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# Discrete Rotations and Symbolic Dynamics

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## Abstract

The aim of this paper is to study local configurations for discrete rotations. The algorithm of discrete rotation we consider is the following: a discretized rotation is defined as the composition of a Euclidean rotation with a rounding operation, as studied in [NR03,NR04,NR05]. It is possible to encode all the information concerning a discrete rotation as two multidimensional words  $C_\alpha$  and  $C'_\alpha$  that we call configurations. We introduce here two discrete dynamical systems defined by a  $\mathbb{Z}^2$ -action on the two-dimensional torus that allow us via a suitable symbolic coding to describe the configurations  $C_\alpha$  and  $C'_\alpha$ ; we then deduce various combinatorial properties for both configurations, and in particular, results concerning densities of occurrence of symbols.

*Key words:* Discrete rotations, discrete geometry, word combinatorics, two-dimensional words, symbolic dynamics.

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## 1 Introduction

Symbolic dynamics and more generally, discrete dynamical systems have natural and deep interactions with combinatorics on words. This interaction is particularly well-illustrated in the Sturmian case, see e.g. [Lot02,Fog02]. The combinatorial objects involved are the Sturmian words, while the dynamical systems are the irrational rotations of the torus  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ . A Sturmian word is indeed a coding with respect to a particular two-interval partition of

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the one-dimensional torus  $\mathbb{T}^1$  of the orbit of a point under the action of an irrational rotation. This point of view allows one to deduce many combinatorial properties of Sturmian words, as discussed in [BFZ05], such as, e.g., the densities of occurrences of factors that can be computed thanks to the equidistribution properties of irrational rotations, or such as powers of factors in Sturmian words [Van00], or the characterization of Sturmian words fixed points of substitutions [BEIR].

Several attempts of generalization of this fruitful interaction have been proposed. For more details, see the survey [BFZ05]. One of the first idea which comes to mind is a rotation of the two-dimensional torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ . As an example, the Tribonacci word, that is, the fixed point of the substitution  $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$  codes the orbit of a point of the torus  $\mathbb{T}^2$  under the action of a translation in  $\mathbb{T}^2$  with respect to a partition of  $\mathbb{T}^2$  into three pieces with fractal boundary [Rau82, Lot05]. More generally, fixed points of Arnoux-Rauzy sequences over  $n$  letters [AR91] code orbits of points of the torus  $\mathbb{T}^{n-1}$  under the action of a translation in  $\mathbb{T}^{n-1}$  with respect to a partition of  $\mathbb{T}^{n-1}$  into  $n$  pieces with fractal boundary [AI01].

A second approach, which is dual to the previous one, consists in working with two rotations of  $\mathbb{T}^1$ . It is indeed convenient to describe arithmetic discrete planes in the sense of [Rev91] by use of the coding with respect to a three-interval partition of a  $\mathbb{Z}^2$ -action by two irrational rotations on  $\mathbb{T}^1$  [BFJP]. One thus gets two-dimensional words over a three-letter alphabet that can be considered as two-dimensional Sturmian words [BV00]. The study of the underlying dynamical system allows one here to obtain a better understanding of the combinatorial and geometric properties of arithmetic discrete planes, such as the enumeration of some local configurations, the so-called  $(m, n)$ -cubes, as well as their densities of occurrence, or their centrosymmetry properties [BFJP].

In all these cases connections between word combinatorics, symbolic dynamics, arithmetics and discrete geometry prove to be natural and enlightening. We consider in the present paper a further generalization motivated by discrete geometry, and more precisely, arithmetic discrete geometry, in the sense of [Rev91]. We study indeed configurations associated with a discrete rotation; there exists several extensions of the the notion of Euclidean rotation in discrete geometry, such as reviewed in [And92]. We consider here discrete rotations defined as the composition of a Euclidean rotation with a rounding operation. It is possible to encode all the information concerning a discrete rotation as two multidimensional words  $C_\alpha$  and  $C'_\alpha$  that we call configurations. These configurations have been introduced and studied in [NR03, NR05, NR05]. The main purpose of the present paper is to prove that both configurations are codings of a  $\mathbb{Z}^2$ -action by two rotations on  $\mathbb{T}^2$  with respect to a partition into a finite number of rectangles. We then deduce in particular results concerning

the density of each symbol in  $C_\alpha$  and  $C'_\alpha$ .

This paper is organized as follows. We introduce the first definitions and conventions in Section 2. Section 3 is devoted to the dynamical study of the configuration  $C_\alpha$ , from which combinatorial properties are deduced in Section 4. A similar study for  $C'_\alpha$  is performed in Section 5. Let us note that results presented here extend those of [BN05].

## 2 Definitions and conventions

We work in the *discrete plane*  $\mathbb{Z}^2$ . For each point  $\mathbf{v} \in \mathbb{Z}^2$ ,  $x_{\mathbf{v}}$  stands for its horizontal coordinate and  $y_{\mathbf{v}}$  for its vertical coordinate.

Let  $x$  be a real number. We recall that the floor function  $x \mapsto \lfloor x \rfloor$  is defined as the greatest integer less or equal to  $x$ . The *rounding function* is defined as  $\lfloor x \rfloor := \lfloor x + 0.5 \rfloor$  and  $\{x\} := x - \lfloor x \rfloor$ . These applications can be extended to vectors in  $\mathbb{Z}^2$ , by independent application on each component.

The *discretization cell* of the point  $\mathbf{v} \in \mathbb{Z}^2$  is defined as the set of elements  $\mathbf{w}$  in  $\mathbb{R}^2$  which have the same image by discretization as  $\mathbf{v}$ , i.e.,  $\lfloor \mathbf{v} \rfloor = \lfloor \mathbf{w} \rfloor$ . Hence the discretization cell of  $\mathbf{v}$  is defined as the half-opened unit square centered at  $\lfloor \mathbf{v} \rfloor$ .

We use the canonical bijection between the torus  $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$  and the square  $\{\mathbf{v} \in \mathbb{R}^2; x_{\mathbf{v}} \in [-\frac{1}{2}, \frac{1}{2}[ \text{ and } y_{\mathbf{v}} \in [-\frac{1}{2}, \frac{1}{2}[ \}$ , i.e., the discretization cell of 0. By abuse of notation, we also denote by  $\{\mathbf{v}\}$  the image under the canonical projection from  $\mathbb{R}^2$  onto  $\mathbb{T}^2$  of a point  $\mathbf{v} \in \mathbb{R}^2$ . Let us stress the fact that the map  $x \mapsto \{x\}$  is thus an additive morphism from  $\mathbb{R}^2$  onto  $\mathbb{T}^2$ .

Without loss of generality, we assume throughout this paper that  $\alpha \in [0, \pi/4]$ : the arguments used here can easily be extended to the case of any other octant. We denote by  $r_\alpha$  the Euclidean rotation of angle  $\alpha$ :

$$r_\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mathbf{v} \mapsto \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \mathbf{v}.$$

The *discrete rotation*  $\lfloor r_\alpha \rfloor$  is defined as

$$\lfloor r_\alpha \rfloor : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2, \mathbf{v} \mapsto \lfloor r_\alpha(\mathbf{v}) \rfloor.$$

By  $\{r_\alpha\}$  we mean the map

$$\{r_\alpha\} : \mathbb{Z}^2 \rightarrow \mathbb{T}^2, \mathbf{v} \mapsto \{r_\alpha(\mathbf{v})\}.$$

We denote by  $(\mathbf{i}, \mathbf{j})$  the canonical basis of the Euclidean space  $\mathbb{R}^2$ . We set  $\mathbf{i}_\alpha := r_\alpha(\mathbf{i})$  and  $\mathbf{j}_\alpha := r_\alpha(\mathbf{j})$ .

Let  $Q$  be a finite set called alphabet. A two-dimensional word in  $Q^{\mathbb{Z}^2}$  is called a *configuration* over  $Q$ . An application from  $\{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\}$  to  $Q$  is called a *pattern* of size  $[m, n]$ . Let  $C$  be a configuration in  $Q^{\mathbb{Z}^2}$ . A pattern  $\chi$  of size  $[m, n]$  occurs at position  $\mathbf{v}$  in  $C$  if  $C(\mathbf{v} + \mathbf{p}) = \chi(\mathbf{p})$ , for all  $\mathbf{p}$  with  $x_{\mathbf{p}}, y_{\mathbf{p}} \in \{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\}$ . The rectangular complexity function of the configuration  $C$  is defined as the function  $p_C: \mathbb{N}^2 \rightarrow \mathbb{N}$ , that counts the number of patterns of size  $[m, n]$  in  $C$ .

The *density* of the symbol  $p \in Q$  in the configuration  $C \in Q^{\mathbb{Z}^2}$  is defined as the following limit (if it exists):

$$\eta_C(p) = \lim_{N \rightarrow \infty} \frac{\text{Card}\{\mathbf{v} \in \mathbb{Z}^2, x_v, y_v \in \{-N, \dots, N\} \text{ and } C(\mathbf{v}) = p\}}{(2N+1)^2}.$$

We similarly define the density of a pattern  $\chi$  in the configuration  $C$  as the following limit (if it exists):

$$\eta_C(\chi) = \lim_{N \rightarrow \infty} \frac{\text{Card}\{\mathbf{v} \in \mathbb{Z}^2, x_v, y_v \in \{-N, \dots, N\} \text{ and } \chi \text{ occurs at position } \mathbf{v}\}}{(2N+1)^2}.$$

A *dynamical system*  $(X, T)$  is defined as the action of a continuous and onto map  $T$  on a compact space  $X$ . Given two continuous and onto maps  $T_1$  and  $T_2$  acting on  $X$  and satisfying  $T_1 \circ T_2 = T_2 \circ T_1$ , the  $\mathbb{Z}^2$ -*action* by  $T_1$  and  $T_2$  on  $X$ , that we denote  $(X, T_1, T_2)$ , is defined by

$$\forall (m, n) \in \mathbb{Z}^2, \forall x \in X, (m, n) \cdot x = T_1^m \circ T_2^n(x).$$

It is natural to associate with two-dimensional symbolic dynamical system to the triple  $(X, T_1, T_2)$  by coding the orbits of the points of  $X$  under the  $\mathbb{Z}^2$ -action as follows: given  $x_0 \in X$  and given a *labelling function*  $l$  defined on  $X$  with values in a finite set  $Q$  that takes constant values on the atoms of a finite partition of  $X$ , the configuration  $C$  defined by

$$\forall (m, n) \in \mathbb{Z}^2, C(m, n) = l(T_1^m \circ T_2^n(x_0))$$

is called the coding of the orbit of  $x_0$  under the  $\mathbb{Z}^2$ -action  $(X, T_1, T_2)$  with respect to the labelling function  $l$ .

### 3 Dynamical system associated with $C_\alpha$

According to [NR03], we associate a first configuration  $C_\alpha$  with the discrete rotation  $[r_\alpha]$  that encodes local information concerning the discrete rotation: the configuration  $C_\alpha$  is defined at point  $\mathbf{v} \in \mathbb{Z}^2$  according to the action of the discrete rotation on the 4-neighbours of  $\mathbf{v}$ ; furthermore, there exists a planar transducer that uses the configuration  $C_\alpha$  as input and gradually computes the action of the discrete rotation [NR05].

More precisely, for a given  $\mathbf{v} \in \mathbb{Z}^2$ , we denote by  $\mathcal{V}_4(\mathbf{v})$  the set of 4-neighbours of  $\mathbf{v}$ , that is,  $\mathcal{V}_4(\mathbf{v}) = \{\mathbf{v} + \mathbf{i}, \mathbf{v} + \mathbf{j}, \mathbf{v} - \mathbf{i}, \mathbf{v} - \mathbf{j}\}$ . The configuration  $C_\alpha$  maps each point  $\mathbf{v}$  of  $\mathbb{Z}^2$  to the set  $[r_\alpha](\mathcal{V}_4(\mathbf{v})) - [r_\alpha][\mathbf{v}]$ , that is,

$$C_\alpha(\mathbf{v}) := \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \text{ with } (a_k = [r_\alpha(\mathbf{v} + r_{\pi/2}^k(\mathbf{i}))] - [r_\alpha(\mathbf{v})], \text{ for } k = 0, \dots, 3).$$

One easily checks that  $C_\alpha$  contains either 3 or 4 non-zero elements; for a detailed proof, see [NR03]. Let  $Q_\alpha$  stand for the finite set of values taken by  $C_\alpha$ .

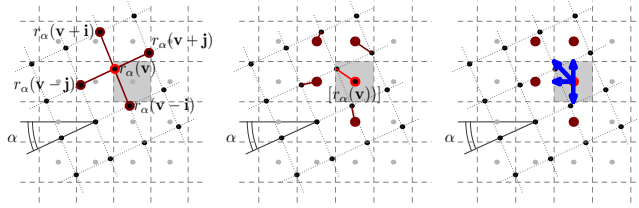


Figure 1. A progressive construction of the configuration  $C_\alpha$ : we represent the set of vectors that leads to the relative positions of the 4-neighbors of  $\mathbf{v}$  after the action of the discrete rotation.

We define a *frame* of the torus  $\mathbb{T}^2 \equiv [-\frac{1}{2}, \frac{1}{2}[ \times [-\frac{1}{2}, \frac{1}{2}[$  as a rectangle of the form  $[a, b[ \times [c, d[$ , with  $-\frac{1}{2} \leq a \leq b < \frac{1}{2}$  and  $-\frac{1}{2} \leq c \leq d < \frac{1}{2}$ . The interpretation of  $C_\alpha$  as a coding a  $\mathbb{Z}^2$ -action is based on the following result:

**Theorem 1 ([NR05])** *There exists a partition  $P_\alpha = \{I_p, p \in Q_\alpha\}$  of the torus  $\mathbb{T}^2$  into a finite number of frames such that*

$$\forall \mathbf{v} \in \mathbb{Z}^2, C_\alpha(\mathbf{v}) = p \iff \{r_\alpha(\mathbf{v})\} \in I_p.$$

More precisely, the partition  $P_\alpha$  is defined as follows: if  $\alpha \in [0, \pi/6[$  (resp.  $[\pi/6, \pi/4[$ ), then the torus is divided into at most 25 frames, delimited by the (at most ) 10 lines with equation  $x = -\frac{1}{2}, x = \frac{1}{2} - \cos(\alpha), x = \sin(\alpha) - \frac{1}{2}, x = \frac{1}{2} - \sin(\alpha), x = \cos(\alpha) - \frac{1}{2}, \frac{1}{2}, y = -\frac{1}{2}, y = \frac{1}{2} - \cos(\alpha), y = \sin(\alpha) - \frac{1}{2}, y = \frac{1}{2} - \sin(\alpha), y = \cos(\alpha) - \frac{1}{2}, \frac{1}{2}$ , (resp.  $x, y = -\frac{1}{2}, \frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}, \frac{1}{2}$ ). More precisely, the alphabet  $Q_\alpha$  has exactly 25 elements if  $\alpha \neq 0, \pi/4, \pi/6$ , 16 elements if  $\alpha = \pi/6$ , and 9, if  $\alpha = \pi/4$ .

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$(0, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 0) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 1) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 2) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	
$(0, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 3) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(2, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(3, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 4) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$
$(0, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(1, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$		$(3, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(4, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$	$(5, 5) \mapsto \begin{array}{ c } \hline \times \\ \hline \end{array}$

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Figure 2. Table describing the action of  $\phi_c$ . The symbols represent the directions of the vectors of  $C_\alpha(\mathbf{v})$ .

Consider now the following two actions

$$T_{\mathbf{i}_\alpha} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, x \mapsto x + \{\mathbf{i}_\alpha\}, T_{\mathbf{j}_\alpha} : \mathbb{T}^2 \rightarrow \mathbb{T}^2, x \mapsto x + \{\mathbf{j}_\alpha\}.$$

One has

$$\forall \mathbf{v} \in \mathbb{Z}^2, \{r_\alpha\}(\mathbf{v}) = T_{\mathbf{i}_\alpha}^{x\mathbf{v}} \circ T_{\mathbf{i}_\alpha}^{y\mathbf{v}}(\mathbf{0}).$$

We then associate with the partition  $P_\alpha$  the labelling function

$$l_{C_\alpha} : \mathbb{T}^2 \rightarrow Q_\alpha, \mathbf{v} \mapsto \phi_c(f_{C_\alpha}(x_{\mathbf{v}}), f_{C_\alpha}(y_{\mathbf{v}})),$$

where  $\phi_c : \{0, 1, 2, 3, 4\}^2 \rightarrow Q_\alpha$  if  $\alpha \in [0, \pi/6]$  (resp.  $\phi_c : \{0, 1, 3, 4, 5\}^2 \rightarrow Q_\alpha$  if  $\alpha \in [\pi/6, \pi/4]$ ) is described in Figure 2, and  $f_{C_\alpha} : [-1/2, 1/2] \rightarrow \{0, 1, 2, 3, 4, 5\}$  is defined by

if  $\alpha \in [0, \pi/6]$ :

if  $\alpha \in [\pi/6, \pi/4]$ :

$$\left[ \begin{array}{l} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[ \quad \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \sin(\alpha) - \frac{1}{2}[ \mapsto 1 \\ [\sin(\alpha) - \frac{1}{2}, \frac{1}{2} - \sin(\alpha)[ \mapsto 2 \\ [\frac{1}{2} - \sin(\alpha), \cos(\alpha) - \frac{1}{2}[ \mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[ \quad \mapsto 4 \end{array} \right. \quad \left[ \begin{array}{l} [-\frac{1}{2}, \frac{1}{2} - \cos(\alpha)[ \quad \mapsto 0 \\ [\frac{1}{2} - \cos(\alpha), \frac{1}{2} - \sin(\alpha)[ \mapsto 1 \\ [\frac{1}{2} - \sin(\alpha), \sin(\alpha) - \frac{1}{2}[ \mapsto 5 \\ [\sin(\alpha) - \frac{1}{2}, \cos(\alpha) - \frac{1}{2}[ \mapsto 3 \\ [\cos(\alpha) - \frac{1}{2}, \frac{1}{2}[ \quad \mapsto 4 \end{array} \right.$$

The values taken by  $C_\alpha$ , i.e., the elements of  $Q_\alpha$  are depicted in Figure 2 according to the directions of the vectors of  $C_\alpha(\mathbf{v})$ , for  $\mathbf{v} \in \mathbb{Z}^2$ .

Theorem 1 can then be reformulated as follows:

**Corollary 2** *Let  $C_\alpha$  be the configuration associated with the discrete rotation  $[r_\alpha]$ . We use the notation introduced above. The configuration  $C_\alpha$  is the coding of the orbit of  $\mathbf{0}$  under the  $\mathbb{Z}^2$ -action  $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{j}_\alpha})$  with respect to the labelling function  $l_{C_\alpha}$ .*

Corollary 2 means that the position, in the discretization cell of a point  $\mathbf{v} \in \mathbb{Z}^2$ , of the point  $\{r_\alpha\}(\mathbf{v})$  of the lattice  $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$  determines the directions of the images of the neighbours of  $\mathbf{v}$  under the action of the discrete rotation.

*Example : the case  $\alpha = \pi/4$*

We detail here the case  $\alpha = \pi/4$ . In this case, the alphabet  $Q_{\pi/4}$  has 9 elements. Consider the sequences in lines of the two-dimensional word  $C_{\pi/4}$ . One has  $m\mathbf{i}_{\pi/4} = m(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ , for  $m \in \mathbb{Z}$ . One easily checks that the one-dimensional words  $(C_{\pi/4}(m, n_0))_{m \in \mathbb{Z}^2}$  are codings of the rotation  $R_{1/\sqrt{2}}: \mathbb{R}/(\sqrt{2}\mathbb{Z}) \rightarrow \mathbb{R}/(\sqrt{2}\mathbb{Z})$ ,  $x \mapsto x + \frac{1}{\sqrt{2}}$ , with respect to the three intervals  $[-1/2, -3/2 + \sqrt{2}[$ ,  $[-3/2 + \sqrt{2}, 1/2[$ ,  $[1/2, \sqrt{2} - 1/2[$ . By renormalizing by  $\frac{1}{\sqrt{2}}$ , one obtains a coding of the rotation by  $1/2$  over  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$  with respect to three intervals of length  $1 - 1/\sqrt{2}$ ,  $\sqrt{2} - 1$ , and  $1 - 1/\sqrt{2}$ . One obtains a similar result for the sequences in columns. Furthermore, the two-dimensional word  $C_{\pi/4}$  presents some intriguing self-similarity properties studied in [Nou]. We plan to explore them by exploiting the self-similarity of the underlying dynamical system provided by Corollary 2, such as illustrated in Fig. 3, and by exhibiting a two-dimensional substitution generating the two-dimensional word  $C_{\pi/4}$ .

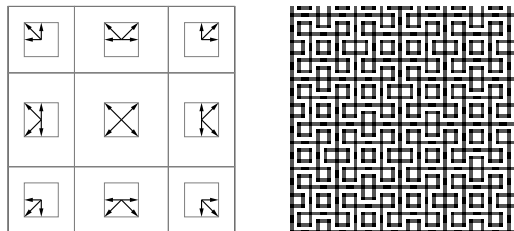


Figure 3. Left: the partition  $P_{\pi/4}$ . Right: an illustration of the self-similarity of  $C_{\pi/4}$ .

#### 4 Distribution of symbols in $C_\alpha$

We can now deduce from the  $\mathbb{Z}^2$ -action introduced in Section 3 combinatorial properties of the two-dimensional word  $C_\alpha$ , and in particular, results concerning densities of symbols, by using classical tools from symbolic dynamics and ergodic theory.

Let  $G_\alpha \subseteq \mathbb{T}^2$  stand for the orbit of  $\mathbf{0}$  under the  $\mathbb{Z}^2$ -action  $(\mathbb{T}^2, T_{\mathbf{i}_\alpha}, T_{\mathbf{j}_\alpha})$  with respect to the labelling function  $l_{C_\alpha}$ : this very orbit is the orbit coded by the configuration  $C_\alpha$ . In other words,  $G_\alpha$  is the image by the canonical projection



$x \mapsto \{x\}$  onto  $\mathbb{T}^2$  of the lattice  $L_\alpha := \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$  of  $\mathbb{R}^2$ ;  $G_\alpha$  is invariant by rotation by  $\pi/2$ .

Let us recall that an angle  $\alpha$  is said *Pythagorean* if  $\cos \alpha$  and  $\sin \alpha$  are both rational. The density of  $G_\alpha$  is a key ingredient of our combinatorial study. Let us distinguish two cases according to the fact that  $\alpha$  is Pythagorean or not.

**Lemma 3** *The group  $G_\alpha$  is dense in  $\mathbb{T}^2$  if and only if  $\alpha$  is not Pythagorean. If  $\alpha$  is not Pythagorean, then the two-dimensional sequence  $(u_{m,n})_{(m,n) \in \mathbb{Z}^2}$ , defined by  $u_{m,n} := T_{\mathbf{i}_\alpha}^m \circ T_{\mathbf{j}_\alpha}^n(\mathbf{0})$  is equidistributed in  $\mathbb{T}^2$ . If  $\alpha$  is Pythagorean, then the configuration  $C_\alpha$  is periodic, and its lattice of periods has dimension two.*

PROOF. Let us assume that  $\alpha$  is not Pythagorean. We prove the equidistribution of the two-dimensional sequence  $(u_{m,n})_{m,n \in \mathbb{Z}^2}$  in  $\mathbb{T}^2$  by using a classical argument on Weyl sums. Indeed, for  $p, q \in \mathbb{Z}^2$ , we set  $f_{p,q}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $(x, y) \mapsto e^{2i\pi(px+qy)}$ . One first checks that  $\iint_{[0,1]^2} f_{p,q}(x, y) dx dy \neq 0$  if and only if  $p = q = 0$ . Furthermore one has

$$\begin{aligned} f_{p,q}(u_{m,n}) &= e^{2i\pi p(m \cos \alpha - n \sin \alpha)} \cdot e^{2i\pi q(m \sin \alpha + n \cos \alpha)} \\ &= e^{2i\pi m(p \cos \alpha + q \sin \alpha)} \cdot e^{2i\pi n(-p \sin \alpha + q \cos \alpha)}. \end{aligned}$$

By hypothesis, one has either  $\cos(\alpha)$  or  $\sin(\alpha)$  irrational. Then one cannot have simultaneously  $p \cos(\alpha) + q \sin(\alpha) \in \mathbb{Z}$  and  $-p \sin(\alpha) + q \cos(\alpha) \in \mathbb{Z}$ . One thus gets that for  $(p, q) \in \mathbb{Z}^2$ ,  $(p, q) \neq (0, 0)$ , then

$$\lim_{n \rightarrow +\infty} \frac{1}{(2N+1)^2} \sum_{|m|, |n| \leq N} f_{p,q}(u_{m,n}) = 0,$$

which yields the equidistribution of  $(u_{m,n})_{m,n \in \mathbb{Z}^2}$ .

We assume now that  $\alpha$  is a Pythagorean angle. There exists a unique prime Pythagorean triple  $(a, b, c) \in \mathbb{N}^3$  that satisfies  $1 \leq b \leq a \leq c$ ,  $\gcd(a, b, c) = 1$ ,  $\cos(\alpha) = \frac{a}{c}$ ,  $\sin(\alpha) = \frac{b}{c}$ , and hence  $a^2 + b^2 = c^2$ . Let  $u, v \in \mathbb{Z}^2$  such that  $ua - bv = \gcd(a, b)$ . The vector  $u\mathbf{i}_\alpha + v\mathbf{j}_\alpha$  generates  $G_\alpha$ , which is hence a finite cyclic group of order  $c$ . Moreover, the vectors  $q\mathbf{i}_\alpha$  and  $q\mathbf{j}_\alpha$  are period vectors for  $C_\alpha$ , hence the lattice of periods of  $C_\alpha$  has dimension two. This ends the proof.  $\square$

Let us note that more information on rotations with Pythagorean angles can be found in [NR04]. We can now deduce from Lemma 3 density results for  $C_\alpha$ .

**Theorem 4** *Let  $C_\alpha$  be the configuration associated with the discrete rotation  $[r_\alpha]$ . For every symbol  $p \in Q_\alpha$ , its density  $\eta_{C_\alpha}(p)$  in  $C_\alpha$  exists and is equal to*

- the area of the frame  $I_p$  defined in Theorem 1, if  $\alpha$  is not Pythagorean,
- and to  $1/c \cdot \text{Card}(G_\alpha \cap I_p)$ , if  $\alpha$  is Pythagorean, where  $c$  stands for the order of the group  $G_\alpha$ .

PROOF. By definition, one has

$$\eta_{C_\alpha}(p) = \lim_{N \rightarrow \infty} (\{r_\alpha\}(\{-N, \dots, N\}^2) \cap I_p) / (2N + 1)^2.$$

If  $\alpha$  is not Pythagorean, then the result comes directly from Lemma 3.

Let us assume now  $\alpha$  Pythagorean. One first checks that  $\eta_{C_\alpha}(p) = \lim_{N \rightarrow \infty} (\{r_\alpha\}(\{-c\lfloor N/c \rfloor, \dots, c\lfloor N/c \rfloor\}^2) \cap I_p) / (2N + 1)^2$ . But as  $G_\alpha$  is cyclic and of order  $c$ , then  $\eta_{C_\alpha}(p) = \frac{\{r_\alpha\}(\{0, \dots, c-1\}^2) \cap I_p}{c^2} = \frac{\text{Card}(G_\alpha \cap I_p)}{c}$ .  $\square$

We can similarly deduce the following combinatorial properties of the two-dimensional word  $C_\alpha$ . Let us note that we have focused here on the statistical properties of repartition of the symbols in  $Q_\alpha$  because of their interest for the study of the discrete rotation  $[r_\alpha]$ .

**Theorem 5** *Let  $C_\alpha$  be the configuration associated with the discrete rotation  $[r_\alpha]$ . The density of rectangular patterns exists in  $C_\alpha$  for every pattern  $\chi$  that occurs in  $C_\alpha$ . The two-dimensional word  $C_\alpha$  is uniformly recurrent, i.e., for every positive integer  $n$ , there exists a positive integer  $N$  such that every square pattern of size  $[N, N]$  of  $C_\alpha$  contains every square pattern of size  $[n, n]$  of  $C_\alpha$ . Furthermore, there exists a positive constant  $A$  such that the rectangular complexity function of  $C_\alpha$  satisfies*

$$\forall m, n, p_{C_\alpha}(m, n) \leq A \cdot mn.$$

PROOF. We first deduce from Corollary 2 that given two positive integers  $m, n$ , there exists a finite partition of  $\mathbb{T}^2$  into finite unions of frames  $P_\alpha^{[m, n]} = \{J_\chi, \chi \text{ pattern of size } [m, n] \text{ of } C_\alpha\}$  such that  $\chi$  occurs at position  $\mathbf{v}$  in  $C_\alpha$  if and only if  $\{r_\alpha\}(\mathbf{v}) \in I_\chi$ . Let us stress the fact that the sets  $J_\chi$  are not necessarily frames, nor even connected sets; indeed, they are obtained as finite intersections of frames  $I_p$  associated with symbols  $p \in Q_\alpha$ . More precisely,  $I_\chi$  is obtained as follows:

$$I_\chi = \bigcap_{0 \leq k \leq m-1, 0 \leq \ell \leq n-1} T_{\mathbf{i}_\alpha}^k \circ T_{\mathbf{j}_\alpha}^\ell I_{\chi(k, \ell)}.$$

This allows us to deduce the existence of densities for all rectangular patterns of  $C_\alpha$ . We thus obtain analogously as for Theorem 4 that they are equal to the measure of  $I_\chi$ , in the non-Pythagorean case, and to the cardinality of the intersection of  $G_\alpha$  with  $I_\chi$ , in the Pythagorean case.

Let us assume that  $\alpha$  is non-Pythagorean. We assume w.l.o.g. that  $\cos(\alpha) \notin \mathbb{Q}$ .

According to [Sla67], given any interval  $I$  of  $\mathbb{T}^1$ , there exists  $n_0$  such that among any finite sequence of points  $\{k \cos(\alpha)\}, \{(k+1) \cos(\alpha)\}, \dots, \{(k+n_0) \cos(\alpha)\}$ , at least of them belongs to  $I$ . Let us fix a pattern  $\chi$  and a position  $\mathbf{v} \in \mathbb{Z}^2$ . We apply the previous result to the interval  $I_\chi \cap [-1/2, 1/2[$ , and to the sequence  $(T_{\mathbf{i}_\alpha}^k(\mathbf{v}) \cap [-1/2, 1/2[)_{k \in \mathbb{Z}} = (x_{\mathbf{v}} + k \cos(\alpha))_{k \in \mathbb{Z}}$ . Hence given any  $\mathbf{v} \in \mathbb{Z}^2$ , the pattern  $\chi$  occurs at position  $\mathbf{v} + k(1, 0)$ , for some  $k$  with  $0 \leq k \leq n_0$ , of the configuration  $C_\alpha$ , which yields the uniform recurrence. If  $\alpha$  is Pythagorean, then the uniform recurrence follows from the fact that  $C_\alpha$  has a lattice of periods of rank 2.

We obtain an upper bound on the complexity function by counting the connected components of the sets obtained by taking intersections of the form  $\bigcap_{0 \leq k \leq m-1, 0 \leq \ell \leq n-1} T_{\mathbf{i}_\alpha}^k \circ T_{\mathbf{j}_\alpha}^\ell I_{\chi(k,\ell)}$ . We thus get  $P_{C_\alpha}(m+1, n) - P_{C_\alpha}(m, n) \leq 5n$ , for all  $n \in \mathbb{N}$ , which yields the desired result by a simple induction.  $\square$

**Remark 6** *Let us note that we deduce from Lemma 3 that the symbols that appear in  $C_\alpha$  at indices of the form  $2\mathbf{v}$ , for  $\mathbf{v} \in \mathbb{Z}^2$  are exactly the elements of  $Q_\alpha$ . Indeed, in the non-Pythagorean case, the sequence  $(u_{2m,2n})_{(m,n) \in \mathbb{Z}^2}$  is still dense. Otherwise, we use the fact that the Pythagorean triple  $(a, b, c)$  introduced in the proof of Lemma 3 is assumed to be a prime triple, i.e.,  $\gcd(a, b, c) = 1$ . We will use this remark hereafter.*

## 5 Distribution of Symbols in $C'_\alpha$

We consider now a second configuration  $C'_\alpha$  studied, e.g., in [NR05]:

$$\forall \mathbf{v} \in \mathbb{Z}^2, C'_\alpha(\mathbf{v}) := \bigcup_{\mathbf{w} \text{ such that } [r_\alpha(\mathbf{w})]=\mathbf{v}} C_\alpha(\mathbf{w}).$$

The configuration  $C'_\alpha$  codes the action of  $[r_\alpha]$  on the 4-neighbours of preimages of points of  $\mathbb{Z}^2$ .

Let  $Q'_\alpha$  stand for the set of values taken by  $C'_\alpha$ . We want to state a result analogous to Theorem 1 in order, first, to interpret the configuration  $C'_\alpha$  as a coding of a symbolic dynamical system, and second, to compute the densities of the symbols in  $C'_\alpha$ . Let us note that Corollary 1 in [NR05] does not directly yield a dynamical interpretation of  $C'_\alpha$ .

Let us note that there exist elements  $\mathbf{v} \in \mathbb{Z}^2$  that have no antecedent by  $[r_\alpha]$ . Such an element is called a *hole*. An example of a hole is depicted in Figure 5 below. According to [NR04], two holes can never be adjacent, i.e., if  $\mathbf{v}$  is a hole, then neither  $\mathbf{v} + \mathbf{i}$ , nor  $\mathbf{v} + \mathbf{j}$  is a hole. Our strategy in order to describe  $C'_\alpha$  as a coding of a  $\mathbb{Z}^2$ -action is thus to create a “block configuration” by working

with patterns of size  $[2, 2]$  that occur in  $C'_\alpha$ . According to Remark 6, there is no restriction in working with even indices, rather than with odd indices.

We then introduce a particular domain of  $\mathbb{R}^2$  that is a fundamental domain for the lattice  $\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$ , such that if we know the projection of a point  $\mathbf{p} \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$  in that domain, then we can recover the symbols that appear in the block configuration; therefore we find out what are the symbols that appear in  $C'_\alpha$ . We thus deduce a symbolic dynamical system for the block configuration. Finally, we use this dynamical system, in order to get the density of the symbols both in the block configuration and in  $C'_\alpha$ .

### 5.1 Dynamical system for $C'_{B_\alpha}$

We denote by  $(Q'_\alpha)^{[2,2]}$  the set of patterns of size  $[2, 2]$  that occur in  $C'_\alpha$ . Let  $(C'_\alpha)^{[2,2]}$  be the configuration with values in the finite alphabet  $(Q'_\alpha)^{[2,2]}$  that maps  $\mathbf{v}$  to the pattern of size  $[2, 2]$  that occurs at position  $2\mathbf{v}$  in  $C'_\alpha$ . Since  $(C'_\alpha)^{[2,2]}(\mathbf{v})$  is an application that returns patterns of size  $[2, 2]$ , then  $C'_\alpha(\mathbf{v})$  is obtained by taking the value at position  $(x_\mathbf{v} \bmod 2, y_\mathbf{v} \bmod 2)$  in the  $[2, 2]$  pattern  $(C'_\alpha)^{[2,2]}(\lfloor x_\mathbf{v}/2 \rfloor, \lfloor y_\mathbf{v}/2 \rfloor)$ .

For any  $\mathbf{v} \in \mathbb{Z}^2$ , one sets

$$F_B(\mathbf{v}) = [x_\mathbf{v} - \frac{1}{2}, x_\mathbf{v} + \frac{3}{2}] \times [y_\mathbf{v} - \frac{1}{2}, y_\mathbf{v} + \frac{3}{2}].$$

Let

$$F_{D_\alpha} := \left( [-\frac{1}{2}, \cos \alpha - \frac{1}{2}] \right)^2 \cup \left( [\cos \alpha - \frac{1}{2}, \cos \alpha + \sin \alpha - \frac{1}{2}] \times [-\frac{1}{2}, \sin \alpha - \frac{1}{2}] \right).$$

The set  $F_{D_\alpha}$  is a fundamental domain for the lattice  $L_\alpha = \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$  (see Figure 4), i.e.,  $\cup_{\gamma \in L_\alpha} F_{D_\alpha} + \gamma$  is a partition of  $\mathbb{R}^2$ . We thus set  $\mathbb{T}_\alpha^2 := \mathbb{R}^2 / (\mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha)$ . Furthermore, we denote by  $\mathbf{v} \mapsto \{\mathbf{v}\}_\alpha$  the canonical projection on  $\mathbb{T}_\alpha^2$ ,  $\mathbb{T}_\alpha^2$  being in one-to-correspondence with  $F_{D_\alpha}$ .

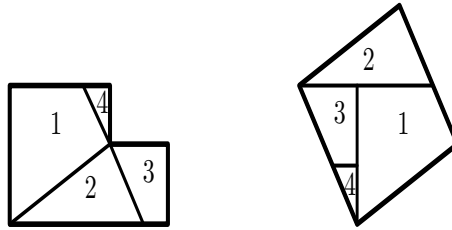


Figure 4. An exchange of pieces between  $F_{D_\alpha}$  and the canonical representation of  $\mathbb{R}^2/L_\alpha$ , obtained by performing translations in  $L_\alpha$ .

**Theorem 7** *Let  $\alpha \in [0, \pi/4]$ . Let  $C'_\alpha$  be the configuration associated with the discrete rotation  $[r_\alpha]$ . There exists a partition  $P'_\alpha = \{J_{p'}, p' \in Q'_\alpha\}$  of  $F_{D_\alpha}$*

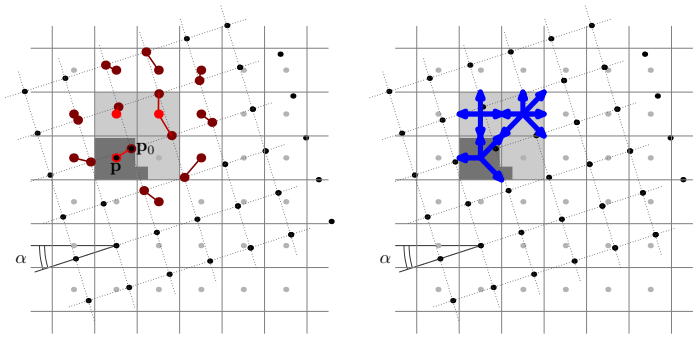


Figure 5. From a point  $\mathbf{p}_0 \in \mathbb{Z}\mathbf{i}_\alpha + \mathbb{Z}\mathbf{j}_\alpha$  contained in the domain  $F_{D_\alpha}(2\mathbf{v})$  (in dark gray), we can recover all the symbols that contribute to the block of size  $[2, 2]$  at position  $2\mathbf{v}$  in  $C'_\alpha$ ;  $F_B(2\mathbf{v})$  is depicted in light gray.

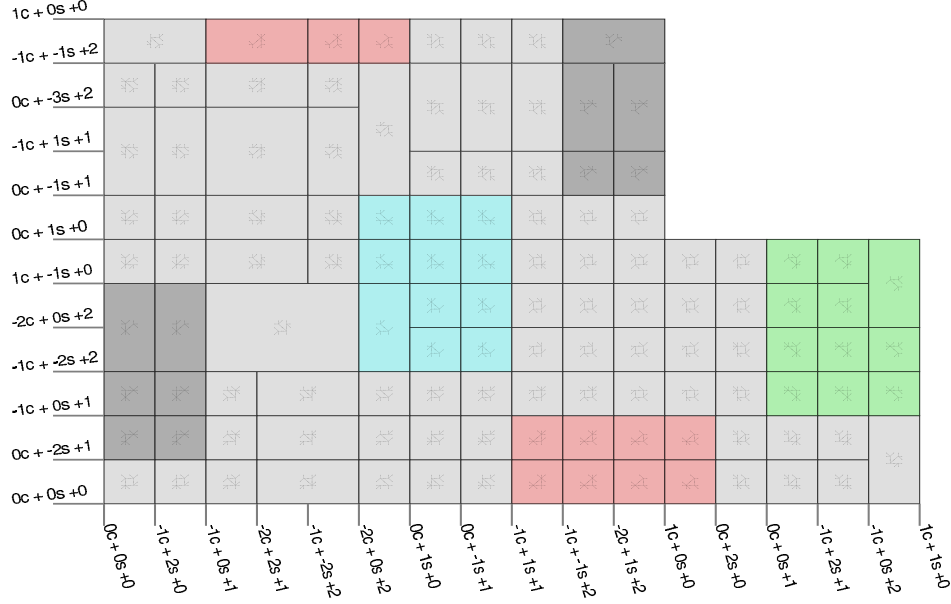


Figure 6. A partition of the domain  $F_{D_\alpha}$ , for  $\alpha \approx 0.464705$  rad. This partition gives the pattern of size  $[2, 2]$  that appears in  $(C'_\alpha)^{[2,2]}(\mathbf{v})$ , according to the position of  $-\{2\mathbf{v}\}_\alpha$  inside that domain. On the axis the positions are labeled by expressions of the form  $kc + k's + k''$ , meaning that the corresponding line is located at  $k \cos(\alpha) + k' \sin(\alpha) + k'' - \frac{1}{2}$  in  $F_{D_\alpha}$ . For readability reasons, the scale is monotone but not linear.

into a finite number of frames such that

$$\forall \mathbf{v} \in \mathbb{Z}^2, (C'_\alpha)^{[2,2]}(\mathbf{v}) = p' \iff -\{2\mathbf{v}\}_\alpha \in J_{p'}.$$

We define by  $l_{(C'_\alpha)^{[2,2]}}: \mathbb{T}_\alpha^2 \rightarrow (Q'_\alpha)^{[2,2]}$  the labelling function that associates with elements of the frame  $J_{p'} \in P'_\alpha$  of  $F_{D_\alpha}$  the corresponding pattern  $p'$  of

size  $[2, 2]$ , i.e.,

$$\forall \mathbf{v} \in \mathbb{Z}^2, (C'_\alpha)^{[2,2]}(\mathbf{v}) = l_{(C'_\alpha)^{[2,2]}}(-\{2\mathbf{v}\}_\alpha).$$

The configuration  $(C'_\alpha)^{[2,2]}$  is thus a coding of the orbit of 0 under the  $\mathbb{Z}^2$ -action  $(\mathbb{T}_\alpha^2, \mathbf{v} \mapsto \mathbf{v} + \{-2\mathbf{i}\}_\alpha, \mathbf{v} \mapsto \mathbf{v} + \{-2\mathbf{j}\}_\alpha)$  with respect to the labelling function  $l_{(C'_\alpha)^{[2,2]}}$ .

PROOF. The proof is based on the following idea: for any  $\mathbf{v} \in \mathbb{Z}^2$ , there exists a unique  $\gamma \in L_\alpha = r_\alpha(\mathbb{Z})$  such that  $-\mathbf{v} \in \gamma + F_{D_\alpha}$ , i.e., for any  $\mathbf{v} \in \mathbb{Z}^2$ , there exists a unique  $\mathbf{w} \in \mathbb{Z}^2$  such that  $-2\mathbf{v} \in -r_\alpha(\mathbf{w}) + F_{D_\alpha}$ . One thus has  $r_\alpha(\mathbf{w}) - 2\mathbf{v} = \{-2\mathbf{v}\}_\alpha = -\{2\mathbf{v}\}_\alpha$ . Let us prove that it is possible to deduce the value of  $(C'_\alpha)^{[2,2]}(\mathbf{v})$  from the location of  $\{-2\mathbf{v}\}_\alpha$  in  $F_{D_\alpha}$ .

For that purpose, we first check that for all points  $\mathbf{w}$  of  $\mathbb{Z}^2$  that have their image by  $r_\alpha$  in  $F_B(2\mathbf{v})$  we can compute  $C_\alpha(\mathbf{w})$ , according to Theorem 1 and Remark 6. Indeed, let  $\mathbf{w}$  be the unique element  $\mathbb{Z}^2$  such that  $r_\alpha(\mathbf{w}) \in \mathbf{v} + F_{D_\alpha}$ ; if  $x_{r_\alpha(\mathbf{w})-2\mathbf{v}} < \frac{1}{2}$ ,  $[r_\alpha(\mathbf{w}) - 2\mathbf{v}] = 0$ , else  $[r_\alpha(\mathbf{w}) - \mathbf{v}] = 1$ ; we thus deduce the value of  $C_\alpha(\mathbf{w})$ , according to Theorem 1. Hence, we get a first partition of  $F_{D_\alpha}$  into a finite number of frames yielding the value of  $C_\alpha(\mathbf{w})$ .

The same argument applies for all points  $\mathbf{w}' = r_\alpha(\mathbf{w})$  of  $\mathbb{Z}_{\mathbf{i}_\alpha} + \mathbb{Z}_{\mathbf{j}_\alpha}$  that are inside  $F_B(2\mathbf{v})$ . We thus refine our first partition by intersecting it by translates by vectors of  $L_\alpha$ , which ends the proof.  $\square$

## 5.2 Application

We can perform the same combinatorial study as in Section 4. In particular, Lemma 3 extends in a natural way. We do not detail here the corresponding results but focus on the following application to density of symbols. We assume in particular that  $\alpha$  is not a Pythagorean angle. Similarly as in the study of  $C_\alpha$ , the orbit of 0 under the  $\mathbb{Z}^2$ -action is dense and uniformly distributed in  $\mathbb{T}_\alpha^2$ . We thus deduce that

$$\forall p \in Q'_\alpha, \eta_{C'_\alpha}(p) = \sum_{p' \in (Q'_\alpha)^{[2,2]}} n(p', p) \mu(f_{p'}),$$

where  $n(p', p)$  is the function that returns the number of occurrences of  $p$  in the pattern  $p'$  of size  $[2, 2]$ , and  $\mu(J_{p'})$  denotes the area of frame  $J_{p'}$  associated with the symbol  $p'$  according to Theorem 7.

However practically, the computations for these symbolic maps are quite tedious. For each symbol  $p$ , there exist 40 patterns  $p'$  of size  $[2, 2]$  to compute. This leads to approximatively 360 inequations... and there are approximatively 25 symbols  $p$  to consider! The results describing the densities of the symbols

in  $C'_\alpha$  have been summarized in Figure 7. In the Pythagorean case, the study is also similar to the one developed for  $C_\alpha$ .

$\alpha$			
$[0, \arctan(\sqrt{2}/4)]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-(\cos(\alpha))^2 + 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) - 2 \sin(\alpha)$
$[\arctan(1/2), \pi/6]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	0
$[\pi/6, \arctan(3/4)]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-2 \cos(\alpha) \sin(\alpha) + 1$
$[\arctan(3/4), \pi/4]$	$2 \cos(\alpha) \sin(\alpha) - 2 \cos(\alpha) - 2 \sin(\alpha) + 2$	$-2 \cos(\alpha) \sin(\alpha) + 1$	0
$\alpha$			
$[0, \arctan(\sqrt{2}/4)]$	$(\cos(\alpha))^2 - 2 \cos(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	$3 \cos(\alpha) \sin(\alpha) - \cos(\alpha) - 3 \sin(\alpha) + 1$
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	$(\cos(\alpha))^2 - 2 \cos(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	0
$[\arctan(1/2), \pi/6]$	$2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) - 2 \sin(\alpha) + 1$	$-2(\sin(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	0
$[\pi/6, \arctan(3/4)]$	0	0	0
$[\arctan(3/4), \pi/4]$	0	0	$2(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) - 3 \cos(\alpha) + \sin(\alpha) + 1$
$\alpha$			
$[0, \arctan(\sqrt{2}/4)]$	0	0	0
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	$-3 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 3 \sin(\alpha) - 1$	0	0
$[\arctan(1/2), \pi/6]$	$-2(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + 3 \cos(\alpha) - \sin(\alpha) - 1$	$(\cos(\alpha))^2 - 2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) + 2 \sin(\alpha)$	0
$[\pi/6, \arctan(3/4)]$	$-2(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + 3 \cos(\alpha) - \sin(\alpha) - 1$	$(\cos(\alpha))^2 - 2 \cos(\alpha) + 1$	$-2(\sin(\alpha))^2 + 2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) + \sin(\alpha)$
$[\arctan(3/4), \pi/4]$	0	$-(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$-2(\sin(\alpha))^2 + 2 \cos(\alpha) \sin(\alpha) - \cos(\alpha) + \sin(\alpha)$
$\alpha$			
$[0, \arctan(\sqrt{2}/4)]$	0	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$	0
$[\arctan(\sqrt{2}/4), \arctan(1/2)]$	0	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$	0
$[\arctan(1/2), \pi/6]$	0	$-(\cos(\alpha))^2 - \cos(\alpha) \sin(\alpha) + 2 \cos(\alpha) + \sin(\alpha) - 1$	0
$[\pi/6, \arctan(3/4)]$	$-2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 2 \sin(\alpha) - 1$	$-(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$4(\sin(\alpha))^2 - 4 \sin(\alpha) + 1$
$[\arctan(3/4), \pi/4]$	$-2 \cos(\alpha) \sin(\alpha) + \cos(\alpha) + 2 \sin(\alpha) - 1$	$-(\cos(\alpha))^2 + \cos(\alpha) \sin(\alpha) + \cos(\alpha) - \sin(\alpha)$	$4(\sin(\alpha))^2 - 4 \sin(\alpha) + 1$

Figure 7. Table describing  $\eta_{C'_\alpha}(p)$  for each symbol  $p$  that appears in  $C'_\alpha$ , with respect to the value of  $\alpha$ .

## References

- [AI01] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001.
- [And92] E. Andrès. *Discrete circles and discrete rotations*. PhD thesis, Université Louis Pasteur, Strasbourg, France, 1992.
- [AR91] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité  $2n + 1$ . *Bull. Soc. Math. France*, 119(2):199–215, 1991.
- [BEIR] V. Berthé, H. Ei, S. Ito, and H. Rao. Invertible substitutions and sturmian words: an application of rauzy fractals. *Theoret. Informatics Appl.*, to appear.
- [BFJP] V. Berthé, C. Fiorio, D. Jamet, and F. Philippe. On some applications of generalized functionality for arithmetic discrete plane. To appear.
- [BFZ05] V. Berthé, S. Ferenczi, and L.Q. Zamboni. Interactions between dynamics, arithmetics, and combinatorics: the good, the bad, and the ugly. In S. Kolyada, Y. Manin, and T. Ward, editors, *Algebraic and Topological Dynamics*, volume 385 of *Contemporary mathematics*, pages 333–364. American Mathematical Society, 2005.

- [BN05] V. Berthé and B. Nouvel. Density of symbols in discretized rotation configurations. In S. Brlek and Reutenauer, editors, *Words 2005, 5th International Conference on Words*, volume 36 of *Publications du LACIM*, pages 163–173, 2005.
- [BV00] V. Berthé and L. Vuillon. Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences. *Discrete Math.*, 223:27–53, 2000.
- [Fog02] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
- [Lot02] M. Lothaire. *Algebraic combinatorics on words*. Cambridge University Press, 2002.
- [Lot05] M. Lothaire. *Applied combinatorics on words*. Cambridge University Press, 2005.
- [Nou] B. Nouvel. Substitutions, low-complexity configurations and Rauzy graphs. Preprint 2006.
- [NR03] B. Nouvel and E. Rémila. On colorations induced by discrete rotations. In *DGCI*, number 2886 in LNCS, pages 174–183, 2003.
- [NR04] B. Nouvel and E. Rémila. Characterization of bijective discretized rotations. In *International Workshop on Combinatorial Images Analysis, 10th International Conference, IW CIA 2004, Auckland, New Zealand, December 1-4, 2004*, number 3322 in LNCS, pages 248–259, 2004.
- [NR05] B. Nouvel and E. Rémila. Configurations induced by discrete rotations: Periodicity and quasiperiodicity properties. *Discrete Applied Mathematics*, 2-3(147):325–343, 2005.
- [Rau82] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.
- [Rev91] J.-P. Reveillès. *Calcul en nombres entiers et algorithmique*. Thèse d'état, Université Louis Pasteur, Strasbourg, France, 1991.
- [Sla67] N. B. Slater. Gaps and steps for the sequence  $n\theta \bmod 1$ . *Proc. Cambridge Philos. Soc.*, 63:1115–1123, 1967.
- [SW] C. Steiner and R. Winkler. Complexity of hartman sequences. J. Théor. Nombres Bordeaux, to appear.
- [Van00] D. Vandeth. Sturmian words and words with a critical exponent. *Theoret. Comput. Science*, 242:283–300, 2000.