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Discrete Rotations and Symbolic Dynamics

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Abstract

The aim of this paper is to study local configurations for discrete rotations. The algorithm of discrete rotation we consider is the following: a discretized rotation is defined as the composition of a Euclidean rotation with a rounding operation, as studied in [NR03,NR04,NR05]. It is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α that we call configurations. We introduce here two discrete dynamical systems defined by a \mathbb{Z}^2 -action on the two-dimensional torus that allow us via a suitable symbolic coding to describe the configurations C_α and C'_α ; we then deduce various combinatorial properties for both configurations, and in particular, results concerning densities of occurrence of symbols.

Key words: Discrete rotations, discrete geometry, word combinatorics, two-dimensional words, symbolic dynamics.

1 Introduction

Symbolic dynamics and more generally, discrete dynamical systems have natural and deep interactions with combinatorics on words. This interaction is particularly well-illustrated in the Sturmian case, see e.g. [Lot02,Fog02]. The combinatorial objects involved are the Sturmian words, while the dynamical systems are the irrational rotations of the torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. A Sturmian word is indeed a coding with respect to a particular two-interval partition of

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the one-dimensional torus \mathbb{T}^1 of the orbit of a point under the action of an irrational rotation. This point of view allows one to deduce many combinatorial properties of Sturmian words, as discussed in [BFZ05], such as, e.g., the densities of occurrences of factors that can be computed thanks to the equidistribution properties of irrational rotations, or such as powers of factors in Sturmian words [Van00], or the characterization of Sturmian words fixed points of substitutions [BEIR].

Several attempts of generalization of this fruitful interaction have been proposed. For more details, see the survey [BFZ05]. One of the first idea which comes to mind is a rotation of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. As an example, the Tribonacci word, that is, the fixed point of the substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ codes the orbit of a point of the torus \mathbb{T}^2 under the action of a translation in \mathbb{T}^2 with respect to a partition of \mathbb{T}^2 into three pieces with fractal boundary [Rau82, Lot05]. More generally, fixed points of Arnoux-Rauzy sequences over n letters [AR91] code orbits of points of the torus \mathbb{T}^{n-1} under the action of a translation in \mathbb{T}^{n-1} with respect to a partition of \mathbb{T}^{n-1} into n pieces with fractal boundary [AI01].

A second approach, which is dual to the previous one, consists in working with two rotations of \mathbb{T}^1 . It is indeed convenient to describe arithmetic discrete planes in the sense of [Rev91] by use of the coding with respect to a three-interval partition of a \mathbb{Z}^2 -action by two irrational rotations on \mathbb{T}^1 [BFJP]. One thus gets two-dimensional words over a three-letter alphabet that can be considered as two-dimensional Sturmian words [BV00]. The study of the underlying dynamical system allows one here to obtain a better understanding of the combinatorial and geometric properties of arithmetic discrete planes, such as the enumeration of some local configurations, the so-called (m, n) -cubes, as well as their densities of occurrence, or their centrosymmetry properties [BFJP].

In all these cases connections between word combinatorics, symbolic dynamics, arithmetics and discrete geometry prove to be natural and enlightening. We consider in the present paper a further generalization motivated by discrete geometry, and more precisely, arithmetic discrete geometry, in the sense of [Rev91]. We study indeed configurations associated with a discrete rotation; there exists several extensions of the the notion of Euclidean rotation in discrete geometry, such as reviewed in [And92]. We consider here discrete rotations defined as the composition of a Euclidean rotation with a rounding operation. It is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α that we call configurations. These configurations have been introduced and studied in [NR03, NR05, NR05]. The main purpose of the present paper is to prove that both configurations are codings of a \mathbb{Z}^2 -action by two rotations on \mathbb{T}^2 with respect to a partition into a finite number of rectangles. We then deduce in particular results concerning

the density of each symbol in C_α and C'_α .

This paper is organized as follows. We introduce the first definitions and conventions in Section 2. Section 3 is devoted to the dynamical study of the configuration C_α , from which combinatorial properties are deduced in Section 4. A similar study for C'_α is performed in Section 5. Let us note that results presented here extend those of [BN05].

2 Definitions and conventions

We work in the *discrete plane* \mathbb{Z}^2 . For each point $\mathbf{v} \in \mathbb{Z}^2$, $x_{\mathbf{v}}$ stands for its horizontal coordinate and $y_{\mathbf{v}}$ for its vertical coordinate.

Let x be a real number. We recall that the floor function $x \mapsto \lfloor x \rfloor$ is defined as the greatest integer less or equal to x . The *rounding function* is defined as $\lfloor x \rfloor := \lfloor x + 0.5 \rfloor$ and $\{x\} := x - \lfloor x \rfloor$. These applications can be extended to vectors in \mathbb{Z}^2 , by independent application on each component.

The *discretization cell* of the point $\mathbf{v} \in \mathbb{Z}^2$ is defined as the set of elements \mathbf{w} in \mathbb{R}^2 which have the same image by discretization as \mathbf{v} , i.e., $\lfloor \mathbf{v} \rfloor = \lfloor \mathbf{w} \rfloor$. Hence the discretization cell of \mathbf{v} is defined as the half-opened unit square centered at $\lfloor \mathbf{v} \rfloor$.

We use the canonical bijection between the torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and the square $\{\mathbf{v} \in \mathbb{R}^2; x_{\mathbf{v}} \in [-\frac{1}{2}, \frac{1}{2}[\text{ and } y_{\mathbf{v}} \in [-\frac{1}{2}, \frac{1}{2}[\}$, i.e., the discretization cell of 0. By abuse of notation, we also denote by $\{\mathbf{v}\}$ the image under the canonical projection from \mathbb{R}^2 onto \mathbb{T}^2 of a point $\mathbf{v} \in \mathbb{R}^2$. Let us stress the fact that the map $x \mapsto \{x\}$ is thus an additive morphism from \mathbb{R}^2 onto \mathbb{T}^2 .

Without loss of generality, we assume throughout this paper that $\alpha \in [0, \pi/4]$: the arguments used here can easily be extended to the case of any other octant. We denote by r_α the Euclidean rotation of angle α :

$$r_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mathbf{v} \mapsto \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \mathbf{v}.$$

The *discrete rotation* $\lfloor r_\alpha \rfloor$ is defined as

$$\lfloor r_\alpha \rfloor : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2, \mathbf{v} \mapsto \lfloor r_\alpha(\mathbf{v}) \rfloor.$$

By $\{r_\alpha\}$ we mean the map

$$\{r_\alpha\} : \mathbb{Z}^2 \rightarrow \mathbb{T}^2, \mathbf{v} \mapsto \{r_\alpha(\mathbf{v})\}.$$

We denote by (\mathbf{i}, \mathbf{j}) the canonical basis of the Euclidean space \mathbb{R}^2 . We set $\mathbf{i}_\alpha := r_\alpha(\mathbf{i})$ and $\mathbf{j}_\alpha := r_\alpha(\mathbf{j})$.

Let Q be a finite set called alphabet. A two-dimensional word in $Q^{\mathbb{Z}^2}$ is called a *configuration* over Q . An application from $\{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\}$ to Q is called a *pattern* of size $[m, n]$. Let C be a configuration in $Q^{\mathbb{Z}^2}$. A pattern χ of size $[m, n]$ occurs at position \mathbf{v} in C if $C(\mathbf{v} + \mathbf{p}) = \chi(\mathbf{p})$, for all \mathbf{p} with $x_{\mathbf{p}}, y_{\mathbf{p}} \in \{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\}$. The rectangular complexity function of the configuration C is defined as the function $p_C: \mathbb{N}^2 \rightarrow \mathbb{N}$, that counts the number of patterns of size $[m, n]$ in C .

The *density* of the symbol $p \in Q$ in the configuration $C \in Q^{\mathbb{Z}^2}$ is defined as the following limit (if it exists):

$$\eta_C(p) = \lim_{N \rightarrow \infty} \frac{\text{Card}\{\mathbf{v} \in \mathbb{Z}^2, x_v, y_v \in \{-N, \dots, N\} \text{ and } C(\mathbf{v}) = p\}}{(2N+1)^2}.$$

We similarly define the density of a pattern χ in the configuration C as the following limit (if it exists):

$$\eta_C(\chi) = \lim_{N \rightarrow \infty} \frac{\text{Card}\{\mathbf{v} \in \mathbb{Z}^2, x_v, y_v \in \{-N, \dots, N\} \text{ and } \chi \text{ occurs at position } \mathbf{v}\}}{(2N+1)^2}.$$

A *dynamical system* (X, T) is defined as the action of a continuous and onto map T on a compact space X . Given two continuous and onto maps T_1 and T_2 acting on X and satisfying $T_1 \circ T_2 = T_2 \circ T_1$, the \mathbb{Z}^2 -*action* by T_1 and T_2 on X , that we denote (X, T_1, T_2) , is defined by

$$\forall (m, n) \in \mathbb{Z}^2, \forall x \in X, (m, n) \cdot x = T_1^m \circ T_2^n(x).$$

It is natural to associate with two-dimensional symbolic dynamical system to the triple (X, T_1, T_2) by coding the orbits of the points of X under the \mathbb{Z}^2 -action as follows: given $x_0 \in X$ and given a *labelling function* l defined on X with values in a finite set Q that takes constant values on the atoms of a finite partition of X , the configuration C defined by

$$\forall (m, n) \in \mathbb{Z}^2, C(m, n) = l(T_1^m \circ T_2^n(x_0))$$

is called the coding of the orbit of x_0 under the \mathbb{Z}^2 -action (X, T_1, T_2) with respect to the labelling function l .

3 Dynamical system associated with C_α

According to [NR03], we associate a first configuration C_α with the discrete rotation $[r_\alpha]$ that encodes local information concerning the discrete rotation: the configuration C_α is defined at point $\mathbf{v} \in \mathbb{Z}^2$ according to the action of the discrete rotation on the 4-neighbours of \mathbf{v} ; furthermore, there exists a planar transducer that uses the configuration C_α as input and gradually computes the action of the discrete rotation [NR05].

More precisely, for a given $\mathbf{v} \in \mathbb{Z}^2$, we denote by $\mathcal{V}_4(\mathbf{v})$ the set of 4-neighbours of \mathbf{v} , that is, $\mathcal{V}_4(\mathbf{v}) = \{\mathbf{v} + \mathbf{i}, \mathbf{v} + \mathbf{j}, \mathbf{v} - \mathbf{i}, \mathbf{v} - \mathbf{j}\}$. The configuration C_α maps each point \mathbf{v} of \mathbb{Z}^2 to the set $[r_\alpha](\mathcal{V}_4(\mathbf{v})) - [r_\alpha][\mathbf{v}]$, that is,

$$C_\alpha(\mathbf{v}) := \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \text{ with } (a_k = [r_\alpha(\mathbf{v} + r_{\pi/2}^k(\mathbf{i}))] - [r_\alpha(\mathbf{v})], \text{ for } k = 0, \dots, 3).$$

One easily checks that C_α contains either 3 or 4 non-zero elements; for a detailed proof, see [NR03]. Let Q_α stand for the finite set of values taken by C_α .

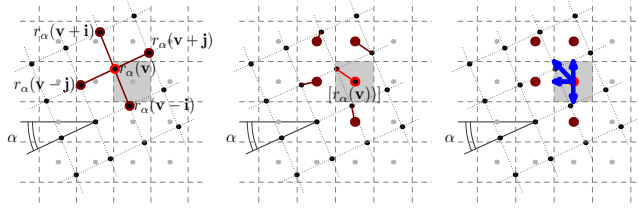


Figure 1. A progressive construction of the configuration C_α : we represent the set of vectors that leads to the relative positions of the 4-neighbors of \mathbf{v} after the action of the discrete rotation.

We define a *frame* of the torus $\mathbb{T}^2 \equiv [-\frac{1}{2}, \frac{1}{2}[\times [-\frac{1}{2}, \frac{1}{2}[$ as a rectangle of the form $[a, b[\times [c, d[$, with $-\frac{1}{2} \leq a \leq b < \frac{1}{2}$ and $-\frac{1}{2} \leq c \leq d < \frac{1}{2}$. The interpretation of C_α as a coding a \mathbb{Z}^2 -action is based on the following result:

Theorem 1 ([NR05]) *There exists a partition $P_\alpha = \{I_p, p \in Q_\alpha\}$ of the torus \mathbb{T}^2 into a finite number of frames such that*

$$\forall \mathbf{v} \in \mathbb{Z}^2, C_\alpha(\mathbf{v}) = p \iff \{r_\alpha(\mathbf{v})\} \in I_p.$$

More precisely, the partition P_α is defined as follows: if $\alpha \in [0, \pi/6[$ (resp. $[\pi/6, \pi/4[$), then the torus is divided into at most 25 frames, delimited by the (at most) 10 lines with equation $x = -\frac{1}{2}, x = \frac{1}{2} - \cos(\alpha), x = \sin(\alpha) - \frac{1}{2}, x = \frac{1}{2} - \sin(\alpha), x = \cos(\alpha) - \frac{1}{2}, \frac{1}{2}, y = -\frac{1}{2}, y = \frac{1}{2} - \cos(\alpha), y = \sin(\alpha) - \frac{1}{2}, y = \frac{1}{2} - \sin(\alpha), y = \cos(\alpha) - \frac{1}{2}, \frac{1}{2}$, (resp. $x, y = -\frac{1}{2}, \frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}, \frac{1}{2}$). More precisely, the alphabet Q_α has exactly 25 elements if $\alpha \neq 0, \pi/4, \pi/6$, 16 elements if $\alpha = \pi/6$, and 9, if $\alpha = \pi/4$.

$(0, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	
$(0, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$		$(3, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$

Figure 2. Table describing the action of ϕ_c . The symbols represent the directions of the vectors of $C_\alpha(\mathbf{v})$.

Consider now the following two actions

$$T_{\mathbf{i}_\alpha} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, x \mapsto x + \{\mathbf{i}_\alpha\}, T_{\mathbf{j}_\alpha} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, x \mapsto x + \{\mathbf{j}_\alpha\}.$$

One has

$$\forall \mathbf{v} \in \mathbb{Z}^2, \{r_\alpha\}(\mathbf{v}) = T_{\mathbf{i}_\alpha}^{x\mathbf{v}} \circ T_{\mathbf{i}_\alpha}^{y\mathbf{v}}(\mathbf{0}).$$

We then associate with the partition P_α the labelling function

$$l_{C_\alpha} : \mathbb{T}^2 \rightarrow Q_\alpha, \mathbf{v} \mapsto \phi_c(f_{C_\alpha}(x_{\mathbf{v}}), f_{C_\alpha}(y_{\mathbf{v}})),$$

where $\phi_c : \{0, 1, 2, 3, 4\}^2 \rightarrow Q_\alpha$ if $\alpha \in [0, \pi/6]$ (resp. $\phi_c : \{0, 1, 3, 4, 5\}^2 \rightarrow Q_\alpha$ if $\alpha \in [\pi/6, \pi/4]$) is described in Figure 2, and $f_{C_\alpha} : [-1/2, 1/2] \rightarrow \{0, 1, 2, 3, 4, 5\}$ is defined by

if $\alpha \in [0, \pi/6]$:

if $\alpha \in [\pi/6, \pi/4]$:

$$\left[\begin{array}{l} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[\quad \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \sin(\alpha) - \frac{1}{2}[\mapsto 1 \\ [\sin(\alpha) - \frac{1}{2}, \frac{1}{2} - \sin(\alpha)[\mapsto 2 \\ [\frac{1}{2} - \sin(\alpha), \cos(\alpha) - \frac{1}{2}[\mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[\quad \mapsto 4 \end{array} \right. \quad \left[\begin{array}{l} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[\quad \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha)[\mapsto 1 \\ [\frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}[\mapsto 5 \\ [\sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}[\mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[\quad \mapsto 4 \end{array} \right.$$

The values taken by C_α , i.e., the elements of Q_α are depicted in Figure 2 according to the directions of the vectors of $C_\alpha(\mathbf{v})$, for $\mathbf{v} \in \mathbb{Z}^2$.

Theorem 1 can then be reformulated as follows:

Corollary 2 *Let C_α be the configuration associated with the discrete rotation $[r_\alpha]$. We use the notation introduced above. The configuration C_α is the coding of the orbit of $\mathbf{0}$ under the \mathbb{Z}^2 -action $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{j}_\alpha})$ with respect to the labelling function l_{C_α} .*

Corollary 2 means that the position, in the discretization cell of a point $\mathbf{v} \in \mathbb{Z}^2$, of the point $\{r_\alpha\}(\mathbf{v})$ of the lattice $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ determines the directions of the images of the neighbours of \mathbf{v} under the action of the discrete rotation.

Example : the case $\alpha = \pi/4$

We detail here the case $\alpha = \pi/4$. In this case, the alphabet $Q_{\pi/4}$ has 9 elements. Consider the sequences in lines of the two-dimensional word $C_{\pi/4}$. One has $m\mathbf{i}_{\pi/4} = m(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, for $m \in \mathbb{Z}$. One easily checks that the one-dimensional words $(C_{\pi/4}(m, n_0))_{m \in \mathbb{Z}^2}$ are codings of the rotation $R_{1/\sqrt{2}}: \mathbb{R}/(\sqrt{2}\mathbb{Z}) \rightarrow \mathbb{R}/(\sqrt{2}\mathbb{Z})$, $x \mapsto x + \frac{1}{\sqrt{2}}$, with respect to the three intervals $[-1/2, -3/2 + \sqrt{2}[$, $[-3/2 + \sqrt{2}, 1/2[$, $[1/2, \sqrt{2} - 1/2[$. By renormalizing by $\frac{1}{\sqrt{2}}$, one obtains a coding of the rotation by $1/2$ over $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ with respect to three intervals of length $1 - 1/\sqrt{2}$, $\sqrt{2} - 1$, and $1 - 1/\sqrt{2}$. One obtains a similar result for the sequences in columns. Furthermore, the two-dimensional word $C_{\pi/4}$ presents some intriguing self-similarity properties studied in [Nou]. We plan to explore them by exploiting the self-similarity of the underlying dynamical system provided by Corollary 2, such as illustrated in Fig. 3, and by exhibiting a two-dimensional substitution generating the two-dimensional word $C_{\pi/4}$.

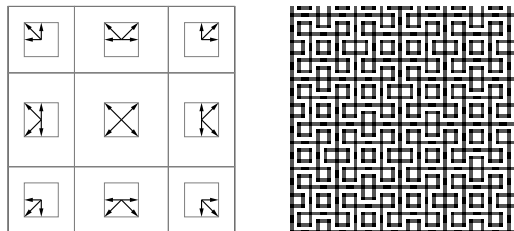


Figure 3. Left: the partition $P_{\pi/4}$. Right: an illustration of the self-similarity of $C_{\pi/4}$.

4 Distribution of symbols in C_α

We can now deduce from the \mathbb{Z}^2 -action introduced in Section 3 combinatorial properties of the two-dimensional word C_α , and in particular, results concerning densities of symbols, by using classical tools from symbolic dynamics and ergodic theory.

Let $G_\alpha \subseteq \mathbb{T}^2$ stand for the orbit of $\mathbf{0}$ under the \mathbb{Z}^2 -action $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{j}_\alpha})$ with respect to the labelling function l_{C_α} : this very orbit is the orbit coded by the configuration C_α . In other words, G_α is the image by the canonical projection

$x \mapsto \{x\}$ onto \mathbb{T}^2 of the lattice $L_\alpha := \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ of \mathbb{R}^2 ; G_α is invariant by rotation by $\pi/2$.

Let us recall that an angle α is said *Pythagorean* if $\cos \alpha$ and $\sin \alpha$ are both rational. The density of G_α is a key ingredient of our combinatorial study. Let us distinguish two cases according to the fact that α is Pythagorean or not.

Lemma 3 *The group G_α is dense in \mathbb{T}^2 if and only if α is not Pythagorean. If α is not Pythagorean, then the two-dimensional sequence $(u_{m,n})_{(m,n) \in \mathbb{Z}^2}$, defined by $u_{m,n} := T_{\mathbf{i}_\alpha}^m \circ T_{\mathbf{j}_\alpha}^n(\mathbf{0})$ is equidistributed in \mathbb{T}^2 . If α is Pythagorean, then the configuration C_α is periodic, and its lattice of periods has dimension two.*

PROOF. Let us assume that α is not Pythagorean. We prove the equidistribution of the two-dimensional sequence $(u_{m,n})_{m,n \in \mathbb{Z}^2}$ in \mathbb{T}^2 by using a classical argument on Weyl sums. Indeed, for $p, q \in \mathbb{Z}^2$, we set $f_{p,q}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $(x, y) \mapsto e^{2i\pi(px+qy)}$. One first checks that $\iint_{[0,1]^2} f_{p,q}(x, y) dx dy \neq 0$ if and only if $p = q = 0$. Furthermore one has

$$\begin{aligned} f_{p,q}(u_{m,n}) &= e^{2i\pi p(m \cos \alpha - n \sin \alpha)} \cdot e^{2i\pi q(m \sin \alpha + n \cos \alpha)} \\ &= e^{2i\pi m(p \cos \alpha + q \sin \alpha)} \cdot e^{2i\pi n(-p \sin \alpha + q \cos \alpha)}. \end{aligned}$$

By hypothesis, one has either $\cos(\alpha)$ or $\sin(\alpha)$ irrational. Then one cannot have simultaneously $p \cos(\alpha) + q \sin(\alpha) \in \mathbb{Z}$ and $-p \sin(\alpha) + q \cos(\alpha) \in \mathbb{Z}$. One thus gets that for $(p, q) \in \mathbb{Z}^2$, $(p, q) \neq (0, 0)$, then

$$\lim_{n \rightarrow +\infty} \frac{1}{(2N+1)^2} \sum_{|m|, |n| \leq N} f_{p,q}(u_{m,n}) = 0,$$

which yields the equidistribution of $(u_{m,n})_{m,n \in \mathbb{Z}^2}$.

We assume now that α is a Pythagorean angle. There exists a unique prime Pythagorean triple $(a, b, c) \in \mathbb{N}^3$ that satisfies $1 \leq b \leq a \leq c$, $\gcd(a, b, c) = 1$, $\cos(\alpha) = \frac{a}{c}$, $\sin(\alpha) = \frac{b}{c}$, and hence $a^2 + b^2 = c^2$. Let $u, v \in \mathbb{Z}^2$ such that $ua - bv = \gcd(a, b)$. The vector $u\mathbf{i}_\alpha + v\mathbf{j}_\alpha$ generates G_α , which is hence a finite cyclic group of order c . Moreover, the vectors $q\mathbf{i}_\alpha$ and $q\mathbf{j}_\alpha$ are period vectors for C_α , hence the lattice of periods of C_α has dimension two. This ends the proof. \square

Let us note that more information on rotations with Pythagorean angles can be found in [NR04]. We can now deduce from Lemma 3 density results for C_α .

Theorem 4 *Let C_α be the configuration associated with the discrete rotation $[r_\alpha]$. For every symbol $p \in Q_\alpha$, its density $\eta_{C_\alpha}(p)$ in C_α exists and is equal to*

- the area of the frame I_p defined in Theorem 1, if α is not Pythagorean,
- and to $1/c \cdot \text{Card}(G_\alpha \cap I_p)$, if α is Pythagorean, where c stands for the order of the group G_α .

PROOF. By definition, one has

$$\eta_{C_\alpha}(p) = \lim_{N \rightarrow \infty} (\{r_\alpha\}(\{-N, \dots, N\}^2) \cap I_p) / (2N + 1)^2.$$

If α is not Pythagorean, then the result comes directly from Lemma 3.

Let us assume now α Pythagorean. One first checks that $\eta_{C_\alpha}(p) = \lim_{N \rightarrow \infty} (\{r_\alpha\}(\{-c\lfloor N/c \rfloor, \dots, c\lfloor N/c \rfloor\}^2) \cap I_p) / (2N + 1)^2$. But as G_α is cyclic and of order c , then $\eta_{C_\alpha}(p) = \frac{\{r_\alpha\}(\{0, \dots, c-1\}^2) \cap I_p}{c^2} = \frac{\text{Card}(G_\alpha \cap I_p)}{c}$. \square

We can similarly deduce the following combinatorial properties of the two-dimensional word C_α . Let us note that we have focused here on the statistical properties of repartition of the symbols in Q_α because of their interest for the study of the discrete rotation $[r_\alpha]$.

Theorem 5 *Let C_α be the configuration associated with the discrete rotation $[r_\alpha]$. The density of rectangular patterns exists in C_α for every pattern χ that occurs in C_α . The two-dimensional word C_α is uniformly recurrent, i.e., for every positive integer n , there exists a positive integer N such that every square pattern of size $[N, N]$ of C_α contains every square pattern of size $[n, n]$ of C_α . Furthermore, there exists a positive constant A such that the rectangular complexity function of C_α satisfies*

$$\forall m, n, p_{C_\alpha}(m, n) \leq A \cdot mn.$$

PROOF. We first deduce from Corollary 2 that given two positive integers m, n , there exists a finite partition of \mathbb{T}^2 into finite unions of frames $P_\alpha^{[m, n]} = \{J_\chi, \chi \text{ pattern of size } [m, n] \text{ of } C_\alpha\}$ such that χ occurs at position \mathbf{v} in C_α if and only if $\{r_\alpha\}(\mathbf{v}) \in I_\chi$. Let us stress the fact that the sets J_χ are not necessarily frames, nor even connected sets; indeed, they are obtained as finite intersections of frames I_p associated with symbols $p \in Q_\alpha$. More precisely, I_χ is obtained as follows:

$$I_\chi = \bigcap_{0 \leq k \leq m-1, 0 \leq \ell \leq n-1} T_{\mathbf{i}_\alpha}^k \circ T_{\mathbf{j}_\alpha}^\ell I_{\chi(k, \ell)}.$$

This allows us to deduce the existence of densities for all rectangular patterns of C_α . We thus obtain analogously as for Theorem 4 that they are equal to the measure of I_χ , in the non-Pythagorean case, and to the cardinality of the intersection of G_α with I_χ , in the Pythagorean case.

Let us assume that α is non-Pythagorean. We assume w.l.o.g. that $\cos(\alpha) \notin \mathbb{Q}$.

According to [Sla67], given any interval I of \mathbb{T}^1 , there exists n_0 such that among any finite sequence of points $\{k \cos(\alpha)\}, \{(k+1) \cos(\alpha)\}, \dots, \{(k+n_0) \cos(\alpha)\}$, at least of them belongs to I . Let us fix a pattern χ and a position $\mathbf{v} \in \mathbb{Z}^2$. We apply the previous result to the interval $I_\chi \cap [-1/2, 1/2[$, and to the sequence $(T_{\mathbf{i}_\alpha}^k(\mathbf{v}) \cap [-1/2, 1/2[)_{k \in \mathbb{Z}} = (x_{\mathbf{v}} + k \cos(\alpha))_{k \in \mathbb{Z}}$. Hence given any $\mathbf{v} \in \mathbb{Z}^2$, the pattern χ occurs at position $\mathbf{v} + k(1, 0)$, for some k with $0 \leq k \leq n_0$, of the configuration C_α , which yields the uniform recurrence. If α is Pythagorean, then the uniform recurrence follows from the fact that C_α has a lattice of periods of rank 2.

We obtain an upper bound on the complexity function by counting the connected components of the sets obtained by taking intersections of the form $\bigcap_{0 \leq k \leq m-1, 0 \leq \ell \leq n-1} T_{\mathbf{i}_\alpha}^k \circ T_{\mathbf{j}_\alpha}^\ell I_{\chi(k,\ell)}$. We thus get $P_{C_\alpha}(m+1, n) - P_{C_\alpha}(m, n) \leq 5n$, for all $n \in \mathbb{N}$, which yields the desired result by a simple induction. \square

Remark 6 *Let us note that we deduce from Lemma 3 that the symbols that appear in C_α at indices of the form $2\mathbf{v}$, for $\mathbf{v} \in \mathbb{Z}^2$ are exactly the elements of Q_α . Indeed, in the non-Pythagorean case, the sequence $(u_{2m,2n})_{(m,n) \in \mathbb{Z}^2}$ is still dense. Otherwise, we use the fact that the Pythagorean triple (a, b, c) introduced in the proof of Lemma 3 is assumed to be a prime triple, i.e., $\gcd(a, b, c) = 1$. We will use this remark hereafter.*

5 Distribution of Symbols in C'_α

We consider now a second configuration C'_α studied, e.g., in [NR05]:

$$\forall \mathbf{v} \in \mathbb{Z}^2, C'_\alpha(\mathbf{v}) := \bigcup_{\mathbf{w} \text{ such that } [r_\alpha(\mathbf{w})]=\mathbf{v}} C_\alpha(\mathbf{w}).$$

The configuration C'_α codes the action of $[r_\alpha]$ on the 4-neighbours of preimages of points of \mathbb{Z}^2 .

Let Q'_α stand for the set of values taken by C'_α . We want to state a result analogous to Theorem 1 in order, first, to interpret the configuration C'_α as a coding of a symbolic dynamical system, and second, to compute the densities of the symbols in C'_α . Let us note that Corollary 1 in [NR05] does not directly yield a dynamical interpretation of C'_α .

Let us note that there exist elements $\mathbf{v} \in \mathbb{Z}^2$ that have no antecedent by $[r_\alpha]$. Such an element is called a *hole*. An example of a hole is depicted in Figure 5 below. According to [NR04], two holes can never be adjacent, i.e., if \mathbf{v} is a hole, then neither $\mathbf{v} + \mathbf{i}$, nor $\mathbf{v} + \mathbf{j}$ is a hole. Our strategy in order to describe C'_α as a coding of a \mathbb{Z}^2 -action is thus to create a “block configuration” by working

with patterns of size $[2, 2]$ that occur in C'_α . According to Remark 6, there is no restriction in working with even indices, rather than with odd indices.

We then introduce a particular domain of \mathbb{R}^2 that is a fundamental domain for the lattice $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$, such that if we know the projection of a point $\mathbf{p} \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ in that domain, then we can recover the symbols that appear in the block configuration; therefore we find out what are the symbols that appear in C'_α . We thus deduce a symbolic dynamical system for the block configuration. Finally, we use this dynamical system, in order to get the density of the symbols both in the block configuration and in C'_α .

5.1 Dynamical system for C'_{B_α}

We denote by $(Q'_\alpha)^{[2,2]}$ the set of patterns of size $[2, 2]$ that occur in C'_α . Let $(C'_\alpha)^{[2,2]}$ be the configuration with values in the finite alphabet $(Q'_\alpha)^{[2,2]}$ that maps \mathbf{v} to the pattern of size $[2, 2]$ that occurs at position $2\mathbf{v}$ in C'_α . Since $(C'_\alpha)^{[2,2]}(\mathbf{v})$ is an application that returns patterns of size $[2, 2]$, then $C'_\alpha(\mathbf{v})$ is obtained by taking the value at position $(x_\mathbf{v} \bmod 2, y_\mathbf{v} \bmod 2)$ in the $[2, 2]$ pattern $(C'_\alpha)^{[2,2]}(\lfloor x_\mathbf{v}/2 \rfloor, \lfloor y_\mathbf{v}/2 \rfloor)$.

For any $\mathbf{v} \in \mathbb{Z}^2$, one sets

$$F_B(\mathbf{v}) = [x_\mathbf{v} - \frac{1}{2}, x_\mathbf{v} + \frac{3}{2}] \times [y_\mathbf{v} - \frac{1}{2}, y_\mathbf{v} + \frac{3}{2}].$$

Let

$$F_{D_\alpha} := \left([-\frac{1}{2}, \cos \alpha - \frac{1}{2}] \right)^2 \cup \left([\cos \alpha - \frac{1}{2}, \cos \alpha + \sin \alpha - \frac{1}{2}] \times [-\frac{1}{2}, \sin \alpha - \frac{1}{2}] \right).$$

The set F_{D_α} is a fundamental domain for the lattice $L_\alpha = \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ (see Figure 4), i.e., $\cup_{\gamma \in L_\alpha} F_{D_\alpha} + \gamma$ is a partition of \mathbb{R}^2 . We thus set $\mathbb{T}_\alpha^2 := \mathbb{R}^2 / (\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha)$. Furthermore, we denote by $\mathbf{v} \mapsto \{\mathbf{v}\}_\alpha$ the canonical projection on \mathbb{T}_α^2 , \mathbb{T}_α^2 being in one-to-correspondence with F_{D_α} .

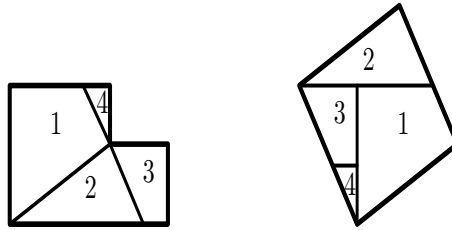


Figure 4. An exchange of pieces between F_{D_α} and the canonical representation of \mathbb{R}^2/L_α , obtained by performing translations in L_α .

Theorem 7 *Let $\alpha \in [0, \pi/4]$. Let C'_α be the configuration associated with the discrete rotation $[r_\alpha]$. There exists a partition $P'_\alpha = \{J_{p'}, p' \in Q'_\alpha\}$ of F_{D_α}*

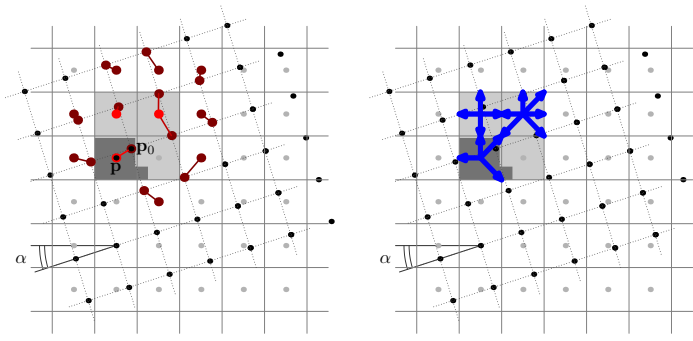


Figure 5. From a point $\mathbf{p}_0 \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ contained in the domain $F_{D_\alpha}(2\mathbf{v})$ (in dark gray), we can recover all the symbols that contribute to the block of size $[2, 2]$ at position $2\mathbf{v}$ in C'_α ; $F_B(2\mathbf{v})$ is depicted in light gray.

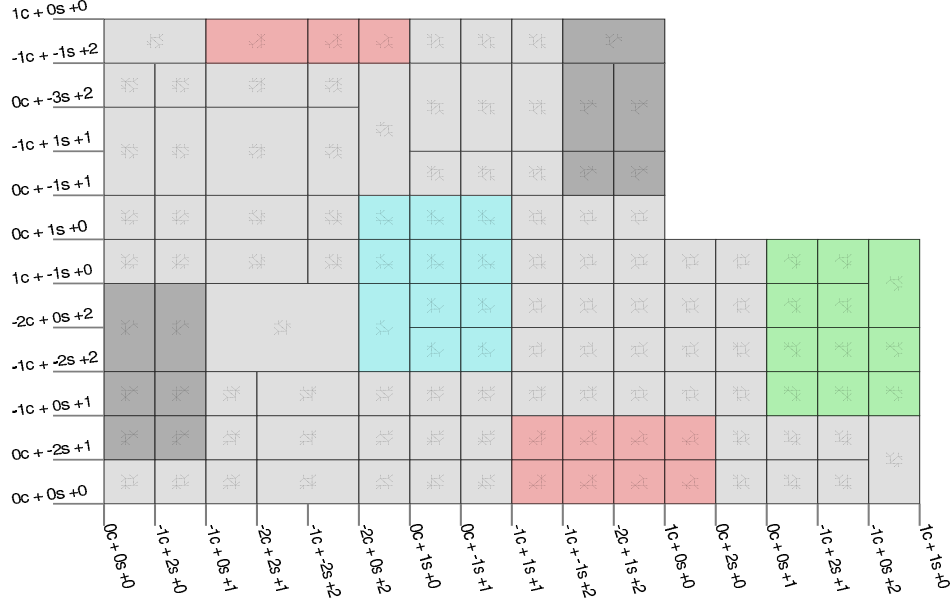


Figure 6. A partition of the domain F_{D_α} , for $\alpha \approx 0.464705$ rad. This partition gives the pattern of size $[2, 2]$ that appears in $(C'_\alpha)^{[2,2]}(\mathbf{v})$, according to the position of $-\{2\mathbf{v}\}_\alpha$ inside that domain. On the axis the positions are labeled by expressions of the form $kc + k's + k''$, meaning that the corresponding line is located at $k \cos(\alpha) + k' \sin(\alpha) + k'' - \frac{1}{2}$ in F_{D_α} . For readability reasons, the scale is monotone but not linear.

into a finite number of frames such that

$$\forall \mathbf{v} \in \mathbb{Z}^2, (C'_\alpha)^{[2,2]}(\mathbf{v}) = p' \iff -\{2\mathbf{v}\}_\alpha \in J_{p'}.$$

We define by $l_{(C'_\alpha)^{[2,2]}}: \mathbb{T}_\alpha^2 \rightarrow (Q'_\alpha)^{[2,2]}$ the labelling function that associates with elements of the frame $J_{p'} \in P'_\alpha$ of F_{D_α} the corresponding pattern p' of

size $[2, 2]$, i.e.,

$$\forall \mathbf{v} \in \mathbb{Z}^2, (C'_\alpha)^{[2,2]}(\mathbf{v}) = l_{(C'_\alpha)^{[2,2]}}(-\{2\mathbf{v}\}_\alpha).$$

The configuration $(C'_\alpha)^{[2,2]}$ is thus a coding of the orbit of 0 under the \mathbb{Z}^2 -action $(\mathbb{T}_\alpha^2, \mathbf{v} \mapsto \mathbf{v} + \{-2\mathbf{i}\}_\alpha, \mathbf{v} \mapsto \mathbf{v} + \{-2\mathbf{j}\}_\alpha)$ with respect to the labelling function $l_{(C'_\alpha)^{[2,2]}}$.

PROOF. The proof is based on the following idea: for any $\mathbf{v} \in \mathbb{Z}^2$, there exists a unique $\gamma \in L_\alpha = r_\alpha(\mathbb{Z})$ such that $-\mathbf{v} \in \gamma + F_{D_\alpha}$, i.e., for any $\mathbf{v} \in \mathbb{Z}^2$, there exists a unique $\mathbf{w} \in \mathbb{Z}^2$ such that $-2\mathbf{v} \in -r_\alpha(\mathbf{w}) + F_{D_\alpha}$. One thus has $r_\alpha(\mathbf{w}) - 2\mathbf{v} = \{-2\mathbf{v}\}_\alpha = -\{2\mathbf{v}\}_\alpha$. Let us prove that it is possible to deduce the value of $(C'_\alpha)^{[2,2]}(\mathbf{v})$ from the location of $\{-2\mathbf{v}\}_\alpha$ in F_{D_α} .

For that purpose, we first check that for all points \mathbf{w} of \mathbb{Z}^2 that have their image by r_α in $F_B(2\mathbf{v})$ we can compute $C_\alpha(\mathbf{w})$, according to Theorem 1 and Remark 6. Indeed, let \mathbf{w} be the unique element \mathbb{Z}^2 such that $r_\alpha(\mathbf{w}) \in \mathbf{v} + F_{D_\alpha}$; if $x_{r_\alpha(\mathbf{w})-2\mathbf{v}} < \frac{1}{2}$, $[r_\alpha(\mathbf{w}) - 2\mathbf{v}] = 0$, else $[r_\alpha(\mathbf{w}) - \mathbf{v}] = 1$; we thus deduce the value of $C_\alpha(\mathbf{w})$, according to Theorem 1. Hence, we get a first partition of F_{D_α} into a finite number of frames yielding the value of $C_\alpha(\mathbf{w})$.

The same argument applies for all points $\mathbf{w}' = r_\alpha(\mathbf{w})$ of $\mathbb{Z}_{\mathbf{i}_\alpha} + \mathbb{Z}_{\mathbf{j}_\alpha}$ that are inside $F_B(2\mathbf{v})$. We thus refine our first partition by intersecting it by translates by vectors of L_α , which ends the proof. \square

5.2 Application

We can perform the same combinatorial study as in Section 4. In particular, Lemma 3 extends in a natural way. We do not detail here the corresponding results but focus on the following application to density of symbols. We assume in particular that α is not a Pythagorean angle. Similarly as in the study of C_α , the orbit of 0 under the \mathbb{Z}^2 -action is dense and uniformly distributed in \mathbb{T}_α^2 . We thus deduce that

$$\forall p \in Q'_\alpha, \eta_{C'_\alpha}(p) = \sum_{p' \in (Q'_\alpha)^{[2,2]}} n(p', p) \mu(f_{p'}),$$

where $n(p', p)$ is the function that returns the number of occurrences of p in the pattern p' of size $[2, 2]$, and $\mu(J_{p'})$ denotes the area of frame $J_{p'}$ associated with the symbol p' according to Theorem 7.

However practically, the computations for these symbolic maps are quite tedious. For each symbol p , there exist 40 patterns p' of size $[2, 2]$ to compute. This leads to approximatively 360 inequations... and there are approximatively 25 symbols p to consider! The results describing the densities of the symbols

in C'_α have been summarized in Figure 7. In the Pythagorean case, the study is also similar to the one developed for C_α .

α			
$[0, \arctan(\sqrt{2}/4)]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-(\cos(\alpha))^2 + 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) - 2 \sin(\alpha)$
$[\arctan(1/2), \pi/6]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	0
$[\pi/6, \arctan(3/4)]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-2 \cos(\alpha) \sin(\alpha) + 1$
$[\arctan(3/4), \pi/4]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-2 \cos(\alpha) \sin(\alpha) + 1$
α			
$[0, \arctan(\sqrt{2}/4)]$	$(\cos(\alpha))^2 - 2 \cos(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	$3 \cos(\alpha) \sin(\alpha) - \cos(\alpha) - 3 \sin(\alpha) + 1$
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	$(\cos(\alpha))^2 - 2 \cos(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	0
$[\arctan(1/2), \pi/6]$	$2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) - 2 \sin(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	0
$[\pi/6, \arctan(3/4)]$	0	0	0
$[\arctan(3/4), \pi/4]$	0	0	$2(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) - 3 \cos(\alpha) + \sin(\alpha) + 1$
α			
$[0, \arctan(\sqrt{2}/4)]$	0	0	0
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	$-3 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	0	0
$[\arctan(1/2), \pi/6]$	$-2(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + 3 \cos(\alpha) - \sin(\alpha) - 1$	$(\cos(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) + 2 \sin(\alpha)$	0
$[\pi/6, \arctan(3/4)]$	$-2(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + 3 \cos(\alpha) - \sin(\alpha) - 1$	$(\cos(\alpha))^2 - 2 \cos(\alpha) + 1$	$-2(\sin(\alpha))^2 + 2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) + \sin(\alpha)$
$[\arctan(3/4), \pi/4]$	0	$-(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$-2(\sin(\alpha))^2 + 2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) + \sin(\alpha)$
α			
$[0, \arctan(\sqrt{2}/4)]$	0	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$	0
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	0	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$	0
$[\arctan(1/2), \pi/6]$	0	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$	0
$[\pi/6, \arctan(3/4)]$	$-2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 2 \sin(\alpha) - 1$	$-(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$4(\sin(\alpha))^2 - 4 \sin(\alpha) + 1$
$[\arctan(3/4), \pi/4]$	$-2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 2 \sin(\alpha) - 1$	$-(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$4(\sin(\alpha))^2 - 4 \sin(\alpha) + 1$

Figure 7. Table describing $\eta_{C'_\alpha}(p)$ for each symbol p that appears in C'_α , with respect to the value of α .

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