications $\Sigma_1$ and $\Sigma_2$ for the lattice in Fig. 1 are $\Sigma_1 = \{b \to ab, d \to cd, e \to cde, f \to cdef, g \to cdg\}$ and $\Sigma_2 = \{acd \to e, abc \to d, acdef \to b, acdg \to b, cdeg \to f\}$.

The following theorem shows that $\Sigma$ is an implicational basis for $\mathcal{C}_L$.

**Theorem 1.** *Let $L$ be a $\wedge$-semidistributive lattice. Then the set of implications $\Sigma$ is an implicational basis for $\mathcal{C}_L$.*

**Proof.** We need to show that $\mathcal{C}_\Sigma = \mathcal{C}_L$.

Let $X \in \mathcal{C}_L$. By Lemma 2, $X$ is $\Sigma_1$-closed. By Lemma 3, $X$ is $\Sigma_2$-closed. Then $X$ is $\Sigma$-closed and $\mathcal{C}_L \subseteq \mathcal{C}_\Sigma$.

Now let us show that $\mathcal{C}_\Sigma \subseteq \mathcal{C}_L$. Let $X \in \mathcal{C}_\Sigma$. Let $x_0 = \bigvee X$, i.e., the least closed set containing $X$. Clearly $X$ is an ideal since it is $\Sigma_1$-closed. Suppose that $X \notin \mathcal{C}_L$ and let $j$ be a minimal element of $J(x_0) \backslash X$. Since $j \leqslant x_0$, we have $x_0 \nleqslant m(j)$. Moreover $X \nsubseteq J(m(j))$, otherwise one would have $\bigvee X \leqslant m(j)$ and then $\bigvee X \neq x_0$. Thus there exists an element $j' \in X$ such that $m(j) \in \gamma(j')$ and therefore $P_{jj'} \to j \in \Sigma$ with $P_{jj'} \subseteq X$ and $j \notin X$. Then $X$ is not $\Sigma$-closed, which concludes the proof. □

**Corollary 1.** *Let $L$ be a $\wedge$-semidistributive lattice. Then there exists an implicational basis for $\mathcal{C}_L$ with at most $|B| + |J(L)|$ implications, where $|B|$ is the number of arcs in the relation $B$.*

**Data**: Let $L$ be a $\wedge$-semidistributive lattice and $\mathcal{M}(C_L)$ the set of meet-irreducible elements of $\mathcal{C}_L$.
**Result**: A minimum basis $\Sigma$ of the closure system $C_L$.
**begin**
  $\Sigma = \emptyset$;
  **for** $j \in J(L)$ **do**
    $\Sigma = \Sigma \cup \{j \to \varphi(j)\}$;
    **for** $j' \in J(L)$ **do**
      $P = (\varphi(j)) \backslash \{j\} \cup \varphi(j')$;
      $\Sigma = \Sigma \cup \{P \to (P)\}$;
  $\Sigma =$ a nonredundant cover of $\Sigma$;
**end**

Algorithm 1. Minimum-Basis($\mathcal{M}(C_L)$).

Clearly the set $\Sigma$ of implications obtained as above is in general not minimum. For instance, for the set $\Sigma$ associated to the lattice in Fig. 1, the implication $acdg \to b$ is redundant[2] and can be removed from $\Sigma$ without changing $\mathcal{C}(\Sigma)$.

In the following we give a polynomial time algorithm to compute a minimum basis for a $\wedge$-semidistributive lattice.

### 3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13]. Indeed, the algorithm in [13] computes a minimum basis (called there a minimum cover) from any given basis in polynomial time.

Let $\mathcal{M}(\mathcal{C}_L)$ be the set of meet-irreducible elements. Consider the closure operator $\varphi : 2^J \to 2^J$, with for $X \subseteq J$, $\varphi(X) = \bigcap\{M \in \mathcal{M}(\mathcal{C}_L) \mid X \subseteq M\}$. The images of the mapping $\varphi$ are said closed sets, and they correspond to the elements of the closure system $\mathcal{C}_L$.

**Remark 1.** We replaced $P \to j$ by $P \to \varphi(P)$ to guarantee the minimality after the calculation of a nonredundant cover of $\Sigma$.

**Remark 2.** Let us note that Algorithm 1 does not compute the same $\Sigma$ as that of Theorem 1. This to avoid the computation of the relation $B$. But like the whole of the implications calculated by Algorithm 1 contains all implications of Theorem 1 (relative with the preceding remark), this guaranteed to us to have a cover of $\mathcal{C}_L$.

**Theorem 2.** *Let $L$ be a $\wedge$-semidistributive lattice. Then Algorithm* 1 *computes a minimum implicational basis $\Sigma$ of $\mathcal{C}_L$ in* $O(|J|^5 + |J|^3|\mathcal{M}(\mathcal{C}_L)|)$ *time complexity. Moreover, the size of $\Sigma$ is at most $|J(L)|^2$ implications.*

**Proof.** Theorem 1 guarantees that $\Sigma$ is a basis for the closure system $\mathcal{C}_L$. Since the conclusions of all implications are closed by the mapping $\varphi$, the result in [13] guarantees that a not redundant basis is minimum.

Computing the closure of a set $X \subseteq J(L)$ by $\varphi$ can be done in $O(|J(L)||\mathcal{M}(\mathcal{C}_L)|)$ time complexity. Thus the total time complexity for computing a basis is in $O(|J(L)|^3|\mathcal{M}(\mathcal{C}_L)|)$. Now computing a not redundant basis can be done in $O(|J(L)||\Sigma|^2)$. Since $\Sigma$ has at most $|J(L)|^2$ implications, we conclude that the time complexity of Algorithm 1 is in $O(|J(L)|^5 + |J(L)|^3|\mathcal{M}(\mathcal{C}_L)|)$. $\square$

## Uncited references

[9]

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