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# Arithmetic distributions of convergents arising from Jacobi-Perron algorithm

Valérie Berthé\*, Hitoshi Nakada and Rie Natsui†

## Abstract

We study the distribution modulo  $m$  of the convergents associated with the  $d$ -dimensional Jacobi-Perron algorithm for a.e. real numbers in  $(0, 1)^d$  by proving the ergodicity of a skew product of the Jacobi-Perron transformation; this skew product was initially introduced in [6] for regular continued fractions.

## 1 Introduction

For an irrational number  $x$ ,  $0 < x < 1$ , we denote by  $\frac{p_n}{q_n}$  the  $n$ -th convergent of  $x$ , which is defined by the regular continued fraction expansion coefficients of  $x$ . In 1988, H. Jager and P. Liardet [6] studied the distribution properties of the pairs  $(p_n, q_n)$  modulo  $m$ . These properties were originally considered by P. Szüsz in [16], and then by R. Moeckel [7] who used the ergodicity of geodesic flows over the modular surfaces: more precisely, they proved that given any positive integer  $m \geq 2$ , for a.e.  $x$ , the sequence  $\{(p_n, q_n) : n \geq 1\}$  is equidistributed modulo  $m$  over the set  $\{(p, q) \in \mathbb{Z}_m^2 : \langle p, q \rangle = \mathbb{Z}_m\}$ , where  $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$  and where the notation  $\langle p_1, \dots, p_k \rangle$  stands for the subgroup of  $\mathbb{Z}_m$  generated by the elements  $p_1, \dots, p_k$ . To prove this property, H. Jager and P. Liardet considered in [6] the group of  $2 \times 2$  matrices with entries from  $\mathbb{Z}_m$  and determinant  $\pm 1$ , that is,

$$G(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_m, ad - bc = \pm 1 \right\}.$$

It is possible to show that  $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : a \in \mathbb{Z}_m \right\}$  generates  $G(m)$ . This fact implies the ergodicity of a  $G(m)$ -extension (a skew product indeed) of the continued fraction transformation. The equidistribution property of  $\{(p_n, q_n) : n \geq 1\}$  modulo  $m$  is then an easy consequence of the individual ergodic theorem.

A natural extension of this skew product was then introduced in [3] to deduce the distribution of the approximation coefficients associated with the

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continued fraction algorithm; these results were also extended to the so-called  $S$ -expansions, in the sense of [4]; see also for connected results [1] and [10].

The aim of the present paper is to generalize these equidistribution results to the  $d$ -dimensional Jacobi-Perron algorithm. Note that the 1-dimensional Jacobi-Perron algorithm reduces to the regular continued fraction algorithm.

Let us start with the definition of the Jacobi-Perron algorithm. We fix a positive integer  $d \geq 2$ . Let  $X = [0, 1)^d$  be endowed with the Borel  $\sigma$ -algebra  $\mathbb{B}$ . We first define the map  $T : X \rightarrow X$  by

$$T(\mathbf{x}) = T((x_1, x_2, \dots, x_d)) = \left( \frac{x_2}{x_1} - \left[ \frac{x_2}{x_1} \right], \dots, \frac{x_d}{x_1} - \left[ \frac{x_d}{x_1} \right], \frac{1}{x_1} - \left[ \frac{1}{x_1} \right] \right)$$

for  $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$  if  $x_1 \neq 0$ , and  $T(\mathbf{x}) = \mathbf{0}$ , otherwise;  $(X, T)$  is called the  $d$ -dimensional Jacobi-Perron algorithm. Notice that there exists a unique absolutely continuous invariant probability measure  $\mu$  for  $T$  which is equivalent to the Lebesgue measure (see for instance [13]).

We put for  $\mathbf{x}$  in  $X$  with  $x_1 \neq 0$

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}^{(0)}(\mathbf{x}) = (k_1, k_2, \dots, k_d) = \left( \left[ \frac{x_2}{x_1} \right], \left[ \frac{x_3}{x_1} \right], \dots, \left[ \frac{x_d}{x_1} \right], \left[ \frac{1}{x_1} \right] \right),$$

if  $x_1 = 0$ , we set  $\mathbf{k}(\mathbf{x}) = \mathbf{0}$ ; we similarly define

$$\mathbf{k}^{(s)}(\mathbf{x}) = \left( k_1^{(s)}, k_2^{(s)}, \dots, k_d^{(s)} \right) = \mathbf{k}(T^{s-1}(\mathbf{x})) \quad \text{for } s \geq 1.$$

We then associate  $(x_1, x_2, \dots, x_d)$  with the column vector  $\begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}$  and consider the following matrix

$$P = \begin{pmatrix} -k_1 & 1 & 0 & \dots & 0 \\ -k_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_d & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (1)$$

Then  $T((x_1, x_2, \dots, x_d))$  corresponds to  $P \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}$ . To construct the sequence

$$\left\{ \left( \frac{p_1^{(k)}}{q^{(k)}}, \dots, \frac{p_d^{(k)}}{q^{(k)}} \right) : k \geq 1 - d \right\}$$

of *simultaneous approximation convergents* of  $\mathbf{x}$  from the  $d$ -dimensional Jacobi-Perron algorithm, we first define  $Q^{(0)}$  as the  $(d+1) \times (d+1)$  identity matrix

$I_{d+1}$ ; we then define recursively  $Q^{(n)}$  for  $n \geq 1$  as

$$Q^{(n)} := Q^{(n-1)} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & k_1^{(n)} \\ 0 & 1 & \dots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_d^{(n)} \end{pmatrix}.$$

We thus set for  $n \geq 1$

$$Q^{(n)} = \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \dots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \dots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \dots & p_d^{(n-1)} & p_d^{(n)} \\ q^{(n-d)} & q^{(n-d+1)} & \dots & q^{(n-1)} & q^{(n)} \end{pmatrix}.$$

Let us observe that

$$Q^{(1)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & k_1 \\ 0 & 1 & \dots & 0 & k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_d \end{pmatrix} = P^{-1}. \quad (2)$$

It is well-known that for any  $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$ ,

$$\lim_{n \rightarrow \infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for } 1 \leq i \leq d$$

holds.

In this paper, we prove that for almost every  $\mathbf{x} \in X$  the sequences of vectors  $\{(q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) : n \geq 1\}$  and  $\{(p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) : n \geq 1\}$  are both equidistributed modulo  $m$  for any integer  $m \geq 2$ .

More precisely we put

$$\tilde{\mathbb{Z}}_m^{d+1} = \{(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1} : \langle \alpha_1, \alpha_2, \dots, \alpha_{d+1} \rangle = \mathbb{Z}_m\}$$

and

$$c_m = \#\tilde{\mathbb{Z}}_m^{d+1} \quad (\text{the cardinality of } \tilde{\mathbb{Z}}_m^{d+1}).$$

One easily sees that

$$\begin{aligned} c_m &= \varphi_{d+1}(m) \\ &= \#\{(a_1, a_2, \dots, a_{d+1}) \in \{1, \dots, m\}^{d+1} : \gcd(a_1, \dots, a_{d+1}, m) = 1\}, \end{aligned} \quad (3)$$

where  $\varphi_{d+1}$  denotes the Jordan totient function of order  $d+1$ ; we thus have (see for instance [15] or [11])

$$c_m = m^{d+1} \prod_{p|m} (1 - p^{-(d+1)}),$$

where the notation  $\prod_{p|m}$  stands in all that follows for the product over the prime numbers  $p$  that divide  $m$ . We then have the following:

**Theorem 1.** *Let  $m \geq 2$  be a nonnegative integer. For almost every  $\mathbf{x} \in X$  and for any  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$ , we have*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \pmod{m}\}}{N} \\ &= \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \pmod{m}\}}{N} \\ &= \frac{1}{c_m} = \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1} \prod_{p|m} (1 - p^{-(d+1)})}. \end{aligned}$$

To prove this theorem, we consider for a given integer  $m \geq 2$ , the group  $G(m)$  defined in a similar way as in [6]:

$$G(m) = \begin{cases} SL(d+1, \mathbb{Z}_m) & \text{if } d \text{ is even,} \\ SL_{\pm}(d+1, \mathbb{Z}_m) & \text{if } d \text{ is odd,} \end{cases}$$

where  $SL(d+1, \mathbb{Z}_m)$  stands for the matrices with entries in  $\mathbb{Z}_m$  with determinant 1, whereas  $SL_{\pm}(d+1, \mathbb{Z}_m)$  stands for the matrices with entries in  $\mathbb{Z}_m$  with determinant  $\pm 1$ . Let us recall that (see for instance [11] or [9]) that

$$\#SL(d+1, \mathbb{Z}_m) = m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) = m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m).$$

Let  $C_m$  denote the cardinality of  $G(m)$ . Since  $SL(d+1, \mathbb{Z}_m)$  is a subgroup of  $SL_{\pm}(d+1, \mathbb{Z}_m)$  of index 2 if  $d$  is odd and  $m \neq 2$ , one thus gets

$$C_m = \begin{cases} \begin{aligned} & m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) \\ &= m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m) \end{aligned} & \text{if } d \text{ is even or } m = 2 \\ \begin{aligned} & 2m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) \\ &= 2m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m) \end{aligned} & \text{if } d \text{ is odd and } m \neq 2. \end{cases} \quad (4)$$

We identify  $Q^{(1)}$  with the matrix with coefficients in  $\mathbb{Z}_m$  obtained by reducing modulo  $m$  its entries. Here we note that  $\det Q^{(1)} = 1$  or  $-1$  if  $d$  is respectively even or odd, which implies that  $Q^{(1)}$  belongs to the group  $G(m)$ , whatever may be the parity of  $d$ .

We define the map  $T_m$  on  $X \times G(m)$  by

$$T_m(\mathbf{x}, A) = (T(\mathbf{x}), AQ^{(1)});$$

$T_m$  is said to be a  $G(m)$ -extension of the map  $T$ .

We define the probability measure  $\delta_m$  on  $G(m)$  by  $(\frac{1}{C_m}, \dots, \frac{1}{C_m})$ . Then it is easy to see that  $\mu \times \delta_m$  is an invariant probability measure for  $T_m$ . Our

question is whether  $(T_m, \mu \times \delta_m)$  is ergodic or not. In Section 2, we show that the set of matrices of the form (2) (reduced modulo  $m$ ) generates  $G(m)$ . Then in Section 3, we prove the ergodicity of  $T_m$ , from which we deduce the following proposition and then Theorem 1 (in the same way as in [6]):

**Proposition 1.** *For a.e.  $\mathbf{x} \in X$  and any  $A \in G(m)$ ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : Q^{(n)} \equiv A \pmod{m}\} = \frac{1}{C_m}.$$

Finally we have the following

**Corollary 1.** *For a.e.  $\mathbf{x} \in X$  and any  $a \in \mathbb{Z}_m$*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : q^{(n)} \equiv a \pmod{m}\} = \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)}.$$

In all that follows, we simply denote by  $0, 1, \dots, m-1$  the elements of  $\mathbb{Z}_m$  if it is clear that the elements are in  $\mathbb{Z}_m$  according to the context. In this case, one has obviously  $m-1 = -1$ .

## 2 Basic properties of $G(m)$

We first define

$$\Gamma_m = \{A \in SL(d+1, \mathbb{Z}) : A \equiv I_{d+1} \pmod{m}\}.$$

Then it is well-known that

$$SL(d+1, \mathbb{Z}_m) \cong \Gamma_m \backslash SL(d+1, \mathbb{Z})$$

e.g., see G. Shimura [15], p. 21. From this property, it easily follows that

$$SL_{\pm}(d+1, \mathbb{Z}_m) \cong \Gamma_m \backslash GL(d+1, \mathbb{Z}).$$

We respectively say that a  $(d+1) \times (d+1)$  matrix with  $\mathbb{Z}$  (or  $\mathbb{Z}_m$ )-entries of the form (2) is a J-P matrix, and that a matrix of the form (1) is a J-P\* matrix; a J-P matrix is the inverse of a J-P\* matrix.

In the sequel of this section, we show that the monoid generated by the set of J-P matrices with  $\mathbb{Z}_m$ -entries is equal to  $G(m)$ :

**Theorem 2.** *For any  $B \in G(m)$ , there exist J-P matrices  $A_1, A_2, \dots, A_s$  such that*

$$B = A_1 A_2 \cdots A_s.$$

For this purpose, we first need some notation and some preliminary lemmas. We put

$$\Delta(k_1, k_2, \dots, k_d) := \begin{pmatrix} k_1 & 1 & 0 & \dots & 0 \\ k_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_d & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix};$$

in particular,

$$\Delta = \Delta(0, 0, \dots, 0) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We thus have

$$(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1})\Delta = (\mathbf{a}_{d+1}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d), \quad (5)$$

where  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1}$  are either elements in  $\mathbb{Z}_m$  or  $(d+1)$ -dimensional vectors with  $\mathbb{Z}_m$ -entries. Let us notice that the matrices  $\Delta(k_1, k_2, \dots, k_d)$  are J-P\* matrices.

**Lemma 1.** *For any  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$ , there exist J-P matrices  $A_1, A_2, \dots, A_s$  with  $\mathbb{Z}_m$ -entries such that*

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) = (0, \dots, 0, 1)A_1A_2 \dots A_s,$$

where  $s$  depends on  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$ .

**Proof of Lemma 1.** We define the following natural order  $\prec$  on  $\mathbb{Z}_m$  by

$$0 \prec 1 \prec \dots \prec m-1.$$

Let  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$ . We denote by  $\alpha^*$  the element in  $\mathbb{Z}_m$  such that  $\langle \alpha^* \rangle = \langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle$ . Let us prove by induction on  $\alpha^*$  (considered then as an element in  $\{1, \dots, m\}$ ) that there exists a finite number of J-P\* matrices  $\Delta_1, \dots, \Delta_t$  and  $(\alpha'_1, \dots, \alpha'_d) \in \mathbb{Z}_m^d$  such that

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \Delta_1 \cdots \Delta_t = (1, \alpha'_1, \dots, \alpha'_d).$$

If  $\alpha^* = 1$ , then  $\langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle = \mathbb{Z}_m$  and there exist  $k_1, \dots, k_d \in \mathbb{Z}_m$  such that  $\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} = 1$ . We thus have

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \Delta(k_1, k_2, \dots, k_d) = (1, \alpha_1, \dots, \alpha_d).$$

Suppose now that  $\alpha^* \neq 1$ . Since  $\langle \alpha_1, \alpha_2, \dots, \alpha_{d+1} \rangle \neq \langle \alpha^* \rangle$ , there exist  $k_1, \dots, k_d \in \mathbb{Z}_m$  such that  $0 \prec \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} \prec \alpha^*$ . We thus have

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \Delta(k_1, k_2, \dots, k_d, 1) = \left( \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \dots, \alpha_d \right).$$

We can now conclude inductively since  $\langle \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \dots, \alpha_d \rangle = \mathbb{Z}_m$ .

Now we have

$$\begin{aligned} (1, \alpha'_1, \dots, \alpha'_d) \cdot \Delta(-\alpha'_d, 0, \dots, 0) \cdot \Delta(0, -\alpha'_{d-1}, 0, \dots, 0) \cdots \Delta(0, \dots, 0, -\alpha'_1) \\ = (0, \dots, 0, 1). \end{aligned}$$

Since a J-P\* matrix is the inverse of a J-P matrix, we get the assertion of this lemma.  $\square$

The following lemmas are essential and easily proved.

**Lemma 2.** For any  $(d+1)$ -dimensional vectors with  $\mathbb{Z}_m$ -entries  $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1})$ , we have

$$\begin{aligned} & (\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}, \dots, \mathbf{a}_d, \mathbf{a}_{d+1}) \cdot \Delta(0, \dots, 0, \overset{i}{\underset{\vee}{-1}}, 0, \dots, 0) \cdot \Delta^{d-i} \\ & \cdot \Delta(0, \dots, 0, \overset{d+1-i}{\underset{\vee}{1}}, 0, \dots, 0) \cdot \Delta^{i-1} \cdot \Delta(0, \dots, 0, \overset{i}{\underset{\vee}{-1}}, 0, \dots, 0) \cdot \Delta^d \\ & = (\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_{d+1}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_d, -\mathbf{a}_i). \end{aligned}$$

**Lemma 3.** We have

$$\begin{aligned} & (\mathbf{a}_1, \dots, \mathbf{a}_{d+1}) \cdot \Delta(0, \dots, 0, 1) \cdot \Delta(-1, 0, \dots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \dots, 0, 1) \\ & = (\mathbf{a}_d, \mathbf{a}_1, \dots, \mathbf{a}_{d-1}, -\mathbf{a}_{d+1}). \end{aligned}$$

In particular, when  $d$  is odd

$$\begin{aligned} & (\mathbf{a}_1, \dots, \mathbf{a}_{d+1}) \cdot [\Delta(0, \dots, 0, 1) \cdot \Delta(-1, 0, \dots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \dots, 0, 1)]^d \\ & = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d, -\mathbf{a}_{d+1}). \end{aligned}$$

**Proof of Theorem 2.** Let us fix

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1(d+1)} \\ b_{21} & b_{22} & \dots & b_{2(d+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(d+1)1} & b_{(d+1)2} & \dots & b_{(d+1)(d+1)} \end{pmatrix}$$

in  $G(m)$ . We want to prove that there exist J-P\* matrices  $\Delta_1, \dots, \Delta_s$  such that

$$B \Delta_1 \cdots \Delta_s = I_{d+1},$$

which implies immediately the desired result. For that purpose, let us prove by induction on  $1 \leq j \leq d$  that there exist J-P\* matrices  $\Delta_1, \dots, \Delta_{s_j}$  such that

$$B_j := B \Delta_1 \cdots \Delta_{s_j} = \left( \begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right), \quad (6)$$

where  $I_j$  is the  $j \times j$  identity matrix. Indeed, if this property holds for  $j = d$ , then we obtain that there exist J-P\* matrices  $\Delta_1, \dots, \Delta_{s_d}$  such that

$$B \Delta_1 \cdots \Delta_{s_d} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \dots & 0 & \pm 1 \end{pmatrix}.$$



If  $d$  is even, then all J-P\* matrices are of determinant 1. Thus the  $(d+1, d+1)$ -entry of the right hand side is equal to 1. If  $d$  is odd and the  $(d+1, d+1)$ -entry of the right hand side is equal to  $-1$ , then by Lemma 3 we can reduce it to 1 by application of J-P\* matrices. In either case, we get the desired result.

It thus remains to prove the induction property. Let us first prove that it holds for  $j = 1$ . Since  $\det B = \pm 1$ , then  $\langle b_{11}, b_{12}, \dots, b_{1(d+1)} \rangle = \mathbb{Z}_m$ , and there thus exist J-P\* matrices  $\Delta_1, \dots, \Delta_{s_1-1}$  such that

$$(b_{11}, b_{12}, \dots, b_{1(d+1)}) \Delta_1 \cdots \Delta_{s_1-1} = (0, \dots, 0, 1)$$

by Lemma 1. Thus

$$B \Delta_1 \cdots \Delta_{s_1-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ b_{11}^{(1)} & \cdots & b_{1d}^{(1)} & b_{1(d+1)}^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ b_{d1}^{(1)} & \cdots & b_{dd}^{(1)} & b_{d(d+1)}^{(1)} \end{pmatrix}.$$

We set

$$B^{(1)} = \begin{pmatrix} b_{11}^{(1)} & \cdots & b_{1d}^{(1)} \\ \vdots & \ddots & \vdots \\ b_{d1}^{(1)} & \cdots & b_{dd}^{(1)} \end{pmatrix}.$$

Since  $\det B^{(1)} = \pm 1$ , then there exist  $k_1, \dots, k_d \in \mathbb{Z}_m$  such that

$$B \Delta_1 \cdots \Delta_{s_1-1} \Delta(k_1, \dots, k_d) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_{11}^{(1)} & \cdots & b_{1d}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{d1}^{(1)} & \cdots & b_{dd}^{(1)} \end{pmatrix}.$$

It remains to set  $\Delta_{s_1} = \Delta(k_1, \dots, k_d)$  to conclude the proof of the induction property for  $j = 1$ .

Let us assume now that the induction property holds for  $1 \leq j \leq d-1$  (if  $d = 1$ , the proof is finished); one thus deduces that the determinant of  $B^{(j)}$  (defined in (6)) is equal to  $\pm 1$ . We set

$$B^{(j)} = \begin{pmatrix} b_{11}^{(j)} & \cdots & b_{1(d+1-j)}^{(j)} \\ \vdots & \ddots & \vdots \\ b_{(d+1-j)1}^{(1)} & \cdots & b_{(d+1-j)(d+1-j)}^{(1)} \end{pmatrix}.$$

Let us divide the induction proof into two steps for clarity issues.

**Step 1.** Let us first prove that we can find J-P\* matrices  $\Delta_{s_j+1}, \dots, \Delta_{s_j+t}$

such that  $B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t}$  is equal to

$$\left( \begin{array}{ccc|c|ccc} & & \mathbf{0} & & & & \\ \hline & 0 & \cdots & 0 & 1 & & \\ \hline g_{1(l+1)}^{(j+1)} & \cdots & g_{1(d-j)}^{(j+1)} & & g_{1(d+1-j)}^{(j+1)} & & \\ \vdots & \ddots & \vdots & & \vdots & & \\ g_{(d-j)(l+1)}^{(j+1)} & \cdots & g_{(d-j)(d-j)}^{(j+1)} & & g_{(d-j)(d+1-j)}^{(j+1)} & & \\ \hline & & & I_j & & & \\ \hline & & & 0 & & & \\ \hline & & & & & \mathbf{0} & \\ \hline & & & & & 0 & \cdots & 0 \\ \hline & & & & & g_{11}^{(j+1)} & \cdots & g_{1l}^{(j+1)} \\ & & & & & \vdots & \ddots & \vdots \\ & & & & & g_{(d-j)1}^{(j+1)} & \cdots & g_{(d-j)l}^{(j+1)} \\ \hline \end{array} \right) \quad (7)$$

for some  $l$ ,  $0 \leq l < d - j + 1$ .

According to the proof of Lemma 1, we can find  $(d - j + 1) \times (d - j + 1)$  J-P\* matrices  $\Delta(k_1^{(1)}, \dots, k_{d-j}^{(1)}), \dots, \Delta(k_1^{(t)}, \dots, k_{d-j}^{(t)})$  such that

$$(b_{11}^{(j)}, \dots, b_{1(d+1-j)}^{(j)}) \Delta(k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdots \Delta(k_1^{(u)}, \dots, k_{d-j}^{(u)}) = (0, \dots, 0, 1).$$

Now

$$\left( \begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, *, \dots, *) = \left( \begin{array}{c|c|c} \mathbf{0} & I_j & \mathbf{0} \\ \hline * & \mathbf{0} & * \end{array} \right),$$

and one checks more generally that for  $0 \leq v \leq d - j$

$$\begin{aligned} & \left( \begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, *, \dots, *) \cdot \Delta(\underbrace{*0, \dots, 0}_j, *, \dots, *) \cdots \\ & \cdots \Delta(\underbrace{*, \dots, *}_v, \underbrace{0, \dots, 0}_j, *, \dots, *) = \left( \begin{array}{c|c|c} \overbrace{\mathbf{0} \cdots \mathbf{0}}^{v+1} & I_j & \mathbf{0} \\ \hline \underbrace{* \cdots *}_{v+1} & \mathbf{0} & * \end{array} \right). \end{aligned}$$

One thus gets that if  $u \leq d - j + 1$ , then

$$\begin{aligned} & \left( \begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdot \Delta(\underbrace{k_1^{(2)}, 0, \dots, 0}_j, k_2^{(2)}, \dots, k_{d-j}^{(2)}) \cdots \\ & \cdots \Delta(k_1^{(u)}, \dots, k_{d-j}^{(u)}, \underbrace{0, \dots, 0}_j) \end{aligned}$$

has the desired form (7).

Now if  $u \geq d - j + 2$ , then using (5), one gets

$$\left( \begin{array}{c|c} \mathbf{0} & I_j \\ \hline * & \mathbf{0} \end{array} \right) \Delta^j = \left( \begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & * \end{array} \right),$$

and suitable insertions of  $\Delta^j$  such as

$$\begin{aligned}
& \left( \begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdot \Delta(k_1^{(2)}, \underbrace{0, \dots, 0}_j, k_2^{(2)}, \dots, k_{d-j}^{(2)}) \\
& \cdots \Delta(k_1^{(d-j+1)}, \dots, k_{d-j}^{(d-j+1)}, \underbrace{0, \dots, 0}_j) \cdot \Delta^j \cdot \Delta(\underbrace{0, \dots, 0}_j, k_1^{(d-j+2)}, \dots, k_{d-j}^{(d-j+2)}) \\
& \cdots \Delta(k_1^{(2(d-j+1))}, \dots, k_{d-l+j}^{(2(d-j+1))}, \underbrace{0, \dots, 0}_j) \cdot \Delta^j \cdot \Delta(\underbrace{0, \dots, 0}_j, k_1^{2(d-j+1)+1}, \dots, k_{d-j}^{2(d-j+1)+1}) \\
& \cdots \Delta(k_1^{(u)}, \dots, k_{d-l+j}^{(u)}, \underbrace{0, \dots, 0}_j, k_{(d-l+j+1)}^{(u)}, \dots, k_{d-j}^{(u)})
\end{aligned}$$

provide the desired form (7), which ends the proof of Step 1.

**Step 2.** By (5),

$$\begin{aligned}
& B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t} \cdot \Delta^{l+j} \\
& = \left( \begin{array}{c|cccc|c|cccc} & & & & & I_j & & & & \mathbf{0} \\ \hline & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & \\ \hline & g_{1(l+1)}^{(j+1)} & \cdots & g_{1(d-j)}^{(j+1)} & g_{1(d+1-j)}^{(j+1)} & & g_{11}^{(j+1)} & \cdots & g_{1l}^{(j+1)} & \\ & \vdots & \ddots & \vdots & \vdots & \mathbf{0} & \vdots & \ddots & \vdots & \\ \hline & g_{(d-j)(l+1)}^{(j+1)} & \cdots & g_{(d-j)(d-j)}^{(j+1)} & g_{(d-j)(d+1-j)}^{(j+1)} & & g_{(d-j)1}^{(j+1)} & \cdots & g_{(d-j)l}^{(j+1)} & \end{array} \right) \cdot \Delta^j \\
& = \left( \begin{array}{c|cccc} I_j & & & & \mathbf{0} \\ \hline & 0 & \cdots & 0 & 1 \\ \hline \mathbf{0} & g_{11} & \cdots & g_{1(d-j)} & g_{1(d-j+1)} \\ & \vdots & \ddots & \vdots & \vdots \\ & g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d-j+1)} \end{array} \right).
\end{aligned}$$

We put

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1(d-j)} \\ \vdots & \ddots & \vdots \\ g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} \end{pmatrix}.$$

Since the determinant of  $G$  is equal  $\pm 1$ , there exist  $k'_1, \dots, k'_{d-j} \in \mathbb{Z}_m$  such that

$$\begin{aligned} & \left( \begin{array}{c|cccc} I_j & & & \mathbf{0} & \\ \hline & 0 & \cdots & 0 & 1 \\ \mathbf{0} & g_{11} & \cdots & g_{1(d-j)} & g_{1(d-j+1)} \\ & \vdots & \ddots & \vdots & \vdots \\ & g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d-j+1)} \end{array} \right) \cdot \Delta(\underbrace{0, \dots, 0}_j, k'_1, \dots, k'_{d-j}) \\ &= \left( \begin{array}{c|cc} 0 & & \\ \vdots & I_j & \mathbf{0} \\ 0 & & \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & & & & & \\ \hline \vdots & \mathbf{0} & & & G & & \\ 0 & & & & & & \end{array} \right). \end{aligned}$$

By applying (5), we get

$$\left( \begin{array}{c|cc} 0 & & \\ \vdots & I_j & \mathbf{0} \\ 0 & & \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & & & & & & \\ \hline \vdots & \mathbf{0} & & & G & & \\ 0 & & & & & & \end{array} \right) \cdot \Delta^d = \left( \begin{array}{c|cc} I_j & \mathbf{0} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \hline \mathbf{0} & & & G & & & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array} \right).$$

We thus have proved that there exist J-P\* matrices  $\Delta_{s_j+t+1}, \dots, \Delta_{s_j+t'}$  such that

$$B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t'} = \left( \begin{array}{c|cc} I_j & \mathbf{0} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \hline \mathbf{0} & & & G & & & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \end{array} \right).$$

It remains to apply Lemma 2; there thus exist J-P\* matrices  $\Delta_{s_j+t'+1}, \dots, \Delta_{s_{j+1}}$  such that

$$B_j \Delta_{s_j+1} \cdots \Delta_{s_{j+1}} = \left( \begin{array}{c|c} I_{j+1} & \mathbf{0} \\ \hline \mathbf{0} & B^{(j+1)} \end{array} \right),$$

where we put

$$B^{(j+1)} = \begin{pmatrix} g_{12} & g_{13} & \cdots & g_{1(d-j)} & -g_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(d-j)2} & g_{(d-j)3} & \cdots & g_{(d-j)(d-j)} & -g_{(d-j)1} \end{pmatrix},$$

which concludes the induction proof.  $\square$

### 3 Ergodicity of $T_m$ and proof of Theorem 1

#### 3.1 Ergodicity

Let us recall some fundamental facts about Jacobi-Perron algorithm. For an integer vector  $\mathbf{a} = (a_1, a_2, \dots, a_d)$  with  $a_i \geq 0$  for  $1 \leq i \leq d$ , we put

$$X_{\mathbf{a}} = \{\mathbf{x} \in X : \mathbf{k}(\mathbf{x}) = \mathbf{a}\}.$$

Then we see that  $X_{\mathbf{a}} \neq \emptyset$  if and only if  $0 \leq a_i \leq a_d$  for any  $1 \leq i \leq d-1$  and  $a_d > 0$ . For a finite sequence of integer vectors  $\{\mathbf{a}^{(l)} = (a_1^{(l)}, a_2^{(l)}, \dots, a_d^{(l)}), 1 \leq l \leq n\}$  such that  $a_i^{(l)} \leq a_d^{(l)}$ ,  $1 \leq i \leq d-1$ , and  $a_d^{(l)} > 0$  for  $1 \leq l \leq n$ , we define the cylinder set of rank  $n$  by

$$X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}} = \{\mathbf{x} \in X : \mathbf{k}^{(l)}(\mathbf{x}) = \mathbf{a}^{(l)} \text{ for } 1 \leq l \leq n\}.$$

A cylinder set  $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}$  is said to be proper (or full) if  $T^n(X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}) = X$ . It is easy to see that  $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}$  is proper if  $a_i^{(l)} < a_d^{(l)}$  for all  $1 \leq l \leq n$  and  $1 \leq i \leq d-1$ . The following is essential.

**Lemma 4.** *For almost every  $\mathbf{x} \in X$  there exists a sequence of positive integers  $n_1 < n_2 < \dots$  such that  $X_{\mathbf{k}^{(1)}(\mathbf{x}) \dots \mathbf{k}^{(n_i)}(\mathbf{x})}$  is proper for any  $i \geq 1$ .*

This shows the exactness of the dynamical system  $(X, T, \mu)$ ; the exactness means here that  $\bigcap_{n=1}^{\infty} T^{-n}\mathbb{B} = \{\emptyset, X\}$  ( $\mu$ -mod 0). In particular,  $(X, T, \mu)$  is ergodic and strong mixing. We refer to F. Schweiger [12] or [14] about the theory of Jacobi-Perron algorithm. Now we will show the ergodicity of  $T_m$ .

**Theorem 3.** *The skew product  $(X \times G(m), T_m, \mu \times \delta_m)$  is ergodic.*

**Proof.** For any non-empty cylinder set  $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}$ , we see from Proposition 2 in [14] that

$$\sup_{\mathbf{x} \in X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}} |DT^n(\mathbf{x})| < (d+1)^{d+1} \inf_{\mathbf{x} \in X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}} |DT^n(\mathbf{x})|, \quad (8)$$

where  $|DT^n|$  is the Jacobian of  $T^n$ . Suppose that  $\mathcal{M}$  is a  $T_m$ -invariant set of  $(\mu \times \delta_m)$ -positive measure. Since  $T_m$  acts as  $T$  on the first coordinate, the ergodicity of  $T$  shows

$$\{\mathbf{x} \in X : (\mathbf{x}, A) \in \mathcal{M} \text{ for some } A \in G(m)\} = X \quad (\mu\text{-mod } 0).$$

Thus there exists  $A \in G(m)$  such that  $(X \times \{A\}) \cap \mathcal{M}$  has positive  $(\mu \times \delta_m)$ -measure. We fix such a set  $A$ . By the density theorem and Lemma 4, for a given sequence  $\varepsilon_i \searrow 0$  there exists a sequence of proper cylinder sets  $W_i$  of rank  $n_i$  and  $B \in G(m)$  such that for all  $i$

$$\frac{(\mu \times \delta_m)((W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(W_i \times \{A\})} > 1 - \varepsilon_i \quad (9)$$

and

$$A \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(1)} \\ 0 & 1 & \dots & 0 & a_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(1)} \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(2)} \\ 0 & 1 & \dots & 0 & a_2^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(2)} \end{pmatrix} \dots \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(n_i)} \\ 0 & 1 & \dots & 0 & a_2^{(n_i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(n_i)} \end{pmatrix} = B \pmod{m},$$

where  $(a_1^{(1)}, a_2^{(1)}, \dots, a_d^{(1)}), \dots, (a_1^{(n_i)}, a_2^{(n_i)}, \dots, a_d^{(n_i)})$  are sequences of integers which define  $W_i$ . From (8) we see that (9) implies

$$\frac{(\mu \times \delta_m)(T_m^{n_i}(W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(T_m^{n_i}(W_i \times \{A\}))} > 1 - (d+1)^{d+1} \varepsilon_i.$$

Since  $W_i$  is proper and  $\mathcal{M}$  is  $T_m$ -invariant, we conclude that

$$(X \times \{B\}) \cap \mathcal{M} = X \times \{B\} \pmod{0}.$$

From Theorem 2, for any  $C \in G(m)$  there exist J-P matrices  $A_1, \dots, A_s$  such that

$$C = BA_1 \dots A_s \pmod{m}$$

with

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(i)} \\ 0 & 1 & \dots & 0 & a_2^{(i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(i)} \end{pmatrix} \text{ for } 1 \leq i \leq s.$$

Moreover we can choose  $A_1, \dots, A_s$  so that the corresponding cylinder set  $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(s)}}$

with  $\mathbf{a}^{(i)} = \begin{pmatrix} a_1^{(i)} \\ \vdots \\ a_d^{(i)} \end{pmatrix}, 1 \leq i \leq s$ , is proper. This means

$$T_m^s(X \times \{B\}) \supset X \times \{C\}$$

and so  $\mathcal{M} = X \times G(m) \pmod{0}$ . Thus we get the assertion of the theorem.  $\square$

### 3.2 Proofs

We are now able to give proofs of Proposition 1, Theorem 1, and Corollary 1.

**Proof of Proposition 1.** Let us recall that  $C_m$  denotes the cardinality of  $G(m)$ . From Theorem 3 and the individual ergodic theorem, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : T_m^n(\mathbf{x}, B) \in X \times \{A\}\} = \frac{1}{C_m}$$

for  $(\mu \times \delta_m)$ -a.e.  $(\mathbf{x}, B)$ . In particular, it holds for  $(\mathbf{x}, I_{d+1})$  for  $\mu$ -a.e.  $\mathbf{x}$ . Since  $T_m^n(\mathbf{x}, I_{d+1}) = (T^n \mathbf{x}, Q^{(n)})$ , we get the assertion.  $\square$

**Proof of Theorem 1.**

For any  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$ , we denote by  $N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})}$  the number of elements in  $G(m)$  such that the  $(d+1)$ th row is  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$ . We will show that

$$N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})} = \mathcal{C}_m \cdot m^d, \quad (10)$$

where  $\mathcal{C}_m$  denotes the cardinality of  $SL(d, \mathbb{Z}_m)$  or  $SL_{\pm}(d, \mathbb{Z}_m)$  if  $d$  is even or odd, respectively. It is easy to see that

$$N_{(0, \dots, 0, 1)} = \mathcal{C}_m \cdot m^d. \quad (11)$$

From Lemma 1, we note that there always exists  $D \in G(m)$  such that the  $(d+1)$ th row is  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$  for any  $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$ .

For any matrix  $E$  of the form

$$\begin{pmatrix} * & & & \\ & \dots & & \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

$ED$  is of the form

$$\begin{pmatrix} * & & & \\ & \dots & & \\ \alpha_1 & \dots & \alpha_d & \alpha_{d+1} \end{pmatrix}.$$

This implies

$$N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})} \geq N_{(0, \dots, 0, 1)}.$$

On the other hand, for any matrix  $D'$  of the form

$$\begin{pmatrix} * & & & \\ & \dots & & \\ \alpha_1 & \dots & \alpha_d & \alpha_{d+1} \end{pmatrix},$$

$D' \cdot D^{-1}$  is of the form

$$\begin{pmatrix} * & & & \\ & \dots & & \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

which implies

$$N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})} \leq N_{(0, \dots, 0, 1)}.$$

Thus we have (10).

From Proposition 1 together with (10), we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \pmod{m}\}}{N} \\ = \frac{\mathcal{C}_m \cdot m^d}{\mathcal{C}_m} = \frac{1}{c_m} \quad \text{for } \mu\text{-a.e. } \mathbf{x}. \end{aligned}$$

Indeed one easily checks according to (3) and (4) that  $\frac{\mathcal{C}_m \cdot m^d}{\mathcal{C}_m} = \frac{1}{\mathcal{C}_m}$  holds. Since  $\mu$  is equivalent to the Lebesgue measure, this holds for a.e.  $\mathbf{x}$  with respect to the Lebesgue measure. If we consider the  $(d+1)$ th column, then the same argument shows the other equality. This completes the proof of Theorem 1.  $\square$

**Proof of Corollary 1.** For a given  $a \in \mathbb{Z}_m$ , let  $\Gamma_a(m)$  denote the cardinality of the subset of  $G(m)$  of matrices whose  $(d+1, d+1)$ -entry is equal to  $a$ .

Let us first assume that  $a$  and  $m$  are coprime. We then deduce from (11) that

$$\begin{aligned}\Gamma_a(m) &= \sum_{(\alpha_1, \dots, \alpha_d): \langle \alpha_1, \dots, \alpha_d, a \rangle = \mathbb{Z}_m} N(\alpha_1, \dots, \alpha_d, a) \\ &= \sum_{(\alpha_1, \dots, \alpha_d)} N(\alpha_1, \dots, \alpha_d, a) \\ &= m^{2d} \cdot \mathcal{C}_m.\end{aligned}$$

Let us assume now that  $m$  is a power of a prime divisor  $p$  of  $a$ . One has  $\gcd(\alpha_1, \dots, \alpha_d, a, m) = 1$  if and only if  $\gcd(\alpha_1, \dots, \alpha_d, m) = 1$ . Hence

$$\begin{aligned}\Gamma_a(m) &= \sum_{(\alpha_1, \dots, \alpha_d): \langle \alpha_1, \dots, \alpha_d, a \rangle = \mathbb{Z}_m} N(\alpha_1, \dots, \alpha_d, a) \\ &= \sum_{(\alpha_1, \dots, \alpha_d): \gcd(\alpha_1, \dots, \alpha_d, m) = 1} N(\alpha_1, \dots, \alpha_d, a) \\ &= m^d \cdot \mathcal{C}_m \cdot \varphi_d(m).\end{aligned}$$

It easily deduced from the Chinese remainder lemma that the functions  $m \mapsto \Gamma_a(m)$ ,  $m \mapsto \varphi_d(m)$ , and  $m \mapsto \mathcal{C}_m$  are arithmetic multiplicative function. Hence one checks that

$$\Gamma_a(m) = \frac{\mathcal{C}_m \cdot \varphi_d(\gcd(m, a)) \cdot m^{2d}}{\gcd(m, a)^d}.$$

It remains now to apply Theorem 1 to obtain the result, that is, for a.e.  $\mathbf{x} \in X$

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : q^{(n)} \equiv a \pmod{m}\} &= \frac{\Gamma_a(m)}{\mathcal{C}_m \cdot m^{2d}} \\ &= \frac{\mathcal{C}_m \cdot m^{2d} \cdot \varphi_d(\gcd(a, m))}{\mathcal{C}_m \cdot \gcd(a, m)^d} \\ &= \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)} \quad \square\end{aligned}$$

**Remark.** Let  $\mathbb{F}_q$  denote the finite field of cardinality  $q$  and let  $\mathbb{F}_q[X]$  be the set of polynomials with  $\mathbb{F}_q$ -coefficients. We denote by  $\mathbb{L}$  the set of formal Laurent power series with negative degree. Since  $\mathbb{L}$  is a compact Abelian group, there exists a unique normalized Haar measure  $m$ . We can define the Jacobi-Perron algorithm on  $\mathbb{L}^d$  for any  $d \geq 1$ . In this case,  $m^d$  is invariant under this algorithm.

Suppose that  $\left(\frac{P_1^{(n)}}{Q^{(n)}}, \dots, \frac{P_d^{(n)}}{Q^{(n)}}\right)$  is the  $n$ -th convergent of  $(f_1, \dots, f_d) \in \mathbb{L}^d$ . For any  $R \in \mathbb{F}_q[X]$ , it is possible to prove the following : for any  $A_1, \dots, A_d, A_{d+1} \in \mathbb{F}_q[X]$  such that  $A_1, \dots, A_d, A_{d+1}, R$  are relatively prime,

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (P_1^{(n)}, \dots, P_d^{(n)}, Q^{(n)}) \equiv (A_1, \dots, A_d, A_{d+1}) \pmod{R}\}}{N} \\ = c_R \quad \text{for } m^d\text{-a.e. } (f_1, \dots, f_d) \in \mathbb{L}^d,\end{aligned}$$



where  $c_R$  is a constant depending only on  $d$  and  $R$ . The proof is essentially the same as that of Theorem 1 of this paper. We refer to K. Inoue and H. Nakada [5] for the study of the rates of convergence for Jacobi-Perron algorithm over  $\mathbb{L}^d$  and to R. Natsui [8] for the  $\mathbb{L}$ -version of Jager-Liardet's result in the case of continued fractions.

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