# On an Involution of Christoffel Words and Sturmian Morphisms

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# On an involution of Christoffel words and Sturmian morphisms

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**Abstract**

There is a natural involution on Christoffel words, originally studied by the second author in [dL2]. We show that it has several equivalent definitions: one of them uses the slope of the word, and changes the numerator and the denominator respectively in their inverses modulo the length; another one uses the cyclic graph allowing the construction of the word, by interpreting it in two ways (one as a permutation and its ascents and descents, coded by the two letters of the word, the other in the setting of the Fine and Wilf periodicity theorem); a third one uses central words and generation through iterated palindromic closure, by reversing the directive word. We show further that this involution extends to Sturmian morphisms, in the sense that it preserves conjugacy classes of these morphisms, which are in bijection with Christoffel words. The involution on morphisms is the restriction of some conjugation of the automorphisms of the free group. Finally, we show that, through the geometrical interpretation of substitutions of Arnoux and Ito, our involution is the same thing as duality of endomorphisms (modulo some conjugation).

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# 1 Introduction

Christoffel words appear as a finitary version of Sturmian sequences. We follow here the original construction of Christoffel [C], who uses a cyclic graph. The latter has some similarity with a graph constructed by Choffrut and Karhumäki [CK], who give a graphical proof of the theorem of Fine and Wilf; their construction defines another Christoffel word, since the graph indicates the equality of letters of some central word (see also [CMR]), from where the Christoffel word is derived by adding an initial and a terminal letter. We call *dual* to each other the two Christoffel words constructed in this way. It appears that the slopes $p/q$ and $p^*/q^*$ of two dual Christoffel words $w$ and $w^*$ satisfy $p+q = p^*+q^*$ and that $p, p^*$ (resp. $q, q^*$) are inverse of each other modulo $p + q$.

Since each Christoffel word $w$, which is *proper* (that is, $\neq x, y$), is of the form $w = xuy$, the duality on Christoffel words extends to central words (which are the words $u$ obtained as above). The duality on central words is intrinsically described by $u = \mathrm{Pal}(v), u^* = \mathrm{Pal}(\tilde{v})$, where Pal is the *right iterated palindromic closure* of [dL1], and $\tilde{v}$ is the *reversal* of $v$.

Some consequences are derived. In particular, if $v$ is mapped onto $\binom{a\ b}{c\ d}$ by the monoid isomorphism which sends $x$ onto $\binom{1\ 1}{0\ 1}$ and $y$ onto $\binom{1\ 0}{1\ 1}$, then $u$ has the relatively prime periods $a+c$ and $b+d$ and the number of occurrences of the letters $x$ and $y$ in $w$ are respectively $a + b$ and $c + d$.

Another consequence is the description of the above duality on the Stern-Brocot tree; from there, we can characterize those couples of numbers whose paths in this tree are mirror each of another.

A striking fact is that the involution on Christoffel words just described extends to Sturmian morphisms. We restrict here to special Sturmian morphisms, that is, those whose incidence matrix has determinant 1. The work of Séébold [S] shows that conjugacy classes of special Sturmian morphisms are in a natural bijection with Christoffel words: to morphism $f$ is associated the unique Christoffel word conjugate to $f(xy)$. Then we define the dual morphism $f^*$ of $f$ by the composition $\omega f^{-1}\omega$, where $\omega$ is the involution on the free group generated by $x$ and $y$ which sends $x$ onto $x^{-1}$ and which fixes $y$. It turns out that $f^*$ is a special Sturmian morphism and that the involution on Christoffel words induced by the involution $f \mapsto f^*$ of Sturmian morphisms is the involution of Christoffel words described above. We give a very precise description of the mapping $f^*$, through the list of conjugate morphisms given by Séébold in [S].

There is a geometrical interpretation of Sturmian morphisms given by Arnoux-Ito [AI]. They associate to $f$ a linear operator $E(f)$ which acts on the free $\mathbb{Z}-$module spanned by the lattice segments in $\mathbb{Z}^2$. They consider also the adjoint operator $E(f)^*$. We show here that $E(f)^*$ may be precisely described, using $E(f^*)$; in particular the translation vector, whose existence is proved in [E] may be easily computed, using the conjugacy class of the Sturmian morphism $f^*$.

*Definitions and notations* Composition of functions will be denoted as a product, except in the last two sections, where the notation $\circ$ is sometimes used to avoid ambiguities. We use the self-evident notation $\{x < y\}$ to define a two-element totally ordered set. For all definitions and notations concerning words not explicitly given in the text, the reader is referred to the book of Lothaire [L]; for Sturmian words and morphims, see [BS2].

# 2 Dual of a Christoffel word

Following closely the construction of [C] (and not its variant given in [BL] and [BdL]), we define *Christoffel words.* Let $p$ and $q$ be positive relatively prime integers and $n = p+q$. Given an ordered two-letter alphabet $\{x < y\}$, the *Christoffel word* $w$ *of slope* $\frac{p}{q}$ on this alphabet is defined as $w = x_1 \cdots x_n$, with

$$x_i = \begin{cases} x & \text{if} \quad ip \mod n > (i-1)p \mod n \\ y & \text{if} \quad ip \mod n < (i-1)p \mod n \end{cases}$$

for $i = 1, \dots, n$, where $k \bmod n$ denotes the remainder of the Euclidean division of $k$ by $n$.

In other words, label the edges of the Cayley graph of $\mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$ with generator $p$ as follows: the label of the edge $h \to k$ (where $h + p \equiv k \mod n$) is $x$ if $h < k$ and $y$ if $h > k$. Then read the word $w$, of length $n$, starting with the label of the edge $0 \to p$. See Figure 1. We call this graph the *Cayley graph* of the Christoffel word $w$.

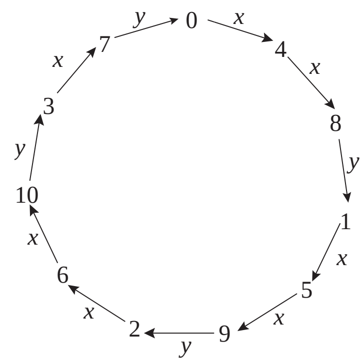

**Figure 1**
The Christoffel word $w = xxyxxyxxyxy$
of slope $\frac{4}{7}$, on the alphabet $\{x < y\}$, $p = 4$, $q = 7$, $n = 11$

For further use, and to understand the definition, note that if $(i-1)p \mod n = k$, then since $n = p + q$, either $ip \mod n = k + p$ and $x_i = x$, or $ip \mod n = k - q$ and $x_i = y$. Hence $x_i = x$ if and only if $(i-1)p \in \{0, 1, \ldots, q-1\}$, and $x_i = y$ if and only if $(i-1)p \in \{q, q+1, \ldots, n-1\}$. Moreover, note that $|w|_x = q$ and $|w|_y = p$, where for any word $m$ and letter $z$, $|m|_z$ denotes the number of occurrences of the letter $z$ in $m$. Thus, if we know the number of occurrences of each letter in $w$, we know its slope, hence $w$. Note that the terminology "slope" comes from the geometric interpretation of Christoffel words [B], [BL].

In addition, the words of length 1, $x$ and $y$, will also be called Christoffel words, of respective slope $\frac{0}{1}$ and $\frac{1}{0}$. The Christoffel words of slope $\frac{p}{q}$ with $p, q \neq 0$ will be called *proper Christoffel words.*

Given the proper Christoffel word of slope $\frac{p}{q}$, we define the *dual Christoffel word* $w^*$ of slope $\frac{p^*}{q^*}$, where $p^*$ and $q^*$ are the respective multiplicative inverses in $\{0, 1, \ldots, n-1\}$ of $p$ and $q$. Note that these inverses exist since $p$ and $q$ are relatively prime, hence are both relatively prime to $p + q = n$, and that $p^*$ and $q^*$ are relatively prime. Note also that a Christoffel word and its dual have the same length; indeed, since $p^*$ is the inverse of $p$ modulo $n$ and $p^* \in \{0, 1, \ldots, n-1\}$, the equality $p + q = n$ implies that $n - p^*$ is the inverse of $q$ and $n - p^* \in \{0, 1, \ldots, n-1\}$, hence $q^* = n - p^*$. For the Christoffel word of Figure 1, its dual is represented in Figure 2, through a graph represented linearly, for further purpose; here we have $p^* = 3, q^* = 8$.

$$0 \xrightarrow{x} 3 \xrightarrow{x} 6 \xrightarrow{x} 9 \xrightarrow{y} 1 \xrightarrow{x} 4 \xrightarrow{x} 7 \xrightarrow{x} 10 \xrightarrow{y} 2 \xrightarrow{x} 5 \xrightarrow{x} 8 \xrightarrow{y} 0$$

**Figure 2**

The dual Christoffel word
of slope $\frac{3}{8}$ on the alphabet $\{x < y\},\ p^* = 3,\ q^* = 8,\ n = 11$

Note that the dual of a Christoffel word $w$ is well-defined: indeed, as said previously, $p$ is the number of $y$'s in $w$, and $q$ is the number of $x$'s in $w$. For completeness, we let $x^* = x$ and $y^* = y$.

Note also that, according to [BL] (see also [BdL]), each Christoffel word $w$ is a *Lyndon word*; as such, if it is proper, it has a *standard factorization* $w = w_1 w_2$, where $w_1, w_2$ are also Christoffel words and $w_1 < w_2$ in lexicographic order [BL].

**Lemma 2.1** *Let $w$ be a proper Christoffel word of slope $\frac{p}{q}$ and $w = w_1 w_2$ its factorization in an increasing product of two Christoffel words. Then $|w_1| = p^*$ and $|w_2| = q^*$. Moreover, $w_1$ (resp. $w_2$) is the label of the path from 0 to 1 (resp. 1 to 0) in the previous Cayley graph.*

For example, let $w = xxy\,xxy\,xxy\,xy$ as in Figure 1. Then $w_1 = xxy$ and $w_2 = xxy\,xxy\,xy$; they are the labels of the paths from 0 to 1 and from 1 to 0 in the graph of Figure 1.

**Proof** We follow a geometrical argument of [BR], Appendix, Part c. According to [BL], the word $w$ corresponds to the path which discretizes from below the segment from $(0,0)$ to $(q,p)$; moreover, the factorization $w = w_1 w_2$ corresponds to the subpaths from $(0,0)$ to $(b,a)$ and from $(b,a)$ to $(q,p)$, where $(b,a)$ is the lattice point on the path closest to the given segment.

Hence $|w_1| = a + b$ and $|w_2| = p + q - a - b$. Now, by definition of the discretization, the triangle constructed on the points $(0,0), (b,a), (q,p)$ has no inner lattice point. Hence the determinant $\begin{vmatrix} b & a \\ q & p \end{vmatrix}$ is equal to 1, from which follows that $bp - aq = 1$; thus $p(a+b) = pa + pb = pa + 1 + aq = 1 + a(p+q) \equiv 1 \mod p+q$. Hence $|w_1| = a + b = p^*$ and likewise $|w_2| = q^*$.

Regarding the Cayley graph, $w$ is by definition the label from 0 to 0. Hence $w_1$ is the label from 0 to $j$, where $j$ is at distance $p^*$ from 0; now, since the labels of the vertices increase of $p$ (modulo $n$) after each edge, we must have $pp^* \equiv j \mod n$. Since $pp^* \equiv 1 \mod n$ it follows $j \equiv 1 \mod n$ so that $j = 1$. Thus $w_1$ is the label of the path from 0 to 1, and similarly for $w_2$. ■

If $w$ is a proper Christoffel word on the alphabet $\{x < y\}$, then $w = xuy$, where $u$ is a palindrome, as observed by Christoffel [C] p. 149: *"ideoque pars* *principalis periodi semper est symmetrica"*. This is easily seen on the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$: indeed, the mapping of the graph sending vertex 0 onto itself, and $k \mapsto n-k$ if $k \in \{1, \ldots, n\}$ and which reverts the edges, fixes the graph, except the labelling of the two edges involving vertex 0; indeed, if $i, j \neq 0$ and $i \xrightarrow{x} j$, hence $i < j$, we have $n-i \xleftarrow{x} n-j$, and similarly for label $y$.

The words $u$ such that $xuy$ are Christoffel words (necessarily proper) on the alphabet $\{x < y\}$ are called *central words*. For further use, we recall the definition of a *standard word*: it is a word on the alphabet $\{x, y\}$ which is either a letter or of the form $uxy$ or $uyx$, where $u$ is a central word. Clearly, conjugacy classes of Christoffel words and those of standard words coincide; recall that two words are *conjugate* if for suitable words $u$ and $v$, one is of the form $uv$ and the other of the form $vu$; this defines an equivalence relation on words whose classes are called *conjugacy classes*.

The central words on the alphabet $\{x, y\}$ have been completely characterized by de Luca and Mignosi [LM]: they are the words which for some relatively prime positive integers $p$ and $q$ are of length $p+q-2$ and have periods $p$ and $q$. Recall that a word $u = y_1 \cdots y_m$, with $y_i \in \{x, y\}, 1 \leq i \leq m$, has the *period* $p$ if $p > 0$ and if whenever $j = i+p$ and $i, j \in \{1, \ldots, m\}$, one has $y_i = y_j$. Note that the set of central words is equal to the set of palindromic prefixes of standard Sturmian sequences, see e.g. [BS2] Corollary 2.2.29.

**Proposition 2.1** *Let $w = xuy$ be the Christoffel word of slope $\frac{p}{q}$ on the alphabet $\{x < y\}$. Then the central word $u$ has the periods $p^*$ and $q^*$ where $pp^*, qq^* \equiv 1 \mod p+q$.*

This result is from [LM]. We give a proof since it is very short. Moreover, this result is interesting for the following consequence.

**Corollary 2.1** *The dual word $w^* = xu^*y$ of a Christoffel word $w = xuy$ of slope $\frac{p}{q}$ on the alphabet $\{x < y\}$ is completely defined by the following condition: $u^*$ has length $p+q-2$, the couple of periods $(p, q)$, and begins by $x$ iff $p^* < q^*$, where $pp^*,\ qq^* \equiv 1 \mod (p+q)$.*

The corollary has the interesting application that the dual of a Christoffel word may be read on the same graph which defines the latter word. For example, take the graph of Figure 2, and remove the vertices 0 and $n-1 = 10$, the labels and the orientation. Then one obtains the graph of Figure 3, reminiscent of the proof of the theorem of Fine and Wilf as given in [CK], see also [CMR].

$$3 \text{ — } 6 \text{ — } 9 \text{ — } 1 \text{ — } 4 \text{ — } 7 \qquad 2 \text{ — } 5 \text{ — } 8$$

**Figure 3**
Equality of positions of a
word of length 9 with periods 3 and 8

This graph expresses the equality of letters according to their positions in a word of length 9 with periods 3 and 8. Hence, by the corollary, the central word of the dual of the Christoffel word of Figure 2 is $xyxxyxxyx$ (since the $x$'$s$ are in positions $3, 6, 9, 1, 4, 7$ and the $y$'$s$ in positions $2, 5, 8$). We thus recover the Christoffel word $xxyxxyxxyxy$ of Figure 1.

Since $u \mapsto xuy$ is a bijection between central words and proper Christoffel words, we see that the involution on Christoffel words $w \mapsto w^*$ induces an involution $u \mapsto u^*$ on central words; we also call $u^*$ the *dual* of $u$. We see that we can define the dual of a central word $u$ directly, either by the corollary, or by taking the dual of the Christoffel word $xuy$ and removing the extreme letters.

The corollary shows that the involution studied here is the involution $\eta^{-1}\zeta = \zeta^{-1}\eta$ of [dL2]; moreover, it is a variant of the involution studied in [CdL2].

**Proof of the proposition** Take the notations of the beginning of the section. It is enough to show that $u$ has the period $p^*$. Indeed, the central word $u'$ of the Christoffel word $w' = xu'y$ of slope $\frac{q}{p}$ is obtained from $u$ by exchanging $x$ and $y$, since the Cayley graph of $w'$ is obtained by reversing the orientation and exchanging $x$ and $y$ in the Cayley graph of $w$.

We have $u = x_2 \cdots x_{n-1}$. It will be enough to show that for $i, j$ in $\{2, \ldots, n-1\}$ and $j = i + p^*$: $x_i = x \Leftrightarrow x_j = x$. Observe that since $pp^* \equiv 1$ (here and below $\equiv$ will be modulo $n$), we have $jp \equiv ip + 1$ and $(j-1)p \equiv (i-1)p + 1$.

Suppose that $x_i = x$. Then $ip \mod n > (i-1)p \mod n$. We have not $ip \mod n = n-1$, otherwise $i = q^*$ (since $pq^* \equiv -1$) and then we would have $j = i + p^* = p^* + q^* = n$, which is excluded. Thus $(ip \mod n) + 1 = (ip+1) \mod n$. Similarly, $(i-1)p \mod n \neq n-1$, otherwise $i - 1 = q^* \Rightarrow j = i + p^* = n + 1$, which is excluded also. Thus $((i-1)p \mod n) + 1 = ((i-1)p + 1) \mod n$. Finally, $jp \mod n = (ip \mod n) + 1 > ((i-1)p \mod n) + 1 = (j-1)p \mod n$, hence $x_j = x$.

Conversely, suppose that $x_j = x$. Then $jp \mod n > (j-1)p \mod n$. Since $j$ and $j-1$ are $\neq n$, we have $jp \mod n$ and $(j-1)p \mod n$ are $\neq 0$. Hence $(jp \mod n) - 1 = (jp-1) \mod n$ and $((j-1)p \mod n) - 1 =$

$((j-1)p-1) \mod n$, and we conclude that $ip \mod n = (jp \mod n) - 1 > ((j-1)p \mod n) - 1 = (i-1)p \mod n$, thus $x_i = x$. ∎

**Remark 2.1** The proof shows that not only $x_2 \cdots x_{n-1}$, but also $x_1 \cdots x_{n-1}$ has the period $p^*$ (since only the condition $i, j \in \{1, \ldots, n-1\}$ has been used); symmetrically, $x_2 \cdots x_n$ has the period $q^*$. Compare [CH] Lemma 5.01.

**Proof of the corollary** This follows because $u^*$ is a word of length $p+q-2$ on the alphabet $\{x, y\}$, having the periods $p$ and $q$; then $u^*$ is completely defined up to exchange of letters, by the theorem of Fine and Wilf [L] (it follows also from a graph as in Figure 3). If we know the first letter of $u^*$, it is therefore completely defined.

It is enough to show that: $u$ begins by $x$ if and only if $p < q$. Now the first letter of $u$ is the second letter of $w$; therefore, it is $x$ if and only if: $2p \mod n < p \mod n$, that is, if and only if $p < q$, by the remark after the definition of the Cayley graph.

# 3 Palindromic closure

As said before, the involution on Christoffel words of the previous section induces an involution on central words.

This involution on central words is completely described by Cor. 2.1. We give now another characterization of this involution. Following [dL1], we define the *right palindromic closure* $w^+$ of a word $w$ as the unique shortest palindrome having $w$ as a prefix. This word exists and is equal to $uv\tilde{u}$, where $w = uv$, $\tilde{u}$ is the mirror image of $u$, and $v$ is the longest palindromic suffix of $w$, see [dL1] Lemma 5. For example, $(xyxxyxx)^+ = xyxxyxx.yx$. The *right iterated palindromic closure* of $w$ is denoted by $\mathrm{Pal}(w)$ and is defined recursively by $\mathrm{Pal}(w) = (\mathrm{Pal}(u)z)^+$, where $w = uz, z$ the last letter of $w$, together with the initial condition $\mathrm{Pal}(1) = 1$ (the empty word), see [dL1] (we use the notation of [J]). Then it is shown in [dL1] Proposition 8, that *the set* $\{\mathrm{Pal}(v), v \in \{x, y\}^*\}$ *coincides with the set of central words.* It is easily verified that if $w = \mathrm{Pal}(v)$, then $v$ is uniquely defined by $w$; $v$ is called the *directive word* of $w$.

Now, we can of course define also the left palindromic closure and the left iterated palindromic closure of $w$. The latter is simply $\mathrm{Pal}(\tilde{w})$, as is easily verified. We can now characterize duality through palindromic closure.

**Proposition 3.1** *Pal*$(\tilde{v})$ *is the dual central word of the central word Pal*$(v)$*.*

Note that obviously, the proposition implies that $|\mathrm{Pal}(\tilde{v})| = |\mathrm{Pal}(v)|$.

**Proof** This result could be proved by using the methods of continued fractions of [CdL2]. We follow another way, inspired by [J].

1. Define the endomorphism $\mu_x : \{x,y\}^* \to \{x,y\}^*$ by $\mu_x(x) = x,\ \mu_x(y) = xy$. Likewise define $\mu_y$ by exchanging $x$ and $y$. Now, for a word $w = x_1 \cdots x_n$ written as a product of letters, let $\mu_w = \mu_{x_1} \cdots \mu_{x_n}$. Then one has the formula $\mathrm{Pal}(zw) = \mu_z(\mathrm{Pal}(w))z$, for any letter $z$ and any word $w$, see [J] Lemma 2.1.

2. We define $M_w$ to be the *incidence matrix* of $\mu_w$, that is, the 2 by 2 matrix with rows and columns indexed by $\{x < y\}$, defined by $(M_w)_{u,v} = |\mu_w(v)|_u$. Note that $M_{w_1 w_2} = M_{w_1} M_{w_2}$, since $\mu_{w_1 w_2} = \mu_{w_1}\mu_{w_2}$.

Likewise $V(w)$ denotes the column vector $\binom{|w|_x}{|w|_y}$. It is well-known that $M_w V(u) = V(\mu_w(u))$ for all words $u$. We show by induction on the length of $w$ that $V(\mathrm{Pal}(w)) = M_w\binom{1}{1} - \binom{1}{1}$.

This is true if $w$ is the empty word. If $w = x$, it reduces to $\binom{1}{0} = \begin{pmatrix}1&1\\0&1\end{pmatrix}\binom{1}{1} - \binom{1}{1}$, which is true; likewise if $w = y$. Now, let $z$ be a letter. Then by 1. and induction $V(\mathrm{Pal}(zw)) = V(\mu_z(\mathrm{Pal}(w))z) = V(\mu_z(\mathrm{Pal}(w))) + V(z) = M_z V(\mathrm{Pal}(w)) + V(z) = M_z\left(M_w\binom{1}{1} - \binom{1}{1}\right) + V(z) = M_{zw}\binom{1}{1} - \binom{1}{1}$, the last equality following from the case of length 1.

3. Let $u = \mathrm{Pal}(v), u' = \mathrm{Pal}(\tilde{v})$ and $M_v = \begin{pmatrix}a&b\\c&d\end{pmatrix}, M_{\tilde{v}} = \begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}$. Then by 2., $|u|_x = a+b-1,\ |u|_y = c+d-1,\ |u'|_x = a'+b'-1,\ |u'|_y = c'+d'-1$. In order to prove that $u'$ is the dual of $u$, it is enough, in view of Section 2, to show that $a+b+c+d = a'+b'+c'+d'(=n)$ and that $a+b$ (resp. $c+d$) is the inverse of $a'+b'$ (resp. $c'+d'$) modulo $n$.

4. Note that $S : \begin{pmatrix}p&q\\r&s\end{pmatrix} \mapsto \begin{pmatrix}s&q\\r&p\end{pmatrix}$ is an anti-automorphism of $SL_2(\mathbb{Z})$, since it can be obtained by composing in $GL_2(\mathbb{Z})$ transposition with conjugation by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. It sends $M_x$ and $M_y$ onto themselves, hence $S(M_v) = M_{\tilde{v}}$. This shows that $\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix} = \begin{pmatrix}d&b\\c&a\end{pmatrix}$. Thus $a'+b' = d+b$ and $c'+d' = c+a$.

Now, we have $ad-bc = 1$, hence $(a+b)(d+b) = ad+ab+bd+b^2 = 1+bc+ab+bd+b^2 = 1+b(a+b+c+d)$ and likewise $(c+d)(c+a) = 1+c(a+b+c+d)$, which ends the proof. ■

Let us remark that the Christoffel word $x\mathrm{Pal}(v)y$ is *self-dual*, that is, a fixpoint of the involution, if and only if the directive word $v$ of $\mathrm{Pal}(v)$ is a palindrome. This implies that $\mathrm{Pal}(v)$ is *harmonic*, see [CdL1]. From the results of Section 2 one derives that a Christoffel word of slope $p/q$ is self-dual if and only if $p^2 \equiv 1 \mod (p+q)$.

The following result is not new [dL1]; we obtain it here as a consequence of Section 2 and of the previous proof.

**Corollary 3.1** *If $u = Pal(v)$ and $M_v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $|u|_x = a + b - 1$, $|u|_y = c + d - 1$ and $u$ has the relatively prime periods $a + c$ and $b + d$. Moreover the Christoffel word $w = xuy$ has slope $\frac{c+d}{a+b}$ and its dual has slope $\frac{c+a}{d+b}$.*

**Corollary 3.2** *Let matrix $M$ be the image of $v \in \{x, y\}^*$ under the multiplicative morphism $\mu : x \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), y \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Let $u = Pal(v)$ and let $w$ be the Christoffel word $xuy$. Let $w = w_1 w_2$ be its decomposition into two Christoffel words with $w_1 < w_2$ in lexicographic order. Then*

$$M = M_v = \begin{pmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{pmatrix}.$$

**Proof** It is easily seen that $M = M_v$, since $M_x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $M_y = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then by Cor. 3.1, we have $|w|_x = a + b, |w|_y = c + d$. Let $|w_1|_x = a', |w_1|_y = c', |w_2|_x = b', |w_2|_y = d'$. Then the parallelogram built on $(a', c'), (b', d')$ is by [BL] positively oriented and contains no integer points. Hence $a'd' - b'c' = 1$; moreover $a + b = |w|_x = a' + b', c + d = |w|_y = c' + d'$.

Now, the following result is stated in [R] and proved in [BdL] (Prop. 6.2): if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ are matrices over $\mathbb{N}$ of determinant 1 with $a+b = a'+b', c+d = c' + d'$, then they are equal. This proves what we want. ∎

To illustrate Cor. 3.1 and 3.2, consider the Christoffel word of Figure 1. It has the factorization $w_1 \cdot w_2 = xxy \cdot xxy\, xxyxy$, where $w_1$ and $w_2$ are Christoffel words with $w_1 < w_2$ in lexicographical order. The associated central word is $u = xy\, xxy\, xxy\, x = \mathrm{Pal}(xyxx)$. We have $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 5 \\ 1 & 3 \end{smallmatrix}\right)$, which is indeed equal to $\left(\begin{smallmatrix} |w_1|_x & |w_2|_x \\ |w_1|_y & |w_2|_y \end{smallmatrix}\right)$.

Recall that positive rational numbers (more precisely: irreducible fractions) are in bijection with the nodes of the Stern-Brocot tree, see for instance [GKP] p. 117.

**Corollary 3.3** *Let $p/q$ and $p'/q'$ be two positive rational numbers, in irreducible form. Then the paths in the Stern-Brocot tree defining the corresponding nodes are mirror each of another if and only if $p + q = p' + q'$ and $pp', qq' \equiv 1 \mod p + q$.*

**Proof** Let $x$ (resp. $y$) replace "left" and "right" in the description of the paths in the Stern-Brocot tree, as it stands in [GKP] p. 119. Let $\mu$ be the homomorphism of Corollary 3.2. Then the rational number $r$ corresponding to some path $v$ in the tree is $\frac{c+d}{a+b}$, by [GKP] Eq. (4.39) p. 121, where $\mu(v) = \binom{a\ b}{c\ d}$.

Hence $r$ is the slope of the Christoffel word $w = x\,\mathrm{Pal}(v)y$, by Corollary 3.1. Write $r = p/q$ in reduced form. Then the slope of the dual Christoffel word $w^*$ is $p^*/q^*$, and we have by definition of the dual: $p + q = p^* + q^*$ and $pp^*$, $qq^* \equiv 1 \mod p+q$. By Proposition 3.1, $w^* = x\,\mathrm{Pal}(\tilde{v})y$, so that the rational number $r^* = p^*/q^*$ corresponds in the Stern-Brocot tree to the path $\tilde{v}$.

Since $v \mapsto r$ is a bijection from the words on $\{x, y\}^*$ onto the positive rational numbers, the corollary follows. ∎

# 4 Sturmian morphisms

A Sturmian morphism is an endomorphism of the free monoid $\{x, y\}^*$ that sends each Sturmian sequence onto a Sturmian sequence.

**Lemma 4.1** *A morphism $f$ is Sturmian if and only if it sends each Christoffel word onto the conjugate of a Christoffel word.*

In the Appendix we shall sharpen the "if part" of this lemma.

**Proof** The "only if" part follows e.g. from the theory of conjugacy classes of Sturmian morphisms, see [BS2] Section 2.3.4, and from the fact that standard words and Christoffel words are conjugate.

Conversely, let $f$ send Christoffel words onto words which are conjugate of Christoffel words. Since morphisms preserve conjugation, we may equivalently say that $f$ sends standard words onto conjugate of standard words. We use four facts: factors of Sturmian sequences and of standard words coincide, a consequence of [BS2] Prop. 2.1.18 and Prop. 2.2.24; any power of a standard word is a factor of a standard word, by the construction of standard pairs and their tree [BS2] p. 64; the conjugate of a word is a factor of the square of this word; balanced words and factors of Sturmian sequences coincide, see e.g. Prop. 2.1.17 in [BS2].

Hence if $v$ is a factor of a Sturmian sequence, it is a factor of some standard word; hence $f(v)$ is by assumption a factor of some conjugate of a standard word, hence of the square of this word, and therefore factor of some standard word, and finally of some Sturmian sequence. We deduce that $f$ sends factors of Sturmian sequences into factors of Stumian sequences. Now $f$ is *acyclic*, that is, $f(x)$ and $f(y)$ are not power of the same word; otherwise $f(xy)$ is not conjugate to a Christoffel word, since such a word is *primitive* (the number of $x$ and $y$ in it being coprime). Note that the word $w_{1,1} = yxxyxxyxyxxyxy$ of [BS1] (see also Exercise 2.3.1 in [BS2]) is *balanced*, as one easily verifies (this is also consequence of a more general result proved in [BS1] p.179). Thus $f(w_{1,1})$ is balanced. This shows by Proposition 4.1 of [BS1] that $f$ is Sturmian. ■

The monoid of Sturmian morphisms is called the *Sturmian monoid* and is denoted by $St$. Each Sturmian morphism $f$ has a *commutative image* (or *incidence matrix*), which is the 2 by 2 matrix $(|f(v)|_u)_{u,v=x,y}$. This matrix has nonnegative coefficients. We shall call *determinant* of a Sturmian morphism the determinant of its incidence matrix. Let $St_0$ denote the *special Sturmian monoid*, which is the submonoid of $St$ of endomorphisms whose determinant is 1. The monoid $St_0$ is generated by the endomorphisms $G, D, \tilde{G}, \tilde{D}$ which are respectively: $G = (x, xy)$, $D = (yx, y)$, $\tilde{G} = (x, yx)$, $\tilde{D} = (xy, y)$, where $f = (u, v)$ means that $f(x) = u$ and $f(y) = v$ (see [KR], Prop.2.1).

It is a consequence of the work of Mignosi-Séébold [MS] and Wen-Wen [WW] that Sturmian morphisms coincide with the positive automorphisms of the free group $F_2$ generated by $x$ and $y$ (an automorphism $f \in F_2$ is *positive* if $f(x),\ f(y) \in \{x, y\}^*$). Hence, the incidence matrix of a Sturmian morphism is in $GL_2(\mathbb{Z})$; in particular, it has determinant 1 or -1, as it occurs for the incidence matrix of any automorphism of $F_2$; note that the mapping $\mathrm{Aut}(F_2) \to GL_2(\mathbb{Z})$ obtained in this way is a group morphism.

Following Séébold [S] (see also [BS2]), we say that given two Sturmian morphisms $f$ and $f'$, $f'$ is a *right conjugate* of $f$ if for some word $w \in \{x, y\}^*$, one has $uw = wu'$, $vw = wv'$, where $f = (u, v)$ and $f' = (u', v')$. Then, we say that $f$ and $f'$ are *conjugate* if one of them is a right conjugate of the other. One shows that $f$ and $f'$ are conjugate if and only if, within $\mathrm{Aut}(F_2)$, $f = \varphi f'$ for some inner automorphism $\varphi$ of $F_2$. Recall the following theorem of Nielsen [N]: given two automorphisms $f$ and $f'$ of the free group, they have the same commutative image, if and only if $f = \varphi f'$ for some inner automorphism of $F_2$. We conclude that, for two Sturmian morphisms $f$ and $f'$, they are conjugate if and only if they have the same commutative image.

Now, take a Sturmian morphism $f \in St_0$ (hence of determinant 1). Then, since $xy$ is a Christoffel word, $f(xy)$ is conjugate to a Christoffel word $w$, necessarily proper (note that this is true also if $f$ is not special, but we shall use this construction only for special Sturmian morphisms). We say that $w$ is the *Christoffel word associated* to $f$. If $f$ and $f'$ are conjugate Sturmian morphisms, then $f(xy)$ and $f'(xy)$ are conjugate words, so that $f$ and $f'$ have the same associated Christoffel word (recall that a Christoffel word is the smallest element of its conjugacy class, for lexicographical order, with $x < y$, see [BL]). Conversely, suppose that $f$ and $f' \in St_0$ have the same

Christoffel word; then $f(xy)$ and $f'(xy)$ are conjugate. Let $\binom{a\ b}{c\ d}$, $\binom{a'\ b'}{c'\ d'}$ be the matrices associated to $f, f'$. Then $a+b = |f(xy)|_x = |f'(xy)|_x = a'+b'$, and similarly, $c+d = c'+d'$. Thus, by a result stated in the proof of Cor. 3.2, these matrices are equal and, by Nielsen's theorem, $f$ and $f'$ are conjugate.

Thus we see that $f \mapsto f(xy)$ induces a bijection between conjugacy classes of Sturmian morphisms of determinant 1, and conjugacy classes of Christoffel words.

Denote by $(x^{-1}, y)$ the automorphism of the free group $F_2$ sending $x$ onto $x^{-1}$ and $y$ onto $y$.

**Proposition 4.1** *The mapping $f \mapsto f^* = (x^{-1}, y)f^{-1}(x^{-1}, y)$ is an involutive anti-automorphism of the special Sturmian monoid, that exchanges $D$ and $\tilde{D}$ and fixes $G$ and $\tilde{G}$. It sends conjugacy classes of morphisms onto conjugacy classes. The involution on Christoffel words that it induces is the same as the one of Section 2.*

**Proof** It is clearly an involutive anti-automorphism, once we have verified that it sends $St_0$ into $St_0$. For this, it suffices to show that it sends the set $\{G, \tilde{G}, D, \tilde{D}\}$ onto itself. We have indeed $D^* = (x^{-1}, y)D^{-1}(x^{-1}, y) = (x^{-1}, y)(y^{-1}x, y)(x^{-1}, y) = (xy, y) = \tilde{D}$; $\tilde{D}^* = D$; $G^* = G$; $\tilde{G}^* = \tilde{G}$.

If $f$ and $f'$ are conjugate Sturmian morphisms, then $f' = \varphi f$ for some inner automorphism $\varphi$ of $F_2$. Then $f'^* = f^* \varphi^*$. Note that

$$\varphi^* = (x^{-1}, y)\varphi^{-1}(x^{-1}, y)$$

so that $\varphi^*$ is an inner automorphism of $F_2$, since the inner automorphisms form a normal subgroup of $\mathrm{Aut}(F_2)$. We have $f'^* = \varphi' f^*$, with $\varphi' = f^* \varphi^* f^{*-1}$, which shows that $\varphi'$ is an inner automorphism. Hence $f'^*$ and $f^*$ are conjugate Sturmian morphisms, since they have the same commutative image.

Now, let $\binom{a\ b}{c\ d}$ be the commutative image of $f$. Then the commutative image of $f^* = (x^{-1}, y)f^{-1}(x^{-1}, y)$ is $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-d & b\\ -c & a\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}d & b\\ c & a\end{pmatrix}$. Let $w$ and $w^*$ be the Christoffel words associated to $f$ and $f^*$, respectively. One has: $|w|_y = |f(xy)|_y = c+d$ and $|w|_x = |f(xy)|_x = a+b$. Likewise, $|w^*|_y = |f^*(xy)|_y = a+c$ and $|w|_x = |f^*(xy)|_x = b+d$. Since the determinants of $f$ and $f^*$ are equal to 1, it follows from Cor. 3.1, that $w^*$ is just the dual of $w$. ■

There is a very precise description of the conjugacy class of a Sturmian morphism by Séébold [S], see also Section 2.3.4 of [BS2]. We recall it now and show how it allows one to describe precisely the involution $*$.

Given the proper Christoffel word $w$ on the alphabet $\{x < y\}$, $w = xuy = x_1 \cdots x_n$, its associated conjugacy class of special Sturmian morphisms has $n - 1 = |w| - 1$ elements $f_1, f_2, \ldots, f_{n-1}$. Among them, $f_1$ is a *standard Sturmian morphism* (that is, a morphism $(u, v)$ such that this pair or its symmetric pair is a standard pair, see [BS2]), and it is the only one. One has $f_i = (u_i, v_i)$ and for $i = 1, \ldots, n-2$, $u_{i+1} = x_{i+1}^{-1} u_i x_{i+1}$, $v_{i+1} = x_{i+1}^{-1} v_i x_{i+1}$, where the central word $u$ is $x_2 \cdots x_{n-1}$. Moreover, $f_i$ is completely defined by $u_i v_i = f_i(xy)$. For example, if $w = xxyxyxy$, then $u = xyxyx$, $n = 7$, and:

- $f_1 = (xyxyx,\ xy)$;
- $f_2 = (yxyxx,\ yx)$;
- $f_3 = (xyxxy,\ xy)$;
- $f_4 = (yxxyx,\ yx)$;
- $f_5 = (xxyxy,\ xy)$;
- $f_6 = (xyxyx,\ yx)$.

Note that $f_5(xy) = w$. Moreover, the first letters of $u_1, \ldots, u_5$ (resp. $v_1, \ldots, v_5$) form the central word $u$.

Define the *conjugation operator* $\gamma$ on words by $\gamma(zm) = mz$, for any letter $z$ and word $m$. Then $f_{i+1}(xy) = \gamma f_i(xy)$, and for $i = 1, \ldots, n-1$, the words $f_i(xy)$ exhaust the conjugacy class of $w$, except one word (which in the example is $yxyxyxx$), cf. [BS2], p.95. Note that one has also $f_{i+1}(x) = \gamma f_i(x)$ and $f_{i+1}(y) = \gamma f_i(y)$.

For further use, we need the following lemma.

**Lemma 4.2** *Let $w$ be a proper Christoffel word of length $n$ and $\{f_1, \ldots, f_{n-1}\}$ its associated conjugacy class of Sturmian morphisms, in the previous description. Let $w = w_1 w_2$, where $w_1, w_2$ are Christoffel words with $w_1 < w_2$ in lexicographical order. Then $(w_1, w_2)$ is a special Sturmian morphism, equal to $f_{p^*}$, where $p/q$ is the slope of $w$ and $pp^* \equiv 1 \mod n$.*

**Proof** The fact that $(w_1, w_2)$ is a special Sturmian morphism follows from [BL], and also from the Christoffel tree of [BdL]. Thus $(w_1, w_2)$ is of the form $f_i$ for suitable $i$, $0 \leq i \leq n-1$. We know by Lemma 2.1 that $|w_1| = p^*$.

To each $j \in \{0, 1, \ldots, n-1\}$ associate the word $s_j$ (resp. $t_j$) which is the label of the simple path from $j$ to $j + 1 \mod n$ (resp. $j + 1 \mod n$ to $j$) in the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generator $p$ (see Section 2). We know that $(w_1, w_2) = (s_0, t_0)$ by Lemma 2.1. Since $s_j t_j = \gamma^r(s_0 t_0)$ for suitable $r$, each $(s_j, t_j)$ is a Sturmian morphism, except one of them, by the remark before the lemma.

We claim that the exceptional couple is $(s_{n-1}, t_{n-1})$.

Taking this claim for granted, we have, on one hand, $\gamma(s_j t_j) = s_k t_k$ with $k \equiv j + p \mod n$; on the other hand $\gamma(f_i(xy)) = f_{i+1}(xy)$. Thus, since $n - 1 + p \equiv p - 1 \mod n$, we must have $s_{p-1}t_{p-1} = \gamma(s_{n-1}t_{n-1})$ and $f_1 = (s_{p-1}, t_{p-1})$, according to the Séébold description of the conjugacy class, together the fact that $f_i$ is determined by $f_i(xy)$. Now, the length of the simple path from $p-1$ to $0$ is $p^*-1$ (since $p-1+(p^*-1)p \equiv 0 \mod n$), thus $\gamma^{p^*-1}(s_{p-1}t_{p-1}) = s_0 t_0$ and we conclude that $(w_1, w_2) = f_{p^*}$.

It remains to prove the claim. By definition of the labels of the edges of the graph, one sees that $s_{n-1}$ (resp. $t_{n-1}$) begins and ends by $y$ (resp. $x$); hence $(s_{n-1}, t_{n-1})$ is not a Sturmian morphism, by [BS2] Lemma 2.3.8. ■

The description of the conjugacy class of a Sturmian morphism given in the previous proof is illustrated in Figure 4: the Christoffel word is $w = xxyxyxy, n = 7, p = 3, q = 4$. Furthermore $p^* = 5, q^* = 2$. The "bad conjugate" is $(s_6, t_6) = (yxyxy, xx)$. One has $f_1 = (s_2, t_2)$ and $f_5 = (s_0, t_0)$.

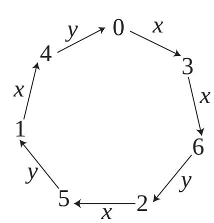


**Figure 4**

**Corollary 4.1** *Let $w = xuy$ be a Christoffel word of length $n$ and slope $\frac{p}{q}$, and $w^*$ the dual word. Let $\{f_1, \ldots, f_{n-1}\}$ (resp. $\{f'_1, \ldots, f'_{n-1}\}$) be the conjugacy class of Sturmian morphisms associated to $w$ (resp. $w^*$), in the previous description. Then $f_i^* = f'_{ip}$, where the subscript is taken modulo $n$.*

**Proof** As observed before, $f_1$ is a special standard morphism. The monoid of standard Sturmian morphisms is $\{E, F\}^*$ [BS2], where $E$ is the *interchange morphism* $(y, x)$ and $F$ is the *Fibonacci morphism* $(xy, x)$. Since $F = ED = GE$, each element of this monoid is a product of morphisms $G$ and $D$, followed or not by $E$, depending whether its determinant is $-1$ or 1. Hence $f_1$ is a product of morphisms $G$ and $D$.

Now, recall that $G^* = G$ and $D^* = \tilde{D}$. Therefore $f_1^*$ is a product of morphisms $\tilde{D}$ and $G$. Hence, by [Ri], $f_1^*$ preserves Lyndon words. Therefore $f_1^*(xy)$ is a Lyndon word, hence a Christoffel word by [BL] and [BdL]. But the previous discussion (cf. Lemma 4.2) shows that the only morphism among $f'_1, \ldots, f'_{n-1}$ such that $f'_i(xy)$ is a Christoffel word (which must be $w^*$) is $f'_p$. This shows that $f_1^* = f'_p$.

Now, we have $f_{i+1}(xy) = \gamma f_i(xy)$, $i = 1, \ldots, n-1$, where $\gamma$ is the conjugation operator. Moreover, $f_{i+1} = \varphi_i f_i$, where $\varphi_i$ is the inner automorphism of $F_2$ defined for $v \in F_2$ as $\varphi_i(v) = z^{-1}vz$, where $z = u_i$ (the $i$-th letter of the central word $u$). Thus $f_{i+1}^* = f_i^* \varphi_i^*$. Note that $\varphi_i^* = \varphi_i$ if $u_i = x$ and $\varphi_i^* = \varphi_i^{-1}$ = the inner automorphism $v \mapsto yvy^{-1}$ if $u_i = y$. Denote $f_i^* = (g_i, h_i)$. Hence we obtain that

- if $u_i = x$, then $g_{i+1} = g_i$, $h_{i+1} = g_i^{-1} h_i g_i$;
- if $u_i = y$, then $g_{i+1} = h_i g_i h_i^{-1}$, $h_{i+1} = h_i$.

In both cases, we have $g_{i+1}h_{i+1} = h_i g_i$. In particular, $f_{i+1}^*(xy) = g_{i+1}h_{i+1} = h_i g_i = \gamma^p(g_i h_i) = \gamma^p(f_i^*(xy))$, since the common length of all words $g_i$ is $p$ (cf. Lemma 4.2). Recall that $f'_{i+1}(xy) = \gamma(f'_i(xy))$, for $i = 1, \ldots, n-1$; thus, the proposition is proved since $f_1^* = f'_p$ and $\gamma^n$ is the identity on words of length $n$. ■

From the proof, one may deduce a precise description of the sequence $f_i^* = (g_i, h_i)$, $i = 1, \ldots, n-1$. We give it in the case where $p < q$. Recall that $w = x_1 \cdots x_n$ and $u = x_2 \cdots x_{n-1}$; thus $u_i$ (in the notation of the previous proof) is $x_{i+1}$. Then, if $x_{i+1} = x$, $g_{i+1} = g_i$, $g_i$ is a prefix of $h_i = g_i h'_i$, and $h_{i+1} = h'_i g_i$; if $x_{i+1} = y$, then $h_{i+1} = h_i$ and $g_i$ is a suffix of $h_i = h'_i g_i$, $g_{i+1} = h'_i g_i h_i'^{-1}$.

We illustrate this by continuing the previous example (recall that $u = x_2 \cdots x_6 = xyxyx$, so that $u^* = yyxyy$):

- $f_1^* = (xyy,\ xyyy) = f'_3$;
- $f_2^* = (xyy,\ yxyy) = f'_6$;
- $f_3^* = (yxy,\ yxyy) = f'_2$;
- $f_4^* = (yxy,\ yyxy) = f'_5$;
- $f_5^* = (yyx,\ yyxy) = f'_1$;
- $f_6^* = (yyx,\ yyyx) = f'_4$.

**Corollary 4.2** *The words $f_i^*(xy)$, $i = 1, \ldots, n-1$, satisfy $w^* = f_1^*(xy) < f_2^*(xy) < \ldots < f_{n-1}^*(xy)$ for the lexicographical order. They exhaust the conjugacy class of $w^*$, except for one word which is the mirror image of $w^*$.*

In the example above, these words are $w^* = xyyxyyy < xyyyxyy < yxyyxyy < yxyyyxy < yyxyyxy < yyxyyyx$. The missing word is $yyyxyyx$. Compare the *Burrows-Wheeler transform* in [MRS], where a tableau is constructed, whose rows are all the conjugates of a Christoffel word in lexicographical order.

**Proof** 1. In the Cayley graph of the Christoffel word $w$, denote by $m_i$, $i = 0, \ldots, n-1$, the label of the simple path from $i$ to $i$, so that $w = m_0$. Clearly, these words exhaust the whole conjugacy class of $w$.

We claim that $m_0 < m_1 < \ldots < m_{n-1}$. Actually, one has for any $i = 0, \ldots, n-2$, a factorization $m_i = uxyv$ and $m_{i+1} = uyxv$. Indeed, for $j = 1, \ldots, n$, the $j$-th letter of $m_i$ (resp. $m_{i+1}$) is $x$ or $y$ depending whether one has the inequality $i+jp > i+(j-1)p$ (resp. $i+1+jp > i+1+(j-1)p$) or the opposite inequality (here and below, we write $k$ for $k \mod n$; recall that the latter is the remainder of the division of $k$ by $n$). Hence the $j$-th letter of $m_i$ is the same that the $j$-th letter of $m_{i+1}$, except if: either $i+jp = n-1$ or $i+(j-1)p = n-1$. In the first case, these letters are $x$ for $m_i$ and $y$ for $m_{i+1}$; in the second, they are $y$ and $x$. Let $j_0$ and $j_1$ the values of $j$ corresponding to the first case and the second one (these values are unique since $p$ is invertible $\mod n$). Then $j_1 = j_0 + 1$, which implies the claim.

In the notations of the proof of Lemma 4.2, one has $m_i = s_i t_i$ and from there, one has: $\gamma(m_i) = m_{i+p}$, where subscripts are taken $\mod n$. Thus $\gamma^{p^*} = m_{i+1}$. Moreover the $m_i$ are exactly all conjugates of $w$; except $m_{n-1}$, they are in the image of the mapping $f \mapsto f(xy)$ from the morphisms in the conjugacy class of Sturmian morphisms associated to $w$ into $\{x, y\}^*$.

2. Switching to $w^*$ and its Cayley graph, let $m'_i$ denote the corresponding label. Then, by 1., $m'_0 = w^*$ and $\gamma^p(m'_i) = m'_{i+1}$. Now in the proof of Cor. 4.1, we have seen that $w^* = f_1^*(xy)$ and that $f_{i+1}^* = \gamma^p(f_i^*(xy))$. Thus the corollary follows. ■

We show now that our involution on Sturmian morphisms may be lifted to the braid group on four strands. It has been proved in [KR] that the special Sturmian monoid $St_0$ is isomorphic to the submonoid of the *braid group* with four strands $B_4$ generated by $\sigma_1$, $\sigma_2^{-1}$, $\sigma_3$, $\sigma_4^{-1}$, where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the standard generators of $B_4$ and $\sigma_4$ is a 4-th (redundant) generator, naturally obtained by drawing the braids on a cylinder. The isomorphism is defined by $\sigma_1 \mapsto G$, $\sigma_2^{-1} \mapsto D$, $\sigma_3 \mapsto \tilde{G}$, $\sigma_4^{-1} \mapsto \tilde{D}$. We therefore identify both monoids in the result below. Recall that the group $B_4$ is generated by $\sigma_1, \sigma_2,\ \sigma_3$ subject to the commutation relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ and the *braid relations* $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$.

**Corollary 4.3** *The involutive anti-automorphism $f \mapsto f^*$ of the special Sturmian monoid extends to an anti-automorphism of $B_4$; it is the anti-automorphism which fixes $\sigma_1$ and $\sigma_3$ and exchanges $\sigma_2$ and $\sigma_4$.*

**Proof** 1. Consider first the involutive mapping $\tau$ of the set $\{\sigma_i, i = 1, 2, 3, 4\}$ into itself that exchanges $\sigma_1$ and $\sigma_3$ and fixes $\sigma_2$ and $\sigma_4$. One has ([KR] Eq.(1.8)): $\sigma_4 = \sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3$ and $\sigma_4 = \sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{-1}$. Thus the group $B_4$ admits the presentation with generators $\sigma_i, i = 1, 2, 3, 4$ and the commutation and braid relations (written before the corollary), together with either of the two previous relations. Now, the mapping $\tau$ preserves the three relations and exchanges the two, if one reverses the products. Hence $\tau$ extends to an involutive anti-automorphism of $B_4$.

2. The mapping $\sigma_i \mapsto \sigma_{i+2},\ i = 1, 2, 3, 4$, where subscripts are taken modulo 4, defines an automorphism of $B_4$, since it is conjugation by $\delta^2$, where $\delta = \sigma_1\sigma_2\sigma_3$, that is, $\sigma_{i+2} = \delta^2\sigma_i\delta^{-2}$, see [KR], Section 1, Eq.(1.4).

3. If we compose the anti-automorphism of 1. and the automorphism of 2., which commute, we obtain an anti-automorphism of $B_4$ which fixes $\sigma_1$ and $\sigma_3$ and exchanges $\sigma_2$ and $\sigma_4$. It induces on the submonoid generated by $\sigma_1, \sigma_2^{-1}, \sigma_3, \sigma_4^{-1}$ the required involution, by the isomorphism between this monoid and $St_0$ described before the corollary. ■

# 5 Geometrical interpretation

In this section, we use a geometrical interpretation of Sturmian morphisms due to Arnoux and Ito [AI]. They associate to each such morphism a linear endomorphism of the free $\mathbb{Z}$-module constructed on the lattice segments in $\mathbb{Z}^2$. Then our $f^*$, in the notations of Section 4, appears really as an adjoint to $f$, modulo a conjugation.

Following [AI], we consider the free $\mathbb{Z}$-module $\mathcal{F}$ with basis the set of lattice segments in $\mathbb{R}^2$: a *lattice segment* is a subset of $\mathbb{R}^2$ of the form $\{W + te_z \,|\, 0 \leq t \leq 1\}$, where $W \in \mathbb{Z}^2$ and $z = x$ or $y$, with $(e_z)_{z=x,y}$ the canonical basis of $\mathbb{Z}^2 = \mathbb{Z}e_x \oplus \mathbb{Z}e_y$. This segment will be denoted by the symbol $(W, z)$. More generally, if $w$ is a word in $\{x, y\}^*$, we define:

$$(W, w) = \sum_{j=1}^{n}(W + V(z_1 \cdots z_{j-1}), z_j) \in \mathcal{F},$$

where $w = z_1 \cdots z_n$, $z_j \in \{x, y\}$, $j = 1, \ldots, n$, and where the function $V$ is the canonical abelianization from $\{x, y\}^*$ into $\mathbb{Z}^2$ (already considered in the proof of Proposition 3.2). Clearly, $(W, w)$ may be identified with a lattice path in $\mathbb{Z}^2$, starting from $W$ and ending at $W + V(w)$, see Figure 5.

Given an endomorphism $f$ of the free monoid $\{x, y\}^*$, we define the endomorphism $E(f)$ of $\mathcal{F}$ by

$$E(f)(W, z) = (M_f W, f(z)),$$

for any $(W, z) \in \mathbb{Z}^2 \times \{x, y\}$; here $M_f$ is the incidence matrix of $f$, or equivalently its commutative image.

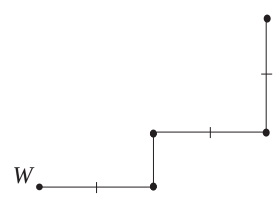


**Figure 5**
The lattice path ($W$, $xxyxxyy$)

Denote as usual by $\mathcal{F}^*$ the dual $\mathbb{Z}$-module of $\mathcal{F}$ and by $(W, z)^*$, $z \in \{x, y\}$, the dual basis. Recall that the adjoint mapping $h^*$ of the endomorphism $h$ of $\mathcal{F}$ is defined by the equation $\langle h^*((W, z)^*), (W', z') \rangle = \langle (W, z)^*, h(W', z') \rangle$, where $\langle \, , \rangle$ is the canonical duality between $\mathcal{F}^*$ and $\mathcal{F}$. The following result is due to [AI].

**Theorem 5.1**

*1. The mapping $f \mapsto E(f)$ is a monoid homomorphism from End$(\{x, y\}^*)$ into End$(\mathcal{F})$.*

*2. If $M_f$ is invertible over $\mathbb{Z}$, then the adjoint mapping $E(f)^*$ satisfies*

$$E(f)^*((W, z)^*) = \sum_{\substack{t = x, y \\ f(t) = uzv}} (M_f^{-1}(W - V(u)), t)^* \; .$$

The following theorem connects the adjoint mapping $E(f)^*$ and $E(f^*)$, where $f^*$ is the dual of $f$, $f$ being here a special Sturmian morphism. It is essentially a simple consequence of a subcase of Th. 2 in [E] (which works in any dimension). We denote by $T(W)$ the "translation" $\mathcal{F} \to \mathcal{F}$, $(U, z) \mapsto (W + U, z)$, for any $U$ in $\mathbb{Z}^2$. Let $H = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$.

**Theorem 5.2** *Let $f$ be a special Sturmian morphism. Modulo a translation, the mappings $E(f)^*$ and $E(f^*)$ are conjugate. More precisely, let $\phi : \mathcal{F} \to \mathcal{F}^*$ be the injective linear mapping defined by $\phi(W,x) = (HW + e_x, y)^*$ and $\phi(W,y) = (HW,x)^*$. Then for any special Sturmian morphism $f$, there exists a unique $W_f \in \mathbb{Z}^2$ such that*

$$E(f)^* \circ \phi = \phi \circ T(W_f) \circ E(f^*).$$

**Remark 5.1** In order to prove this result, one may follow the lines of [Ei]: first show that the result holds for the generators of the special Sturmian monoid; then show that the result is preserved under composition of morphisms. This may be extracted from the proof below of Prop. 5.1.

The rest of this section is devoted to the precise description of vector $W_f$, using the structure of Sturmian morphisms as described by Séébold (see [BS2]).

If $f$ is a special Sturmian morphism, by taking the notation of Cor. 4.1, $f_1$ is a (special) *standard morphism*, and therefore a product of $G$'s and $D$'s. Likewise, $f_{n-1}$ is *antistandard* (that is, the mirror image of a standard morphism), hence a product of $\tilde{G}$'s and $\tilde{D}$'s: indeed, it is known (see [BS2]) that $f_{n-1}$ is the only morphism of the conjugation class of $f$ whose image of $x$ and $y$ begin by a different letter; moreover, $G(x)$ and $G(y)$ (resp. $D(x)$ and $D(y)$) begin by the same letter; thus in any product involving the four morphims $G$, $D$, $\tilde{G}$ and $\tilde{D}$, and involving at least once $G$ or $D$, the result is a morphism whose images of $x$ and $y$ begin by the same letter; we conclude that $f_{n-1}$ is a product of $\tilde{G}$'s and $\tilde{D}$'s.

Since $(n-1)p \equiv q = n - p \mod n$, we have by Cor. 4.1, $f^*_{n-1} = f'_q$. Denote by $\gamma_g$ the inner automorphism $u \mapsto g^{-1}ug$ of $F_2$. Recall that the morphisms $f^*_i$, $i = 1, \ldots, n-1$, are all conjugate. We still denote by $V$ the canonical abelianization from $F_2$ into $\mathbb{Z}^2$.

**Proposition 5.1** *(i) If $f$ is an antistandard Sturmian morphism, then $W_f = 0$.*
*(ii) If $f$ is any special Sturmian morphism, let $g \in F_2$ be such that $f^* = \gamma_g \circ f^*_{n-1}$, with the previous notations. Then $W_f = V(g)$, the commutative image of $g$.*

We continue the example of Section 4. Here, $f_6$ is antistandard, $f^*_6 = f'_4 = (yyx, yyyx)$ and $W_{f_6} = 0$. Then $f^*_1 = f'_3 = (xyy, xyyy) = \gamma_{x^{-1}} \circ (yyx, yyyx) = \gamma_{x^{-1}} \circ f'_4 = \gamma_{x^{-1}} \circ f^*_6$, hence $W_{f_1} = -e_x$. Moreover, we have

$f_2^* = f_6' = (xyy,\, yxyy) = \gamma_{yy} \circ (yyx,\, yyyx)$, hence $W_{f_2} = 2e_y$. Likewise, one has $W_{f_3} = -e_x - e_y$, $W_{f_4} = e_y$, $W_{f_5} = -e_x - 2e_y$.

**Proof** 1. We have $\tilde{G} = (x, yx)$, hence $M_{\tilde{G}} = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $M_{\tilde{G}^{-1}} = M_{\tilde{G}}^{-1} = \left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)$. Thus, by the formula of Th. 5.1, $E(\tilde{G})^*$ maps $(W, x)^*$ onto

$$\left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} W, x\right)^* + \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}(W - V(y)), y\right)^*$$

$$= \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} W, x\right)^* + \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} W - \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}, y\right)^*$$

$$= \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} W, x\right)^* + \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} W + e_x - e_y, y\right)^*.$$

Moreover, $E(\tilde{G})^*$ maps $(V, y)^*$ onto $\left(\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right) W, y\right)^*$. We already know from Section 4 that $\tilde{G}^* = \tilde{G}$, hence $E(\tilde{G}^*)$ maps $(W, x)$ onto $\left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W, x\right)$ and $(W, y)$ onto $\left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W, yx\right) = \left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W, y\right) + \left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W + e_y, x\right)$.

Taking $\phi$ as in Th. 5.2, we have $E(\tilde{G})^* \circ \phi(W, x) = E(\tilde{G})^* \, (HW + e_x, y)^* = \left(\left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)(HW + e_x), y\right)^* = \left(\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right) W + e_x, y\right)^*$.

Furthermore $\phi \circ E(\tilde{G}^*)(W, x) = \phi\left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W, x\right) = \left(H\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W + e_x, y\right)^*$ $= \left(\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right) W + e_x, y\right)^*$, so that $E(\tilde{G})^* \circ \phi(W, x) = \phi \circ E(\tilde{G}^*)(W, x)$.

Similarly $E(\tilde{G})^* \circ \phi(W, y) = E(\tilde{G})^* \, (HW, x)^*$

$$= \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} HW, x\right)^* + \left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} HW + e_x - e_y, y\right)^*$$

$$= \left(\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} W, x\right)^* + \left(\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} W + e_x - e_y, y\right)^*.$$

Moreover $\phi \circ E(\tilde{G}^*)(W, y) = \phi\left(\left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W, y\right) + \left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) W + e_y, x\right)\right)$

$$= \left(H\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} W, x\right)^* + \left(H\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} W + e_y\right) + e_x, y\right)^*$$

$$= \left(\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} W, x\right)^* + \left(\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} W - e_y + e_x, y\right)^*,$$

so that $E(\tilde{G})^* \circ \phi(W, y) = \phi \circ E(\tilde{G}^*)(W, y)$. We conclude that $W_{\tilde{G}} = 0$.

2. A similar calculation, which is left to the reader, shows that $W_{\tilde{D}} = 0$.

3. Now, note that if the formula in Th. 5.2 is true for $f$ and $g$, then it is true for $gf$, with $W_{gf} = W_f + M_{f^*}(W_g)$. Indeed, one verifies first that $E(f)T(W) = T(M_f W)E(f)$, and a simple calculation allows one to conclude.

4. Now, an antistandard morphism $f$ is in the monoid generated by $\tilde{G}$ and $\tilde{D}$. Hence, by 3. we see that $W_f = 0$.

5. Now, let $f^* = \gamma_g \circ f_{n-1}^*$ as in the statement. Then $f = f_{n-1}\gamma_g^*$ and from 3. we conclude that $W_f = W_{\gamma_g^*} + M_{\gamma_g} W_{f_{n-1}} = W_{\gamma_g^*}$ since, $f_{n-1}$ being antistandard, $W_{f_{n-1}} = 0$. Hence it is enough to show that $W_{\gamma_g^*} = V(g)$.

It is easy to verify that $\gamma_x^* = (x^{-1}, y)(x, xyx^{-1})(x^{-1}, y) = \gamma_x$ and $\gamma_y^* = \gamma_{y^{-1}}$. Hence, if we write $g = z_1 \cdots z_n$, then $\gamma_g = \gamma_{z_n} \cdots \gamma_{z_1}$ and $\gamma_g^* = \gamma_{z_1}^* \cdots \gamma_{z_n}^*$ so that, since the incidence matrix of an inner automorphism is the identity, by 3. we conclude that $W_{\gamma_g^*} = W_{\gamma_{z_1}^*} + \ldots + W_{\gamma_{z_n}^*}$.

Thus, it is enough to show that $W_{\gamma_x^*} = e_x$ and $W_{\gamma_y^*} = e_y$. One has $\gamma_x^* = \gamma_x = \tilde{G}G^{-1}$ and $\gamma_y^* = \gamma_{y^{-1}} = D\tilde{D}^{-1}$. Hence by 3., we obtain $W_{\gamma_x^*} = W_{G^{-1}} + M_{G^{-1*}} W_{\tilde{G}} = W_{G^{-1}}$ (since $W_{\tilde{G}} = 0$) and $W_{\gamma_y^*} = W_{\tilde{D}^{-1}} + M_{\tilde{D}^{-1*}}(W_D)$. Now, it follows from 3. that $W_{f^{-1}} = -M_{f^{-1*}}(W_f)$. Thus $W_{\tilde{D}^{-1}} = 0$ (since $W_{\tilde{D}} = 0$) and $W_{\gamma_x^*} = -M_{G^{-1*}}(W_G) = -\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) W_G$; furthermore, $W_{\gamma_y^*} = M_{\tilde{D}^{-1*}}(W_D) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) W_D$. We leave to the reader the task to compute $W_G = -e_x$ and $W_D = e_y$. Thus finally $W_{\gamma_x^*} = e_x$ and $W_{\gamma_y^*} = e_y$. ■

# 6 Appendix

We prove here the following generalization of Lemma 4.1.

**Theorem 6.1** *An endomorphism of the free monoid on $x$ and $y$ is Sturmian if and only if it sends the three Christoffel words $xy$, $xxy$ and $xyy$ onto conjugates of Christoffel words.*

**Proof** In view of Lemma 4.1, it is enough to prove the "if" part. Suppose that endomorphism $f$ satisfies the hypothesis. Note first that $f$ is not erasing since otherwise $f(xxy)$ or $f(xyy)$ is not primitive, hence not a Christoffel word. We may assume that $f$ is not the identity, nor the morphism $E$ which exchanges the letters

We show that the words $f(x)$ and $f(y)$ both start or end with the same letter. First, we may suppose that $f(x)$ starts with $x$ (since $E$ is a Sturmian morphism). If we assume that the result is not true, then $f(y)$ starts with $y$; if moreover $f(x)$ ends with $y$, then $f(y)$ ends with $x$; thus $f(xy) = f(x)f(y)$

contains the factor $yy$, and its conjugate $f(yx)$ contains $xx$: this contradicts the hypothesis, since a Christoffel word, as well as its square, is balanced (see [BS2]); hence the associated circular word cannot contain both factors $xx$ and $yy$. We conclude that $f(x)$ must end with $x$ and $f(y)$ with $y$. Then $f(xxy) = f(x)f(x)f(y)$ contains $xx$ (there is an $x$ at the end and at the beginning of $f(x)$), so that neither $f(x)$ nor $f(y)$ contains $yy$. Similarly, by considering $f(xyy)$, we see that $f(x)$ and $f(y)$ do not contain $xx$. We conclude that $f(x) = x(yx)^i$ and $f(y) = y(xy)^j$; hence $f(xy) = x(yx)^{i+j}y$, which is conjugate to a Christoffel word only if $i = j = 0$, a contradiction, since $f$ would be the identity.

In what follows we use the following fact: a nonempty word is in the submonoid $\{x, xy\}^*$ (resp. $\{x, yx\}^*$) if and only if it starts (resp. ends) by $x$ and has not the factor $yy$. Similarly for the two submonoids $\{yx, y\}^*$ and $\{xy, y\}^*$, where the roles of $x$ and $y$ are exchanged.

Now, since $f(xy)$ is conjugate to a Christoffel word, it may contain as factor $xx$ or $yy$, but not both; hence we may suppose that it does not contain $yy$; hence $f(x)$ and $f(y)$ both do not contain $yy$.

Suppose first that they both start with the same letter, and we consider the case where it is $x$; then $f(x)$ and $f(y)$ are in $\{x, xy\}^*$ and therefore we may find an endomorphism $g$ such that $f = G \circ g$. If on the other hand, they both start with $y$, suppose that one of them contains $xx$; then, we see that $f(x)$ and $f(y)$ do not end with $y$ (otherwise $f(xy)$ or $f(yx)$ will contain both $xx$ and $yy$, a contradiction); hence they both end by $x$, and they belong to $\{x, yx\}^*$; therefore we may find $g$ such that $f = \tilde{G} \circ g$. If however $f(x)$ and $f(y)$ do not contain $xx$, then they are in $\{yx, y\}^*$; thus $f = D \circ g$.

Suppose now, if they both end with the same letter, we argue symmetrically. From this one derives that $f$ can be factorized, besides one of the preceding forms, also as $f = \tilde{G} \circ g$, where $g$ is a suitable endomorphism.

We shall give an inductive proof of our assertion. First we show that if $g$ is an endomorphism and if $H$ is among the four endomorphisms $G$, $\tilde{G}$, $D$, or $\tilde{D}$, if moreover $H \circ g$ sends the three Christoffel words $xy$, $xxy$ and $xyy$ onto conjugate of Christoffel words, then so does $g$. This may be done directly. But we use the following algebraic argument: the conjugates of Christoffel words are shown to coincide with primitive elements of the free group $F_2$ generated by $x$ and $y$ which are in the free monoid $\{x, y\}^*$, see [KR], Cor. 3.3; recall that an element $u \in F_2$ is *primitive* if there exists $v \in F_2$ such that $(u, v)$ is a basis of $F_2$ (this notion of primitivity has not to be confused with that used usually in combinatorics on words). An automorphism of a free group sends bases onto bases, hence primitive elements onto primitive elements. Since the inverse of $H$ as above is an automorphism of $F_2$, we are done.

To conclude the induction, note that the sum of the lengths of the images of $x$ and $y$ by $g$ is strictly less than the same sum for $f$, since the image by $f$ involve both letters (otherwise $f$ projects onto $\{x\}^*$ or $\{y\}^*$ and $f(xy)$ is not primitive). ■

Note that there is an analogue characterization of Sturmian morphisms by Berstel and Séébold [BS1]: $f$ is a Sturmian morphism if and only if $f$ is acyclic and the word $f(yxxyxxyxyxxyxy)$ is balanced; another characterization of Sturmian morphisms may be found in [TW]. Note also that in [Sh] is proved the result that an endomorphism of $F_2$ is an automorphism if and only if it sends primitive elements onto primitive elements.

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