Characterization of the Best Discrete Approximation of a Line in the 3-Dimentional Space

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Abstract. The present paper deals with discrete lines in the 3-dimensional space. In particular, we focus on the minimal 0-connected set of closest integer points to a Euclidean line. We propose a definition which leads to geometric, arithmetic and algorithmic characterizations of naive discrete lines in the 3-dimensional space.

1 Introduction

In discrete geometry, as in Euclidean one, linear objects are essential. All the other discrete objects can be approximated as soon as the linear ones have been characterized. The only well understood class of linear discrete objects is the one of $(d-1)$-dimensional objects in the d-dimensional space, namely, the discrete hyperplanes [1, 2]. Numerous works also exist on 1-dimensional linear discrete objects. They tackle this problem algorithmically [3, 4] or arithmetically [5–7]. Moreover, discretization models, such as the standard [8] and the supercover [9] ones, define 2-connected discrete lines. Nevertheless, as far as we know, many problems are still open. For instance, we are currently unable to characterize the minimal 0-connected set of closest integer points to a Euclidean line in the 3-dimensional space.

In the present paper, our purpose is to introduce a modeling of discrete lines in the 3-dimensional space such that topological properties and the relationship with the closest integer points to the Euclidean line with same parameter are easily determined. We propose a representation of the 1-dimensional linear discrete object inspired by the notion of functionality [10, 11]. Indeed, a connected discrete line in the 3-dimensional space should verify some conditions similar to this notion: we can define subsets of $\mathbb{Z}^3$ such that the discrete line is connected only if it contains at least a point of each of them.

The paper is organized as follows. In the next section, we recall some basic notions of discrete geometry useful to understand the remainder of the paper. Then, in the third section, we focus on already known naive discrete line. First we present the usual 2-dimensional definition which extends in higher dimensions to discrete hyperplanes. Secondly, the 3-dimensional current definition based on projections on particular planes is detailed. In the fourth section, we propose a definition related to the closest integer points to a Euclidean line and we introduce its geometric, arithmetic, and algorithmic characterizations in the 3-dimensional space.

2 Basic Notions

The aim of this section is to introduce the basic notions of discrete geometry used throughout the present paper. Let d be an integer greater than 1 and let $\{\mathbf{e_1}, \ldots, \mathbf{e_d}\}$ be the canonical basis of the Euclidean vector space $\mathbb{R}^d$. Let us call *discrete set* any subset of the *discrete space* $\mathbb{Z}^d$. The point $\mathbf{x} = \sum_{i=1}^{d} x_i \mathbf{e_i} \in \mathbb{R}^d$, with $x_i \in \mathbb{R}$ for each $i \in \{1, \ldots, d\}$, is represented by $(x_1, \ldots, x_d)$. A point $\mathbf{v} \in \mathbb{Z}^d$ is called a *voxel* in a d-dimensional space or a *pixel* in a 2-dimensional space.

Definition 1 (k-Adjacency or k-Neighborhood). *Let d be the dimension of the discrete space and $k \in \mathbb{N}$ such that $k < d$. Two voxels $\mathbf{v} = (v_1, \ldots, v_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ are* k-neighbors *or* k-adjacent *if and only if:*

$$\|\mathbf{v}-\mathbf{w}\|_\infty = \max\{|v_1-w_1|, \ldots, |v_d-w_d|\} = 1 \text{ and } \|\mathbf{v}-\mathbf{w}\|_1 = \sum_{i=1}^{d} |v_i-w_i| \leq d-k.$$

Let $k \in \{0, \ldots, d-1\}$. A discrete set E is said to be *k-connected* if for each pair of voxels $(\mathbf{v}, \mathbf{w}) \in \mathrm{E}^2$, there exists a finite sequence of voxels $(\mathbf{s_1}, \ldots, \mathbf{s_p}) \in \mathrm{E}^p$ such that $\mathbf{v} = \mathbf{s_1}$, $\mathbf{w} = \mathbf{s_p}$ and the voxels $\mathbf{s_j}$ and $\mathbf{s_{j+1}}$ are k-neighbors, for each $j \in \{1, \ldots, p-1\}$.

Let E be a discrete set, $\mathbf{v} \in \mathrm{E}$ and $k \in \{0, \ldots, d-1\}$. The *k-connected component* of $\mathbf{v}$ in E is the maximal k-connected subset of E (with respect to set inclusion) containing $\mathbf{v}$.

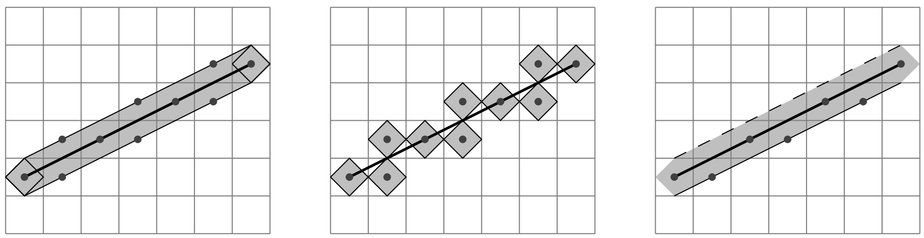
Definition 2 (k-Separatingness). *A discrete set* E *is* k-separating *in a discrete set* F *if its complement in* F, $\overline{\mathrm{E}} = \mathrm{F} \setminus \mathrm{E}$, *has two distinct k-connected components.* E *is called a* separator *of* F.

Definition 3 (k-Simple Point, k-Minimality). *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. Let also* F *and* E *be two discrete sets such that* E *is k-separating in F. A voxel $\mathbf{v} \in \mathrm{E}$ is said to be* k-simple *if* $\mathrm{E} \setminus \{\mathbf{v}\}$ *remains k-separating in* F. *Moreover, a k-separating discrete set in* F *without k-simple points is said to be* k-minimal *in* F.

3 Discrete Lines

Lines are elementary objects in geometry. They have been widely studied in discrete geometry [1] and are the best known discrete objects. However we understand them in the 2-dimensional space as $(d-1)$-dimensional linear objects and not as 1-dimensional linear objects. Consequently, results on discrete lines in the 2-dimensional space extend in higher dimension to discrete hyperplanes and not to discrete lines in d-dimensional spaces. Definitions of discrete lines in the 3-dimensional space exist, but none are equivalent to the minimal 0-connected set of closest integer points to a Euclidean line.

3.1 The 2-Dimensional Space: Discrete Lines as Discrete Hyperplanes



(a) (b) (c)

Fig. 1. (a) First definition of the closed naive representation of a line, (b) Second definition of the closed naive representation of a line, (c) The associated naive discrete line

First, discrete line drawing algorithms were designed to provide for the needs of digital plotters [12]. Later, arithmetic and geometric characterizations have been proposed [1, 2]. The minimal 0-connected set of closest integer points to a Euclidean line is its closed naive representation [13]. The closed naive model introduced by E. Andres associates a Euclidean object $\mathcal{O}$ with the representation $\overline{\mathrm{N}}(\mathcal{O})$ defined as follows:

$$\overline{\mathrm{N}}(\mathcal{O}) = \left(\mathcal{B}^1\left(\frac{1}{2}\right) \oplus \mathcal{O}\right) \cap \mathbb{Z}^d, \tag{1}$$

$$= \left\{\mathbf{p} \in \mathbb{Z}^d; \left(\mathcal{B}^1\left(\frac{1}{2}\right) \oplus \mathbf{p}\right) \cap \mathcal{O} \neq \varnothing\right\}, \tag{2}$$

where $\mathcal{B}^1\left(\frac{1}{2}\right)$ is the ball of radius $\frac{1}{2}$ based on $\|\cdot\|_1$, and $\oplus$ denote the Minkowski sum:

$$A \oplus B = \{\mathbf{a} + \mathbf{b}; \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}.$$

In Figure 1(a), an example of definition (1) is shown. The selected points are the ones contained in the band described by the translation of $\mathcal{B}_1\left(\frac{1}{2}\right)$ along the discrete line. In Figure 1(b), an example of definition (2) is shown. The selected discrete points are the ones for which the intersection between the ball $\mathcal{B}_1\left(\frac{1}{2}\right)$ centered on them and the Euclidean line is not empty. Both definitions are equivalent.

From an arithmetic point of view, the closed naive representation $\overline{\mathrm{N}}(\mathcal{D}(\mathbf{n}, \mu))$ of the Euclidean line $\mathcal{D}(\mathbf{n}, \mu)$ with normal vector $\mathbf{n} = (a, b) \in \mathbb{Z}^2$ and translation parameter μ is defined as follows:

$$\overline{\mathrm{N}}\left(\mathcal{D}(\mathbf{n})\right) = \left\{\mathbf{p} = (i, j) \in \mathbb{Z}^2; -\frac{\|\mathbf{n}\|_\infty}{2} \leq ai + bj + \mu \leq \frac{\|\mathbf{n}\|_\infty}{2}\right\} \tag{3}$$

Such an arithmetic representation is well adapted to the deduction of properties, such as the membership of a discrete point to a discrete line, and the definition of drawing and recognition algorithms.

However, the closed naive representation of an object can contain 0-simple points. A simple way to avoid such configuration is to restrict one of the inequalities in (3). By doing so, we obtain the naive line $\mathrm{N}\left(\mathcal{D}(\mathbf{n},\mu)\right)$ introduced by J.-P. Reveillès [1], defined as follows:

$$\mathrm{N}\left(\mathcal{D}(\mathbf{n},\mu)\right) = \left\{\mathbf{p} = (i,j) \in \mathbb{Z}^2; -\frac{\|\mathbf{n}\|_\infty}{2} \leq ai + bj + \mu < \frac{\|\mathbf{n}\|_\infty}{2}\right\} \tag{4}$$

Definition (4) is the common definition of discrete lines because it leads to the minimal 0-connected discrete set without simple points, as shown in Figure 1(c). J. Bresenham's line [12] is, in particular, a naive discrete line.

The above mentioned models of discrete lines easily extend in higher dimensions to discrete hyperplanes [1, 2]. In particular, we have the following naive model of $\mathcal{P}(\mathbf{n},\mu)$, the hyperplane with normal vector $\mathbf{n} \in \mathbb{Z}^d$ and $\mu \in \mathbb{Z}$ its translation parameter:

$$\mathrm{N}\left(\mathcal{P}(\mathbf{n},\mu)\right) = \left\{\mathbf{v} = (v_1 ..., v_d) \in \mathbb{Z}^d; -\frac{\|\mathbf{n}\|_\infty}{2} \leq \sum_{i=1}^{d} n_i v_i + \mu < \frac{\|\mathbf{n}\|_\infty}{2}\right\} \tag{5}$$

In the 2-dimensional space, discrete lines are defined by their normal vector. This is not possible in higher dimensions: a line is then defined by its direction vector or its normal hyperplane. Another approach is necessary to understand them.

3.2 3-Dimensional Space: Discrete Lines and Projections

The closed naive description of a Euclidean line in the 3-dimensional space does not share all the properties of a Euclidean line in the 2-dimensional space. In particular, such a discrete set is not 0-connected.

Example 1. Let $\mathbf{v} = (1, 1, 2)$ be the direction vector of the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$ through the origin. Then, $\overline{\mathrm{N}}\left(\mathcal{D}_{3D}(\mathbf{v})\right)$, its closed naive representation is defined as follows:

$$\overline{\mathrm{N}}\left(\mathcal{D}_{3D}(\mathbf{v})\right) = \{p.(1, 1, 2); p \in \mathbb{Z}\}\,.$$

This set is obviously not connected.

Another definition have been proposed to characterize naive discrete lines in the 3-dimensional space. It was first introduced by A. Kaufman and E. Shimony [3] with an algorithm computing incrementally the set of its points. This algorithm is a generalization to the 3-dimensional space of the J. Bresenham's classical 2-dimensional one [12]. Considering that this naive line is provided with a direction vector $\mathbf{v} = (a, b, c)$ such that $\|\mathbf{v}\|_\infty = c \neq 0$ and $\gcd(a, b, c) = 1$, its projections on the planes normal to $\mathbf{e_1}$ and $\mathbf{e_2}$ (both equivalent to the 2-dimensional space $\mathbb{Z}^2$) are naive discrete lines. This simplification is the key point of the approach.

Later, I. Debled-Rennesson [7], O. Figueiredo and J.-P. Reveillès [5, 6] proposed an arithmetic characterization of naive discrete lines in the 3-dimensional space. $\mathrm{N}(\mathcal{D}_{3D}(\mathbf{v}))$, the naive representation of the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$ through the origin and directed by $\mathbf{v} = (a, b, c)$, such that $\|\mathbf{v}\|_\infty = c \neq 0$ and $\gcd(a, b, c) = 1$, is the set of discrete points $\mathbf{n} = (i, j, k) \in \mathbb{Z}^3$ verifying:

$$\begin{cases} -\frac{c}{2} \leq bk - cj < \frac{c}{2} \\ -\frac{c}{2} \leq ci - ak < \frac{c}{2} \end{cases} \tag{6}$$

This arithmetic definition characterizes the same set as A. Kaufman and E. Shimony's algorithm [6].

In [6], O. Figueiredo and J.-P. Reveillès notice that the resulting set is different from the one of the closest discrete points to the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$.

Example 2. Let $\mathbf{v} = (1, 2, 4)$ be the direction vector of the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$. Then, $\mathrm{N}(\mathcal{D}_{3D}(\mathbf{v}))$, the naive discrete representation of $\mathcal{D}_{3D}(\mathbf{v})$ is the set of points:

$$\mathrm{N}(\mathcal{D}_{3D}(\mathbf{v})) = \{p\mathbf{v} \oplus \{(0,0,0), (0,1,1), (1,1,2), (1,2,3)\}\,;\ p \in \mathbb{Z}\}\,.$$

The Euclidean distance between the points $\{p.\mathbf{v} \oplus (0, 1, 1);\ p \in \mathbb{Z}\}$ and the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$ is of 0.535, whereas the points $\{p\mathbf{v} \oplus (0, 0, 1);\ p \in \mathbb{Z}\}$ are only at a distance of 0.487 from the line. So the set of the closest points to the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$ is:

$$\{p.\mathbf{v} \oplus \{(0,0,0), (\mathbf{0},\mathbf{0},\mathbf{1}), (1,1,2), (1,2,3)\}\,;\ p \in \mathbb{Z}\}\,.$$

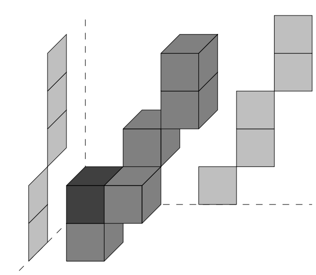


Fig. 2. The naive discrete line directed by $(1, 2, 4)$, for which projections are 2-dimensional naive discrete lines, contains an error in the point selection

Another definition of discrete lines was introduced by V. Brimkov and R. Barneva in [14]. Graceful lines are seen as intersection of particular discrete planes, the graceful ones. They are 0-connected but not minimal sets.

4 Discrete Lines as Sets of Closest Integer Points

The usual definition of discrete lines in 3-dimensional space is not satisfactory because it it not equivalent to the set of the closest integer points to the Euclidean line with same parameters and because geometric properties are lost. In the sequel, we propose a definition which overcomes these limitations and leads to geometric, arithmetic and algorithmic characterizations.

4.1 Minimal 0-Connected Set of Closest Integer Points

We are interested in the thinnest discrete line $\mathrm{D}_{3D}(\mathbf{v})$ through the origin and directed by $\mathbf{v} = (a, b, c) \in \mathbb{N}^3$ such that $\|\mathbf{v}\|_\infty = c \neq 0$ and $\gcd(a, b, c) = 1$. By thinnest, we mean the minimal (with respect to set inclusion) 0-connected set constituted by the closest discrete points to the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$.

Theorem 1. *The discrete line* $\mathrm{D}_{3D}(\mathbf{v})$ *is* 0*-connected if and only if :*

$$\forall k \in \mathbb{Z}, \exists (i, j) \in \mathbb{Z}^2;\ (i, j, k) \in \mathrm{D}_{3D}(\mathbf{v}).$$

Proof (Sketch). If $\exists k \in \mathbb{Z}$ such that $\nexists (i, j) \in \mathbb{Z}^2, (i, j, k) \in \mathrm{D}_{3D}(\mathbf{v})$ then $\mathrm{D}_{3D}(\mathbf{v})$ is not 0-connected since the discrete plane $\mathrm{P}(\mathbf{e_3}, k, 1)$ is 0-separating in $\mathbb{Z}^3$ and points of $\mathrm{D}_{3D}(\mathbf{v})$ belongs to both sides of it. Thus, if $\mathrm{D}_{3D}(\mathbf{v})$ is 0-connected, then $\forall k \in \mathbb{Z}, \exists (i, j) \in \mathbb{Z}^2, (i, j, k) \in \mathrm{D}_{3D}(\mathbf{v})$.

The discrete line $\mathrm{D}_{3D}(\mathbf{v})$ is the set of closest discrete points to the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$. Let us assume that $\mathbf{n}$ and $\mathbf{m} \in \mathbb{Z}^3$ belong to $\mathrm{D}_{3D}(\mathbf{v})$ such that $k_m = k_n + 1$. From the initial conditions on the direction vector $\mathbf{v}$ ($\|\mathbf{v}\|_\infty = c \neq 0$), the intersection between $\mathrm{D}_{3D}(\mathbf{v})$ and $\mathcal{P}(\mathbf{e_3}, k_m)$, the plane normal to $\mathbf{e_3}$ containing $\mathbf{m}$, is the intersection between $\mathrm{D}_{3D}(\mathbf{v})$ and $\mathcal{P}(\mathbf{e_3}, k_n)$ translated by vector $(\frac{a}{c}, \frac{b}{c}, 1)$. As the criterion is the distance to the line, in the worst case, $\mathbf{m} = \mathbf{n} + (1, 1, 1)$ and finally for each k_n, $\mathbf{n}$ and $\mathbf{m}$ are always 0-adjacent. Thus, if $\forall k \in \mathbb{Z}, \exists (i, j) \in \mathbb{Z}^2, (i, j, k) \in \mathrm{D}_{3D}(\mathbf{v})$, then $\mathrm{D}_{3D}(\mathbf{v})$ is 0-connected. $\square$

So for each $k \in \mathbb{Z}$, we look for the closest discrete points $\mathbf{n} = (i, j, k) \in \mathbb{Z}^3$ to $\mathrm{D}_{3D}(\mathbf{v})$. Let us now define $\mathcal{V}_{3D}$ (Figure 3(a)), a subset of $\mathbb{R}^3$ we use to determine them.

Definition 4. *Let* $\mathcal{P}(\mathbf{e_3}, 0)$ *be the Euclidean plane of normal vector* $\mathbf{e_3}$ *and with translation parameter* 0*. Then,* $\mathcal{V}_{3D}$ *is defined as follows:*

$$\mathcal{V}_{3D} = \left(\mathcal{B}^\infty\left(\frac{1}{2}\right) \setminus \left\{\mathbf{x} \in \mathbb{R}^3; \|\mathbf{x}\|_\infty = \frac{1}{2} \text{ and } sgn(x_1) = -sgn(x_2)\right\}\right) \cap \mathcal{P}(\mathbf{e_3}, 0).$$

Let $k \in \mathbb{Z}$. Let $\mathcal{P}(\mathbf{e_3}, k)$ be the Euclidean plane of normal vector $\mathbf{e_3}$ and with translation parameter k. Let $\mathcal{D}_{3D}(\mathbf{v})$ be the Euclidean line through the origin and directed by $\mathbf{v} = (a, b, c) \in \mathbb{N}^3$ such that $\|\mathbf{v}\|_\infty = c \neq 0$ and $\gcd(a, b, c) = 1$. Let $\mathbf{x} = \mathcal{D}_{3D}(\mathbf{v}) \cap \mathcal{P}(\mathbf{e_3}, k)$ be the intersection between the line $\mathcal{D}_{3D}(\mathbf{v})$ and the plane $\mathcal{P}(\mathbf{e_3}, k)$.

$\mathcal{V}_{3D}$ centered on $\mathbf{x}$ obviously contains at least one discrete point:

$$(\mathcal{V}_{3D} \oplus \mathbf{x}) \cap \mathbb{Z}^3 \neq \varnothing,$$

and:

Proposition 1. *$\mathcal{V}_{3D}$ centered on $\mathbf{x}$ contains the closest discrete points, included in $\mathcal{P}(\mathbf{e_3}, k)$, to $\mathcal{D}_{3D}(\mathbf{v})$:*

$$\forall \mathbf{n} \in (\mathcal{V}_{3D} \oplus \mathbf{x}) \cap \mathbb{Z}^3, d_2\left(\mathbf{n}, \mathcal{D}_{3D}(\mathbf{v})\right) = \min_{m \in \mathcal{P}(\mathbf{e_3},k) \cap \mathbb{Z}^3} \left\{ d_2\left(\mathbf{m}, \mathcal{D}_{3D}(\mathbf{v})\right) \right\} \quad (7)$$

Proof (Sketch). In order to prove this proposition, we have to evaluate the distance from the point $\mathbf{n}$ to the line $\mathcal{D}_{3D}(\mathbf{v})$. The cross product $\mathbf{v} \times \mathbf{n}$ is useful since its norm 2 is equal to this distance multiplied by $\|\mathbf{n}\|_2$. Then, points $\mathbf{x}$ are of the form $\left(\frac{ka}{c}, \frac{kb}{c}, k\right)$ and points $\mathbf{n}$ are of the form $\left(\frac{ka}{c} + \varepsilon_1, \frac{kb}{c} + \varepsilon_2, k\right)$. Those considerations are the key points to demonstrate the proposition. □

This result allows to geometrically characterize the minimal 0-connected set $\mathrm{D}_{3D}(\mathbf{v})$ of the closest discrete points to the euclidean line $\mathcal{D}_{3D}(\mathbf{v})$.

Theorem 2. *The minimal 0-connected set $\mathrm{D}_{3D}(\mathbf{v})$ of the closest discrete points to the Euclidean line $\mathcal{D}_{3D}(\mathbf{v})$ with normal vector $\mathbf{v} = (a, b, c) \in \mathbb{N}^3$ such that $\|\mathbf{v}\|_\infty = c \neq 0$ and $\gcd(a, b, c) = 1$ is:*

$$\begin{aligned} \mathrm{D}_{3D}(\mathbf{v}) &= (\mathcal{V}_{3D} \oplus \mathcal{D}_{3D}(\mathbf{v})) \cap \mathbb{Z}^3 & (8)\\ &= \left\{\mathbf{p} \in \mathbb{Z}^3; (\mathcal{V}_{3D} \oplus \mathbf{p}) \cap \mathcal{D}_{3D}(\mathbf{v}) \neq \varnothing\right\} & (9) \end{aligned}$$

Proof. Theorem 2 is a direct consequence of Proposition 1. $\mathcal{V}_{3D}$ is normal to $\mathbf{e_3}$. It can contains discrete points only if its component relative to $\mathbf{e_3}$ is an integer. In this particular case, $\mathcal{V}_{3D}$ contains only the discrete points of $\mathcal{P}(\mathbf{e_3}, k)$ for which the euclidean distance to the line $\mathcal{D}_{3D}(\mathbf{v})$ is minimal. Thus, for each $k \in \mathbb{Z}$, we select the points, at least one, closest to the line and obtain the minimal 0-connected set we are looking for. □

4.2 Naive Discrete Lines in the 3-Dimensional Space

In order to obtain a naive discrete line in the 3-dimensional space, we arbitrarily select one discrete point when several are possible in $\mathrm{D}_{3D}(\mathbf{v})$. From the solution shown in Figure 3(b) and the cross product $\mathbf{v} \times \mathbf{n}$ we deduce the following arithmetic definition. Only two components of the cross product are evaluated since the third one depends on them.

Definition 5. *Let $\mathbf{v} = (a, b, c) \in \mathbb{N}^3$ such that $\|\mathbf{v}\|_\infty = c \neq 0$ and $\gcd(a, b, c) = 1$. The naive discrete line through the origin and directed by $\mathbf{v}$ is the set of discrete points $\mathbf{n} = (i, j, k) \in \mathbb{Z}^3$ such that:*

$$\begin{cases} -\frac{c}{2} \leq & \operatorname{sgn}(ci - ak)\ (bk - cj) & < \frac{c}{2}, \\ -\frac{c}{2} \leq & \operatorname{sgn}(bk - cj)\ (ci - ak) & < \frac{c}{2}, \\ & (ci - ak, bk - cj) \neq \left(\frac{c}{2}, \frac{c}{2}\right) & \end{cases} \quad (10)$$

An example of the resulting set is shown in Figure 3(c) for the line directed by $(2, 3, 6)$. Its orthogonal projections on planes with normal vector $\mathbf{e_1}$ or $\mathbf{e_2}$ are not discrete lines.

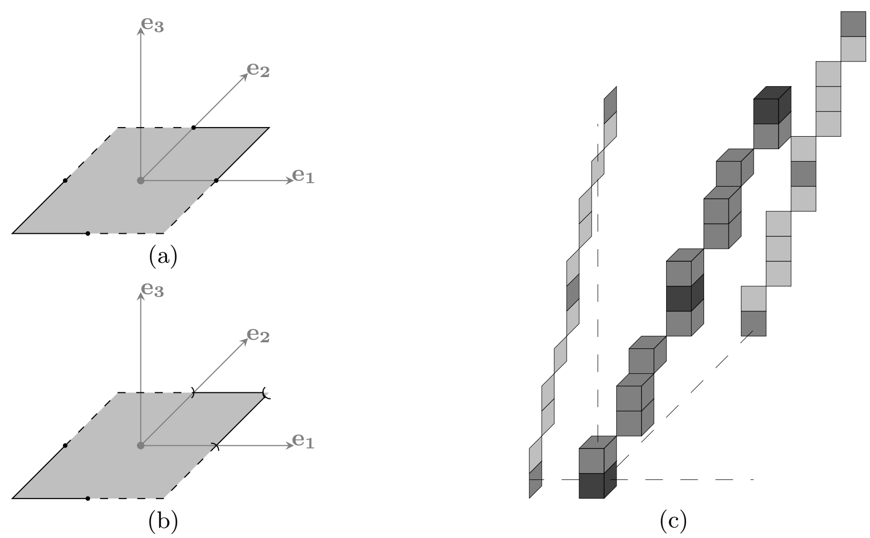


Fig. 3. (a) $\mathcal{V}_{3D}$, (b) The arbitrary point selection which leads to inequalities in (10) , (c) The resulting naive representation of the line through the origin and directed by $(2, 3, 6)$

From the arithmetic definition (10), we design a simple drawing algorithm described in Algorithm 1. We use the three components (p_1, p_2, p_3) of the cross product $\mathbf{v} \times \mathbf{n}$ to evaluate the distance from $\mathbf{n} = (i, j, k)$ to the line, and not only p_1 and p_2 as in the arithmetic definition. p_3 allows us to determine incrementally which of the inequality should be considered large without studying the sign of p_1 and p_2. First, the algorithm is initiated with a trivial point of the discrete set, $(0, 0, 0)$ for which $\mathbf{v} \times \mathbf{n} = \mathbf{0}$. Then k will be incremented until it reaches the value c. At each step, four conditions are successively evaluated. Both first are true if the bounds of inequalities in (10) are not concerned. They check if the current point (i, j, k) is close to the line or if it has to be updated by incrementing either i or j or the both. The two last conditions concern the bound of the inequalities in (10). To choose between them without studying the sign of p_1 or p_2, we just study the distance to the line. When $|p_1| = |p_1 - c|$, changing p_1 has no influence on the distance to the line. Moreover, p_2 is fixed and so do not change the distance either. So, in order to minimize the distance, it is then sufficient to minimize p_3. That's what is done with the two last conditions. The exclusion of the point $\mathbf{n} = (i, j, k) \in \mathbb{Z}^3$ such that $(bk - cj, ci - ak) = (\frac{c}{2}, \frac{c}{2})$ is a consequence of the strict inequalities $|p_3| > |p_3 + a|$ and $|p_3| > |p_3 - b|$.

Algorithm 1 Naive 3-dimensional discrete line drawing.

Input : $\mathbf{v} = (a, b, c) \in \mathbb{N}^3$, $0 \leq a, b \leq c$, $c \neq 0$ and $gcd(a, b, c) = 1$.
Output : $\mathrm{D}_{3D}(\mathbf{v})$, the naive 3-dimensional discrete line trough the origin and directed by $\mathbf{v}$.

```
i = 0, j = 0, k = 0;
p1 = 0, p2 = 0, p3 = 0;
select(i, j, k);
for k = 1 to c do
  p1 = p1 + b;
  p2 = p2 − a;
  if |p1| > |p1 − c| then
    p1 = p1 − c;
    p3 = p3 + a;
    j + +;
  end if
  if |p2| > |p2 + c| then
    p2 = p2 + c;
    p3 = p3 − b;
    i + +;
  end if
  if |p1| = |p1 − c| and |p3| > |p3 + a| then
    p1 = p1 − c;
    p3 = p3 + a;
    j + +;
  end if
  if |p2| = |p2 + c| and |p3| > |p3 − b| then
    p2 = p2 + c;
    p3 = p3 − b;
    i + +;
  end if
  select(i, j, k);
end for
```

5 Conclusion

In the present paper, we have proposed a definition of naive discrete lines in the 3-dimensional space and given geometric, arithmetic and algorithmic characterizations. The resulting set is the minimal 0-connected set of the closest integer points to a Euclidean line. This is a significant property since we expect from a discretization that it approximates as close as possible its Euclidean equivalent. Previous definitions are unable to fulfill this requirement. Indeed, in order to simplify the original 3-dimensional problem, they reduce it to the determination of the discrete points belonging to two naive lines in the 2-dimensional space and thus loose relationship between the different directions of the space. Our definition provide naive 3-dimensional discrete line with new geometric properties. We recover the intrinsic symmetry of line in case where $\mathrm{D}_{3D}(\mathbf{v})$ does not contain simple points. At the opposite, the projections on the planes normal to the vectors of the basis do not correspond to any particular discrete sets. Consequently, the representation of discrete lines as intersections of discrete planes do not seem compatible with our approach.

The study of 3-dimensional discrete line not only concern naive ones. The determination of the best k-connected approximation of a Euclidean line is also of interest. In the same way, trying to extend results to the d-dimensional case could confirm or invalidate our approach.

The closed naive model allows discretizations of hyperplanes with geometric and topological properties. It seems that it is also the case for the largest class of $(d-1)$-dimensional objects as hyperspheres. At the opposite, it leads to nothing when apply on objects of other dimensions. The appropriated discretization model certainly depends on the dimension of the considered object. It would be interesting to applied our discretization scheme to other planar objects like circles.

References

1. Reveillès, J.P.: Géométrie discrète, calcul en nombres entiers et algorithmique. Thèse d'Etat, Université Louis Pasteur, Strasbourg (1991)
2. Andres, E., Acharya, R., Sibata, C.: Discrete analytical hyperplanes. CVGIP: Graphical Models and Image Processing **59** (1997) 302–309
3. Kaufman, A., Shimony, E.: 3d scan-conversion algorithms for voxel-based graphics. In: SI3D '86: Proceedings of the 1986 workshop on Interactive 3D graphics, New York, NY, USA, ACM Press (1987) 45–75
4. Cohen-Or, D., Kaufman, A.: 3d line voxelization and connectivity control. IEEE Comput. Graph. Appl. **17** (1997) 80–87
5. Figueiredo, O., Reveilles, J.: A contribution to 3d digital lines (1995)
6. Figueiredo, O., Reveillès, J.P.: New results about 3D digital lines. In Melter, R.A., Wu, A.Y., Latecki, L., eds.: Vision Geometry V. Volume 2826. (1996) 98–108
7. Debled-Rennesson, I.: Etude et reconnaissance des droites et plans discrets. Thèse de Doctorat, Universit Louis Pasteur, Strasbourg. (1995)
8. Andres, E.: Discrete linear objects in dimension n: the standard model. Graphical Models **65** (2003) 92–111
9. Brimkov, V., Andres, E., Barneva, R.: Object discretizations in higher dimensions. Pattern Recognition Letters **23** (2002) 623–636
10. Debled-Rennesson, I., Reveillès, J.P.: A new approach to digital planes. In: Vision Geometry III, Proc. SPIE. Volume 2356., Boston, USA (1994)
11. Berthe, V., Fiorio, C., Jamet, D.: Generalized functionality for arithmetic discrete planes. In: DGCI'05. (2005)
12. Bresenham, J.: Algorithm for computer control of a digital plotter. IBM Systems Journal **4** (1965) 25–30
13. Andres, E.: Modélisation Analytique Discrète d'Objets Géométriques. Habilitation à diriger des recherches, UFR Sciences Fondamentale et Appliquées - Université de Poitiers (France) (2000)
14. Brimkov, V., Barneva, R.: Graceful planes and lines. Theoritical Computer Science **283** (2002) 151–170