Arithmetic Distributions of Convergents Arising from Jacobi-Perron Algorithm

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Abstract

We study the distribution modulo m of the convergents associated with the d-dimensional Jacobi-Perron algorithm for a.e. real numbers in $(0,1)^d$ by proving the ergodicity of a skew product of the Jacobi-Perron transformation; this skew product was initially introdued in [6] for regular continued fractions.

1 Introduction

For an irrational number x, $0 < x < 1$, we denote by $\frac{p_n}{q_n}$ the n-th convergent of x, which is defined by the regular continued fraction expansion coefficients of x. In 1988, H. Jager and P. Liardet [6] studied the distribution properties of the pairs (p_n, q_n) modulo m. These properties were originally considered by P. Szüsz in [16], and then by R. Moeckel [7] who used the ergodicity of geodesic flows over the modular surfaces: more precisely, they proved that given any positive integer $m \geq 2$, for a.e.x, the sequence $\{(p_n, q_n) : n \geq 1\}$ is equidistributed modulo m over the set $\{(p, q) \in \mathbb{Z}_m^2 : \langle p, q \rangle = \mathbb{Z}_m\}$, where $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ and where the notation $\langle p_1, \ldots, p_k \rangle$ stands for the subgroup of $\mathbb{Z}_m$ generated by the elements $p_1, \ldots, p_k$. To prove this property, H. Jager and P. Liardet considered in [6] the group of 2×2 matrices with entries from $\mathbb{Z}_m$ and determinant ± 1, that is,

$$G(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_m,\ ad - bc = \pm 1 \right\}.$$

It is possible to show that $\{\left(\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right) : a \in \mathbb{Z}_m\}$ generates $G(m)$. This fact implies the ergodicity of a $G(m)$-extension (a skew product indeed) of the continued fraction transformation. The equidistribution property of $\{(p_n, q_n) : n \geq 1\}$ modulo m is then an easy consequence of the individual ergodic theorem.

A natural extension of this skew product was then introduced in [3] to deduce the distribution of the approximation coefficients associated with the

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continued fraction algorithm; these results were also extended to the so-called S-expansions, in the sense of [4]; see also for connected results [1] and [10].

The aim of the present paper is to generalize these equidistribution results to the d-dimensional Jacobi-Perron algorithm. Note that the 1-dimensional Jacobi-Perron algorithm reduces to the regular continued fraction algorithm.

Let us start with the definition of the Jacobi-Perron algorithm. We fix a positive integer $d \geq 2$. Let $X = [0,1)^d$ be endowed with the Borel σ-algebra $\mathbb{B}$. We first define the map $T : X \to X$ by

$$T(\mathbf{x}) = T\left((x_1, x_2, \ldots, x_d)\right) = \left(\frac{x_2}{x_1} - \left[\frac{x_2}{x_1}\right], \ldots, \frac{x_d}{x_1} - \left[\frac{x_d}{x_1}\right], \frac{1}{x_1} - \left[\frac{1}{x_1}\right]\right)$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in X$ if $x_1 \neq 0$, and $T(\mathbf{x}) = \mathbf{0}$, otherwise; (X, T) is called the d-dimensional Jacobi-Perron algorithm. Notice that there exists a unique absolutely continuous invariant probability measure μ for T which is equivalent to the Lebesgue measure (see for instance [13]).

We put for $\mathbf{x}$ in X with $x_1 \neq 0$

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}^{(0)}(\mathbf{x}) = (k_1, k_2, \ldots, k_d) = \left(\left[\frac{x_2}{x_1}\right], \left[\frac{x_3}{x_1}\right], \ldots, \left[\frac{x_d}{x_1}\right], \left[\frac{1}{x_1}\right]\right),$$

if $x_1 = 0$, we set $\mathbf{k}(\mathbf{x}) = \mathbf{0}$; we similarly define

$$\mathbf{k}^{(s)}(\mathbf{x}) = \left(k_1^{(s)}, k_2^{(s)}, \ldots, k_d^{(s)}\right) = \mathbf{k}\left(T^{s-1}(\mathbf{x})\right) \quad \text{for} \quad s \geq 1.$$

We then associate $(x_1, x_2, \ldots, x_d)$ with the column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}$ and consider the following matrix

$$P = \begin{pmatrix} -k_1 & 1 & 0 & \ldots & 0 \\ -k_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_d & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{1}$$

Then $T((x_1, x_2, \ldots, x_d))$ corresponds to $P \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}$. To construct the sequence

$$\left\{ \left(\frac{p_1^{(k)}}{q^{(k)}}, \ldots, \frac{p_d^{(k)}}{q^{(k)}} \right) : \; k \geq 1 - d \right\}$$

of *simultaneous approximation convergents* of $\mathbf{x}$ from the d-dimensional Jacobi-Perron algorithm, we first define $Q^{(0)}$ as the $(d+1) \times (d+1)$ identity matrix

I_{d+1}; we then define recursively $Q^{(n)}$ for $n \geq 1$ as

$$Q^{(n)} := Q^{(n-1)} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & k_1^{(n)} \\ 0 & 1 & \dots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_d^{(n)} \end{pmatrix}.$$

We thus set for $n \geq 1$

$$Q^{(n)} = \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \dots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \dots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \dots & p_d^{(n-1)} & p_d^{(n)} \\ q^{(n-d)} & q^{(n-d+1)} & \dots & q^{(n-1)} & q^{(n)} \end{pmatrix}.$$

Let us observe that

$$Q^{(1)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & k_1 \\ 0 & 1 & \dots & 0 & k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_d \end{pmatrix} = P^{-1}. \tag{2}$$

It is well-known that for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$,

$$\lim_{n\to\infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for } 1 \leq i \leq d$$

holds.

In this paper, we prove that for almost every $\mathbf{x} \in X$ the sequences of vectors $\{(q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) : n \geq 1\}$ and $\{(p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) : n \geq 1\}$ are both equidistributed modulo m for any integer $m \geq 2$.

More precisely we put

$$\widetilde{\mathbb{Z}}_m^{d+1} = \{(\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1} : \langle \alpha_1, \alpha_2 \dots, \alpha_{d+1} \rangle = \mathbb{Z}_m\}$$

and

$$c_m = \sharp\, \widetilde{\mathbb{Z}}_m^{d+1} \quad \text{(the cardinality of } \widetilde{\mathbb{Z}}_m^{d+1}).$$

One easily sees that

$$\begin{aligned} c_m &= \varphi_{d+1}(m) \\ &= \sharp\{(a_1, a_2 \dots, a_{d+1}) \in \{1, \dots, m\}^{d+1} : \gcd(a_1, \dots, a_{d+1}, m) = 1\}, \end{aligned} \tag{3}$$

where φ_{d+1} denotes the Jordan totient function of order $d+1$; we thus have (see for instance [15] or [11])

$$c_m = m^{d+1} \prod_{p|m} (1 - p^{-(d+1)}),$$

where the notation $\prod_{p|m}$ stands in all that follows for the product over the prime numbers p that divide m. We then have the following:

Theorem 1. *Let $m \geq 2$ be a nonnegative integer. For almost every $\mathbf{x} \in X$ and for any $(\alpha_1, \alpha_2 \ldots, \alpha_{d+1}) \in \widetilde{\mathbb{Z}}_m^{d+1}$, we have*

$$\lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)}) \equiv (\alpha_1, \alpha_2 \ldots, \alpha_{d+1})\ (mod\ m)\}}{N}$$
$$= \lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N : (p_1^{(n)}, p_2^{(n)}, \ldots, p_d^{(n)}, q^{(n)}) \equiv (\alpha_1, \alpha_2 \ldots, \alpha_{d+1})\ (mod\ m)\}}{N}$$
$$= \frac{1}{c_m} = \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1}\prod_{p|m}(1-p^{-(d+1)})}.$$

To prove this theorem, we consider for a given integer $m \geq 2$, the group $G(m)$ defined in a similar way as in [6]:

$$G(m) = \begin{cases} SL(d+1,\, \mathbb{Z}_m) & \text{if } d \text{ is even,} \\ SL_{\pm}(d+1,\, \mathbb{Z}_m) & \text{if } d \text{ is odd,} \end{cases}$$

where $SL(d+1,\, \mathbb{Z}_m)$ stands for the matrices with entries in $\mathbb{Z}_m$ with determinant 1, whereas $SL_{\pm}(d+1,\, \mathbb{Z}_m)$ stands for the matrices with entries in $\mathbb{Z}_m$ with determinant ± 1. Let us recall that (see for instance [11] or [9]) that

$$\sharp\, SL(d+1,\, \mathbb{Z}_m) = m^{(d+1)^2-1}\prod_{i=2}^{d+1}\prod_{p|n}(1-p^{-i}) = m^{d(d+1)/2}\prod_{i=2}^{d+1}\varphi_i(m).$$

Let C_m denote the cardinality of $G(m)$. Since $SL(d+1,\, \mathbb{Z}_m)$ is a subgroup of $SL_{\pm}(d+1,\, \mathbb{Z}_m)$ of index 2 if d is odd and $m \neq 2$, one thus gets

$$C_m = \begin{cases} m^{(d+1)^2-1}\prod_{i=2}^{d+1}\prod_{p|n}(1-p^{-i}) & \\ = m^{d(d+1)/2}\prod_{i=2}^{d+1}\varphi_i(m) & \text{if } d \text{ is even or } m = 2 \\ 2m^{(d+1)^2-1}\prod_{i=2}^{d+1}\prod_{p|n}(1-p^{-i}) & \\ = 2m^{d(d+1)/2}\prod_{i=2}^{d+1}\varphi_i(m) & \text{if } d \text{ is odd and } m \neq 2. \end{cases} \tag{4}$$

We identify $Q^{(1)}$ with the matrix with coefficients in $\mathbb{Z}_m$ obtained by reducing modulo m its entries. Here we note that $\det Q^{(1)} = 1$ or -1 if d is respectively even or odd, which implies that $Q^{(1)}$ belongs to the group $G(m)$, whatever may be the parity of d.

We define the map T_m on $X \times G(m)$ by

$$T_m(\mathbf{x}, A) = (T(\mathbf{x}),\, AQ^{(1)});$$

T_m is said to be a $G(m)$-extension of the map T.

We define the probability measure δ_m on $G(m)$ by $(\frac{1}{C_m}, \ldots, \frac{1}{C_m})$. Then it is easy to see that $\mu \times \delta_m$ is an invariant probability measure for T_m. Our question is whether $(T_m,\, \mu \times \delta_m)$ is ergodic or not. In Section 2, we show that the set of matrices of the form (2) (reduced modulo m) generates $G(m)$. Then in Section 3, we prove the ergodicity of T_m, from which we deduce the following proposition and then Theorem 1 (in the same way as in [6]):

Proposition 1. *For a.e. $\mathbf{x} \in X$ and any $A \in G(m)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \le n \le N : Q^{(n)} \equiv A \ (mod\ m)\} = \frac{1}{C_m}\cdot$$

Finally we have the following

Corollary 1. *For a.e. $\mathbf{x} \in X$ and any $a \in \mathbb{Z}_m$*

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \le n \le N : q^{(n)} \equiv a \ (mod\ m)\} = \frac{m^d \cdot \varphi_d(\gcd(a,m))}{\gcd(a,m) \cdot \varphi_{d+1}(m)}\cdot$$

In all that follows, we simply denote by $0, 1, \ldots, m-1$ the elements of $\mathbb{Z}_m$ if it is clear that the elements are in $\mathbb{Z}_m$ according to the context. In this case, one has obviously $m - 1 = -1$.

2 Basic properties of G(m)

We first define

$$\Gamma_m = \{A \in SL(d+1, \mathbb{Z}) : A \equiv I_{d+1} (\text{mod } m)\}.$$

Then it is well-known that

$$SL(d+1,\, \mathbb{Z}_m) \cong \Gamma_m \setminus SL(d+1, \mathbb{Z})$$

e.g., see G. Shimura [15], p. 21. ¿From this property, it easily follows that

$$SL_{\pm}(d+1,\, \mathbb{Z}_m) \cong \Gamma_m \setminus GL(d+1, \mathbb{Z}).$$

We respectively say that a $(d+1)\times(d+1)$ matrix with $\mathbb{Z}$ (or $\mathbb{Z}_m$)-entries of the form (2) is a J-P matrix, and that a matrix of the form (1) is a J-P* matrix; a J-P matrix is the inverse of a J-P* matrix.

In the sequel of this section, we show that the monoid generated by the set of J-P matrices with $\mathbb{Z}_m$-entries is equal to $G(m)$:

Theorem 2. *For any $B \in G(m)$, there exist J-P matrices $A_1, A_2, \ldots, A_s$ such that*

$$B = A_1 A_2 \cdots A_s.$$

For this purpose, we first need some notation and some preliminary lemmas. We put

$$\Delta(k_1, k_2, \ldots, k_d) := \begin{pmatrix} k_1 & 1 & 0 & \ldots & 0 \\ k_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_d & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix};$$

in particular,

$$\Delta = \Delta(0,0,\dots,0) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We thus have

$$(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1})\Delta = (\mathbf{a}_{d+1}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d), \tag{5}$$

where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1}$ are either elements in $\mathbb{Z}_m$ or $(d+1)$-dimensional vectors with $\mathbb{Z}_m$-entries. Let us notice that the matrices $\Delta(k_1, k_2, \dots, k_d)$ are J-P* matrices.

Lemma 1. *For any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, there exist J-P matrices A_1, A_2, $\dots$,A_s with $\mathbb{Z}_m$-entries such that*

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) = (0, \dots, 0, 1)A_1 A_2 \dots A_s,$$

where s depends on $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$.

Proof of Lemma 1. We define the following natural order $\prec$ on $\mathbb{Z}_m$ by

$$0 \prec 1 \prec \dots \prec m-1.$$

Let $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{(d+1)}$. We denote by α^* the element in $\mathbb{Z}_m$ such that $\langle \alpha^* >=< \alpha_1, \alpha_2, \dots, \alpha_d \rangle$. Let us prove by induction on α^* (considered then as an element in $\{1, \dots, m\}$) that there exists a finite number of J-P* matrices $\Delta_1, \dots, \Delta_t$ and $(\alpha'_1, \dots, \alpha'_d) \in \mathbb{Z}_m^d$ such that

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1})\, \Delta_1 \cdots \Delta_t = (1, \alpha'_1, \dots, \alpha'_d).$$

If $\alpha^* = 1$, then $\langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle = \mathbb{Z}_m$ and there exist $k_1, \dots, k_d \in \mathbb{Z}_m$ such that $\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} = 1$. We thus have

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1})\, \Delta(k_1, k_2, \dots, k_d) = (1, \alpha_1, \dots, \alpha_d).$$

Suppose now that $\alpha^* \neq 1$. Since $\langle \alpha_1, \alpha_2, \dots, \alpha_{d+1} \rangle \neq \langle \alpha^* \rangle$, there exist $k_1, \dots, k_d \in \mathbb{Z}_m$ such that $0 \prec \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} \prec \alpha_*$. We thus have

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1})\, \Delta(k_1, k_2, \dots, k_d, 1) = (\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \dots, \alpha_d).$$

We can now conclude inductively since $\langle \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \dots, \alpha_d \rangle = \mathbb{Z}_m$.

Now we have

$$\begin{aligned}(1, \alpha'_1, \dots, \alpha'_d) \cdot \Delta(-\alpha'_d, 0, \dots, 0) \cdot \Delta(0, -\alpha'_{d-1}, 0, \dots, 0) \cdots \Delta(0, \dots, 0, -\alpha'_1) \\ = (0, \dots, 0, 1).\end{aligned}$$

Since a J-P* matrix is the inverse of a J-P matrix, we get the assertion of this lemma. $\square$

The following lemmas are essential and easily proved.

Lemma 2. *For any $(d+1)$-dimensional vectors with $\mathbb{Z}_m$-entries $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{d+1})$, we have*

$$\begin{aligned}
&(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_d, \mathbf{a}_{d+1}) \cdot \Delta(0, \ldots, 0, \overset{\overset{i}{\vee}}{-1}, 0, \ldots, 0) \cdot \Delta^{d-i} \\
&\cdot \Delta(0, \ldots, 0, \overset{\overset{d+1-i}{\vee}}{1}, 0, \ldots, 0) \cdot \Delta^{i-1} \cdot \Delta(0, \ldots, 0, \overset{\overset{i}{\vee}}{-1}, 0, \ldots, 0) \cdot \Delta^{d} \\
&= (\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{d+1}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_d, -\mathbf{a}_i).
\end{aligned}$$

Lemma 3. *We have*

$$\begin{aligned}
&(\mathbf{a}_1, \ldots, \mathbf{a}_{d+1}) \cdot \Delta(0, \ldots, 0, 1) \cdot \Delta(-1, 0, \ldots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \ldots, 0, 1) \\
&= (\mathbf{a}_d, \mathbf{a}_1, \ldots, \mathbf{a}_{d-1}, -\mathbf{a}_{d+1}).
\end{aligned}$$

In particular, when d is odd

$$\begin{aligned}
&(\mathbf{a}_1, \ldots, \mathbf{a}_{d+1}) \cdot [\Delta(0, \ldots, 0, 1) \cdot \Delta(-1, 0, \ldots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \ldots, 0, 1)]^d \\
&= (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d, -\mathbf{a}_{d+1}).
\end{aligned}$$

Proof of Theorem 2. Let us fix

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1(d+1)} \\ b_{21} & b_{22} & \ldots & b_{2(d+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(d+1)1} & b_{(d+1)2} & \ldots & b_{(d+1)(d+1)} \end{pmatrix}$$

in $G(m)$. We want to prove that there exist J-P* matrices $\Delta_1, \ldots, \Delta_s$ such that

$$B\,\Delta_1 \cdots \Delta_s = I_{d+1},$$

which implies immediately the desired result. For that purpose, let us prove by induction on $1 \le j \le d$ that there exist J-P* matrices $\Delta_1, \ldots, \Delta_{s_j}$ such that

$$B_j := B\,\Delta_1 \cdots \Delta_{s_j} = \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \mathbf{0} & B^{(j)} \end{array}\right), \tag{6}$$

where I_j is the $j \times j$ identity matrix. Indeed, if this property holds for $j = d$, then we obtain that there exist J-P* matrices $\Delta_1, \ldots, \Delta_{s_d}$ such that

$$B\,\Delta_1 \cdots \Delta_{s_d} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \ldots & 0 & \pm 1 \end{pmatrix}.$$

If d is even, then all J-P* matrices are of determinant 1. Thus the $(d+1,d+1)$-entry of the right hand side is equal to 1. If d is odd and the $(d+1,d+1)$-entry of the right hand side is equal to -1, then by Lemma 3 we can reduce it to 1 by application of J-P* matrices. In either case, we get the desired result.

It thus remains to prove the induction property. Let us first prove that it holds for $j=1$. Since $\det B = \pm 1$, then $\langle b_{11}, b_{12}, \ldots, b_{1(d+1)}\rangle = \mathbb{Z}_m$, and there thus exist J-P* matrices $\Delta_1, \ldots, \Delta_{s_1-1}$ such that

$$(b_{11}, b_{12}, \ldots, b_{1(d+1)})\, \Delta_1 \cdots \Delta_{s_1-1} = (0, \ldots, 0, 1)$$

by Lemma 1. Thus

$$B\,\Delta_1 \cdots \Delta_{s_1-1} = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ b^{(1)}_{11} & \ldots & b^{(1)}_{1d} & b^{(1)}_{1(d+1)} \\ \vdots & \ddots & \vdots & \vdots \\ b^{(1)}_{d1} & \ldots & b^{(1)}_{dd} & b^{(1)}_{d(d+1)} \end{pmatrix}.$$

We set

$$B^{(1)} = \begin{pmatrix} b^{(1)}_{11} & \ldots & b^{(1)}_{1d} \\ \vdots & \ddots & \vdots \\ b^{(1)}_{d1} & \ldots & b^{(1)}_{dd} \end{pmatrix}.$$

Since $\det B^{(1)} = \pm 1$, then there exist $k_1, \ldots, k_d \in \mathbb{Z}_m$ such that

$$B\Delta_1 \cdots \Delta_{s_1-1}\Delta(k_1, \ldots, k_d) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & b^{(1)}_{11} & \ldots & b^{(1)}_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b^{(1)}_{d1} & \ldots & b^{(1)}_{dd} \end{pmatrix}.$$

It remains to set $\Delta_{s_1} = \Delta(k_1, \ldots, k_d)$ to conclude the proof of the induction property for $j=1$.

Let us assume now that the induction property holds for $1 \le j \le d-1$ (if $d=1$, the proof is finished); one thus deduces that the determinant of $B^{(j)}$ (defined in (6)) is equal to ± 1. We set

$$B^{(j)} = \begin{pmatrix} b^{(j)}_{11} & \ldots & b^{(j)}_{1(d+1-j)} \\ \vdots & \ddots & \vdots \\ b^{(1)}_{(d+1-j)1} & \ldots & b^{(1)}_{(d+1-j)(d+1-j)} \end{pmatrix}.$$

Let us divide the induction proof into two steps for clarity issues.

Step 1. Let us first prove that we can find J-P* matrices $\Delta_{s_j+1}, \ldots, \Delta_{s_j+t}$

such that $B_j\, \Delta_{s_j+1} \cdots \Delta_{s_j+t}$ is equal to

$$\left(\begin{array}{cccc|c|ccc} \multicolumn{4}{c|}{\mathbf{0}} & I_j & \multicolumn{3}{c}{\mathbf{0}} \\ \hline 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ \hline g^{(j+1)}_{1(l+1)} & \dots & g^{(j+1)}_{1(d-j)} & g^{(j+1)}_{1(d+1-j)} & & g^{(j+1)}_{11} & \dots & g^{(j+1)}_{1l} \\ \vdots & \ddots & \vdots & \vdots & \mathbf{0} & \vdots & \ddots & \vdots \\ g^{(j+1)}_{(d-j)(l+1)} & \dots & g^{(j+1)}_{(d-j)(d-j)} & g^{(j+1)}_{(d-j)(d+1-j)} & & g^{(j+1)}_{(d-j)1} & \dots & g^{(j+1)}_{(d-j)l} \end{array}\right) \tag{7}$$

for some l, $0 \le l < d-j+1$.

According to the proof of Lemma 1, we can find $(d-j+1) \times (d-j+1)$ J-P* matrices $\Delta(k_1^{(1)}, \dots, k_{d-j}^{(1)}), \dots, \Delta(k_1^{(t)}, \dots, k_{d-j}^{(t)})$ such that

$$(b_{11}^{(j)}, \dots, b_{1(d+1-j)}^{(j)})\, \Delta(k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdots \Delta(k_1^{(u)}, \dots, k_{d-j}^{(u)}) = (0, \cdots, 0, 1).$$

Now

$$\left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array}\right) \Delta(\underbrace{0, \dots, 0}_{j}, *, \dots, *) = \left(\begin{array}{c|c|c} \mathbf{0} & I_j & \mathbf{0} \\ \hline * & \mathbf{0} & * \end{array}\right),$$

and one checks more generally that for $0 \le v \le d-j$

$$\left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array}\right) \Delta(\underbrace{0, \dots, 0}_{j}, *, \dots, *) \cdot \Delta(* \underbrace{0, \dots, 0}_{j}, *, \dots, *) \cdots$$
$$\cdots \Delta(\underbrace{*, \dots, *}_{v}, \underbrace{0, \dots, 0}_{j}, *, \dots, *) = \left(\begin{array}{c|c|c} \overbrace{\mathbf{0} \cdots \mathbf{0}}^{v+1} & I_j & \mathbf{0} \\ \hline \underbrace{* \cdots *}_{v+1} & \mathbf{0} & * \end{array}\right).$$

One thus gets that if $u \le d-j+1$, then

$$\left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array}\right) \Delta(\underbrace{0, \dots, 0}_{j}, k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdot \Delta(k_1^{(2)}, \underbrace{0, \dots, 0}_{j}, k_2^{(2)}, \dots, k_{d-j}^{(2)}) \cdots$$
$$\cdots \Delta(k_1^{(u)}, \dots, k_{d-j}^{(u)}, \underbrace{0, \dots, 0}_{j})$$

has the desired form (7).

Now if $u \ge d-j+2$, then using (5), one gets

$$\left(\begin{array}{c|c} \mathbf{0} & I_j \\ \hline * & \mathbf{0} \end{array}\right) \Delta^j = \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & * \end{array}\right),$$

and suitable insertions of Δ^j such as

$$\left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array}\right) \Delta(\underbrace{0,\ldots,0}_{j}, k_1^{(1)},\ldots,k_{d-j}^{(1)}) \cdot \Delta(k_1^{(2)}, \underbrace{0,\ldots,0}_{j}, k_2^{(2)},\ldots,k_{d-j}^{(2)})$$

$$\cdots \Delta(k_1^{(d-j+1)},\ldots,k_{d-j}^{(d-j+1)}, \underbrace{0,\ldots,0}_{j}) \cdot \Delta^j \cdot \Delta(\underbrace{0,\ldots,0}_{j}, k_1^{(d-j+2)},\ldots,k_{d-j}^{(d-j+2)})$$

$$\cdots \Delta(k_1^{(2(d-j+1))},\ldots,k_{d-l+j}^{(2(d-j+1))}, \underbrace{0,\ldots,0}_{j}) \cdot \Delta^j \cdot \Delta(\underbrace{0,\ldots,0}_{j}, k_1^{2(d-j+1)+1},\ldots,k_{d-j}^{2(d-j+1)+1})$$

$$\cdots \Delta(k_1^{(u)},\ldots,k_{d-l+j}^{(u)}, \underbrace{0,\ldots,0}_{j}, k_{(d-l+j+1)}^{(u)},\ldots,k_{d-j}^{(u)})$$

provide the desired form (7), which ends the proof of Step 1.

Step 2. By (5),

$$\begin{aligned}
&B_j\,\Delta_{s_j+1}\cdots\Delta_{s_j+t}\cdot\Delta^{l+j} \\
&= \left(\begin{array}{cccc|c|ccc}
 & & \mathbf{0} & & I_j & & \mathbf{0} & \\ \hline
0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 \\ \hline
g_{1(l+1)}^{(j+1)} & \cdots & g_{1(d-j)}^{(j+1)} & g_{1(d+1-j)}^{(j+1)} & & g_{11}^{(j+1)} & \cdots & g_{1l}^{(j+1)} \\
\vdots & \ddots & \vdots & \vdots & \mathbf{0} & \vdots & \ddots & \vdots \\
g_{(d-j)(l+1)}^{(j+1)} & \cdots & g_{(d-j)(d-j)}^{(j+1)} & g_{(d-j)(d+1-j)}^{(j+1)} & & g_{(d-j)1}^{(j+1)} & \cdots & g_{(d-j)l}^{(j+1)}
\end{array}\right)\cdot\Delta^j \\
&= \left(\begin{array}{c|cccc}
I_j & & & \mathbf{0} & \\ \hline
 & 0 & \cdots & 0 & 1 \\
\mathbf{0} & g_{11} & \cdots & g_{1(d-j)} & g_{1(d-j+1)} \\
 & \vdots & \ddots & \vdots & \vdots \\
 & g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d-j+1)}
\end{array}\right).
\end{aligned}$$

We put

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1(d-j)} \\ \vdots & \ddots & \vdots \\ g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} \end{pmatrix}.$$

Since the determinant of G is equal ± 1, there exist $k'_1, \ldots, k'_{d-j} \in \mathbb{Z}_m$ such that

$$\left(\begin{array}{c|cccc} I_j & & & \mathbf{0} & \\ \hline & 0 & \cdots & 0 & 1 \\ \mathbf{0} & g_{11} & \cdots & g_{1(d-j)} & g_{1(d-j+1)} \\ & \vdots & \ddots & \vdots & \vdots \\ & g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d-j+1)} \end{array}\right) \cdot \Delta(\underbrace{0,\ldots,0}_{j}, k'_1, \ldots, k'_{d-j})$$

$$= \left(\begin{array}{c|ccc|ccc} 0 & & & & & & \\ \vdots & & I_j & & & \mathbf{0} & \\ 0 & & & & & & \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & & \\ \vdots & & \mathbf{0} & & & G & \\ 0 & & & & & & \end{array}\right).$$

By applying (5), we get

$$\left(\begin{array}{c|ccc|ccc} 0 & & & & & & \\ \vdots & & I_j & & & \mathbf{0} & \\ 0 & & & & & & \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & & \\ \vdots & & \mathbf{0} & & & G & \\ 0 & & & & & & \end{array}\right) \cdot \Delta^d = \left(\begin{array}{ccc|ccc|c} & & & & & & 0 \\ & I_j & & & \mathbf{0} & & \vdots \\ & & & & & & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \hline & & & & & & 0 \\ & \mathbf{0} & & & G & & \vdots \\ & & & & & & 0 \end{array}\right).$$

We thus have proved that there exist J-P* matrices $\Delta_{s_j+t+1}, \ldots, \Delta_{s_{j+t'}}$ such that

$$B_j\, \Delta_{s_j+1} \ldots \Delta_{s_j+t'} = \left(\begin{array}{ccc|ccc|c} & & & & & & 0 \\ & I_j & & & \mathbf{0} & & \vdots \\ & & & & & & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\ \hline & & & & & & 0 \\ & \mathbf{0} & & & G & & \vdots \\ & & & & & & 0 \end{array}\right).$$

It remains to apply Lemma 2; there thus exist J-P* matrices $\Delta_{s_j+t'+1}, \ldots, \Delta_{s_{j+1}}$ such that

$$B_j\, \Delta_{s_j+1} \cdots \Delta_{s_{j+1}} = \left(\begin{array}{c|c} I_{j+1} & \mathbf{0} \\ \hline \mathbf{0} & B^{(j+1)} \end{array}\right),$$

where we put

$$B^{(j+1)} = \begin{pmatrix} g_{12} & g_{13} & \cdots & g_{1(d-j)} & -g_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(d-j)2} & g_{(d-j)3} & \cdots & g_{(d-j)(d-j)} & -g_{(d-j)1} \end{pmatrix},$$

which concludes the induction proof. $\square$

3 Ergodicity of T_m and proof of Theorem 1

3.1 Ergodicity

Let us recall some fundamental facts about Jacobi-Perron algorithm. For an integer vector $\mathbf{a} = (a_1, a_2, \ldots, a_d)$ with $a_i \geq 0$ for $1 \leq i \leq d$, we put

$$X_{\mathbf{a}} = \{\mathbf{x} \in X : \mathbf{k}(\mathbf{x}) = \mathbf{a}\}.$$

Then we see that $X_{\mathbf{a}} \neq \emptyset$ if and only if $0 \leq a_i \leq a_d$ for any $1 \leq i \leq d-1$ and $a_d > 0$. For a finite sequence of integer vectors $\{\mathbf{a}^{(l)} = (a_1^{(l)}, a_2^{(l)}, \ldots, a_d^{(l)}), 1 \leq l \leq n\}$ such that $a_i^{(l)} \leq a_d^{(l)}$, $1 \leq i \leq d-1$, and $a_d^{(l)} > 0$ for $1 \leq l \leq n$, we define the cylinder set of rank n by

$$X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}} = \{\mathbf{x} \in X : \mathbf{k}^{(l)}(\mathbf{x}) = \mathbf{a}^{(l)} \text{ for } 1 \leq l \leq n\}.$$

A cylinder set $X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}}$ is said to be proper (or full) if $T^n(X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}}) = X$. It is easy to see that $X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}}$ is proper if $a_i^{(l)} < a_d^{(l)}$ for all $1 \leq l \leq n$ and $1 \leq i \leq d-1$. The following is essential.

Lemma 4. *For almost every $\mathbf{x} \in X$ there exists a sequence of positive integers $n_1 < n_2 < \ldots$ such that $X_{\mathbf{k}^{(1)}(\mathbf{x}) \ldots \mathbf{k}^{(n_i)}(\mathbf{x})}$ is proper for any $i \geq 1$.*

This shows the exactness of the dynamical system (X, T, μ); the exactness means here that $\bigcap_{n=1}^{\infty} T^{-n}\mathbb{B} = \{\emptyset, X\}$ (μ-mod 0). In particular, (X, T, μ) is ergodic and strong mixing. We refer to F. Schweiger [12] or [14] about the theory of Jacobi-Perron algorithm. Now we will show the ergodicity of T_m.

Theorem 3. *The skew product $(X \times G(m), T_m, \mu \times \delta_m)$ is ergodic.*

Proof. For any non-empty cylinder set $X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}}$, we see from Proposition 2 in [14] that

$$\sup_{\mathbf{x} \in X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}}} |DT^n(\mathbf{x})| < (d+1)^{d+1} \inf_{\mathbf{x} \in X_{\mathbf{a}^{(1)} \ldots \mathbf{a}^{(n)}}} |DT^n(\mathbf{x})|, \tag{8}$$

where $|DT^n|$ is the Jacobian of T^n. Suppose that $\mathcal{M}$ is a T_m-invariant set of $(\mu \times \delta_m)$-positive measure. Since T_m acts as T on the first coordinate, the ergodicity of T shows

$$\{\mathbf{x} \in X : (\mathbf{x}, A) \in \mathcal{M} \text{ for some } A \in G(m)\} = X \quad (\mu\text{-mod } 0).$$

Thus there exists $A \in G(m)$ such that $(X \times \{A\}) \cap \mathcal{M}$ has positive $(\mu \times \delta_m)$-measure. We fix such a set A. By the density theorem and Lemma 4, for a given sequence $\varepsilon_i \searrow 0$ there exists a sequence of proper cylinder sets W_i of rank n_i and $B \in G(m)$ such that for all i

$$\frac{(\mu \times \delta_m)((W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(W_i \times \{A\})} > 1 - \varepsilon_i \tag{9}$$

and

$$A\begin{pmatrix}0 & 0 & \dots & 0 & 1\\ 1 & 0 & \dots & 0 & a_1^{(1)}\\ 0 & 1 & \dots & 0 & a_2^{(1)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & a_d^{(1)}\end{pmatrix}\begin{pmatrix}0 & 0 & \dots & 0 & 1\\ 1 & 0 & \dots & 0 & a_1^{(2)}\\ 0 & 1 & \dots & 0 & a_2^{(2)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & a_d^{(2)}\end{pmatrix}\cdots\begin{pmatrix}0 & 0 & \dots & 0 & 1\\ 1 & 0 & \dots & 0 & a_1^{(n_i)}\\ 0 & 1 & \dots & 0 & a_2^{(n_i)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & a_d^{(n_i)}\end{pmatrix} = B \pmod m,$$

where $(a_1^{(1)}, a_2^{(1)}, \dots, a_d^{(1)}), \dots, (a_1^{(n_i)}, a_2^{(n_i)}, \dots, a_d^{(n_i)})$ are sequences of integers which define W_i. ¿From (8) we see that (9) implies

$$\frac{(\mu \times \delta_m)(T_m^{n_i}(W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(T_m^{n_i}(W_i \times \{A\}))} > 1 - (d+1)^{d+1}\varepsilon_i.$$

Since W_i is proper and $\mathcal{M}$ is T_m-invariant, we conclude that

$$(X \times \{B\}) \cap \mathcal{M} = X \times \{B\} \quad ((\mu \times \delta_m)\text{-mod } 0).$$

¿From Theorem 2, for any $C \in G(m)$ there exist J-P matrices $A_1, \dots, A_s$ such that

$$C = BA_1 \cdots A_s \pmod m$$

with

$$A_i = \begin{pmatrix}0 & 0 & \dots & 0 & 1\\ 1 & 0 & \dots & 0 & a_1^{(i)}\\ 0 & 1 & \dots & 0 & a_2^{(i)}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & a_d^{(i)}\end{pmatrix} \quad \text{for } 1 \le i \le s.$$

Moreover we can choose $A_1, \dots, A_s$ so that the corresponding cylinder set $X_{\mathbf{a}^{(1)}\dots\mathbf{a}^{(s)}}$ with $\mathbf{a}^{(i)} = \begin{pmatrix}a_1^{(i)}\\ \vdots\\ a_d^{(i)}\end{pmatrix}$, $1 \le i \le s$, is proper. This means

$$T_m^s(X \times \{B\}) \supset X \times \{C\}$$

and so $\mathcal{M} = X \times G(m)$ $((\mu \times \delta_m)$-mod $0)$. Thus we get the assertion of the theorem. □

3.2 Proofs

We are now able to give proofs of Proposition 1, Theorem 1, and Corollary 1.

Proof of Proposition 1. Let us recall that C_m denotes the cardinality of $G(m)$. ¿From Theorem 3 and the individual ergodic theorem, we have

$$\lim_{N\to\infty} \frac{1}{N} \sharp\{1 \le n \le N : T_m^n(\mathbf{x}, B) \in X \times \{A\}\} = \frac{1}{C_m}$$

for $(\mu \times \delta_m)$-a.e. $(\mathbf{x},\, B)$. In particular, it holds for $(\mathbf{x},\, I_{d+1})$ for μ-a.e. $\mathbf{x}$. Since $T_m^n(\mathbf{x},\, I_{d+1}) = (T^n\mathbf{x},\, Q^{(n)})$, we get the assertion. $\square$

Proof of Theorem 1.
For any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, we denote by $N_{(\alpha_1,\alpha_2,\ldots,\alpha_{d+1})}$ the number of elements in $G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$. We will show that

$$N_{(\alpha_1,\alpha_2,\ldots,\alpha_{d+1})} = \mathcal{C}_m \cdot m^d, \tag{10}$$

where $\mathcal{C}_m$ denotes the cardinality of $SL(d,\, \mathbb{Z}_m)$ or $SL_{\pm}(d,\, \mathbb{Z}_m)$ if d is even or odd, respectively. It is easy to see that

$$N_{(0,\ldots,0,1)} = \mathcal{C}_m \cdot m^d. \tag{11}$$

¿From Lemma 1, we note that there always exists $D \in G(m)$ such that the $(d+1)$th row is $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1})$ for any $(\alpha_1, \alpha_2, \ldots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$.

For any matrix E of the form

$$\begin{pmatrix} & * & & \\ 0 & \ldots & 0 & 1 \end{pmatrix},$$

ED is of the form

$$\begin{pmatrix} & * & & \\ \alpha_1 & \ldots & \alpha_d & \alpha_{d+1} \end{pmatrix}.$$

This implies

$$N_{(\alpha_1,\alpha_2,\ldots,\alpha_{d+1})} \geq N_{(0,\ldots,0,1)}.$$

On the other hand, for any matrix D' of the form

$$\begin{pmatrix} & * & & \\ \alpha_1 & \ldots & \alpha_d & \alpha_{d+1} \end{pmatrix},$$

$D' \cdot D^{-1}$ is of the form

$$\begin{pmatrix} & * & & \\ 0 & \ldots & 0 & 1 \end{pmatrix},$$

which implies

$$N_{(\alpha_1,\alpha_2,\ldots,\alpha_{d+1})} \leq N_{(0,\ldots,0,1)}.$$

Thus we have (10).

¿From Proposition 1 together with (10), we have

$$\lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \ldots, q^{(n)}) \equiv (\alpha_1, \alpha_2 \ldots, \alpha_{d+1}) \pmod{m}\}}{N}$$
$$= \frac{\mathcal{C}_m \cdot m^d}{C_m} = \frac{1}{c_m} \quad \text{for } \mu\text{-a.e. } \mathbf{x}.$$

Indeed one easily checks according to (3) and (4) that $\frac{\mathcal{C}_m \cdot m^d}{C_m} = \frac{1}{c_m}$ holds. Since μ is equivalent to the Lebesgue measure, this holds for a.e. $\mathbf{x}$ with respect to the Lebesgue measure. If we consider the $(d+1)$th column, then the same argument shows the other equality. This completes the proof of Theorem 1. $\square$

Proof of Corollary 1. For a given $a \in \mathbb{Z}_m$, let $\Gamma_a(m)$ denote the cardinality of the subset of $G(m)$ of matrices whose $(d+1, d+1)$-entry is equal to a.

Let us first assume that a and m are coprime. We then deduce from (11) that

$$\begin{aligned} \Gamma_a(m) &= \textstyle\sum_{(\alpha_1,\ldots,\alpha_d):\langle \alpha_1,\ldots,\alpha_d,a\rangle=\mathbb{Z}_m} N(\alpha_1,\ldots,\alpha_d,a) \\ &= \textstyle\sum_{(\alpha_1,\ldots,\alpha_d)} N(\alpha_1,\ldots,\alpha_d,a) \\ &= m^{2d}.\mathcal{C}_m \,. \end{aligned}$$

Let us assume now that m is a power of a prime divisor p of a. One has $\gcd(\alpha_1,\ldots,\alpha_d,a,m) = 1$ if and only if $\gcd(\alpha_1,\ldots,\alpha_d,m) = 1$. Hence

$$\begin{aligned} \Gamma_a(m) &= \textstyle\sum_{(\alpha_1,\ldots,\alpha_d):\langle \alpha_1,\ldots,\alpha_d,a\rangle=\mathbb{Z}_m} N(\alpha_1,\ldots,\alpha_d,a) \\ &= \textstyle\sum_{(\alpha_1,\ldots,\alpha_d):\ \gcd(\alpha_1,\ldots,\alpha_d,m)=1} N(\alpha_1,\ldots,\alpha_d,a) \\ &= m^d.\mathcal{C}_m.\,\varphi_d(m). \end{aligned}$$

It easily deduced from the Chinese remainder lemma that the functions $m \mapsto \Gamma_a(m)$, $m \mapsto \varphi_d(m)$, and $m \mapsto \mathcal{C}_m$ are arithmetic multiplicative function. Hence one checks that

$$\Gamma_a(m) = \frac{\mathcal{C}_m.\,\varphi_d(\gcd(m,a)).\,m^{2d}}{\gcd(m,a)^d}.$$

It remains now to apply Theorem 1 to obtain the result, that is, for a.e. $\mathbf{x} \in X$

$$\begin{aligned} \lim_{N\to\infty} \tfrac{1}{N}\,\sharp\{1 \le n \le N : q^{(n)} \equiv a \pmod{m}\} &= \tfrac{\Gamma_a(m)}{C_m} \\ &= \tfrac{\mathcal{C}_m \cdot m^{2d} \cdot \varphi_d(\gcd(a,m))}{C_m \cdot \gcd(a,m)^d} \\ &= \tfrac{m^d \cdot \varphi_d(\gcd(a,m))}{\gcd(a,m)\cdot \varphi_{d+1}(m)} \quad \square \end{aligned}$$

Remark. Let $\mathbb{F}_q$ denote the finite field of cardinality q and let $\mathbb{F}_q[X]$ be the set of polynomials with $\mathbb{F}_q$-coefficients. We denote by $\mathbb{L}$ the set of formal Laurent power series with negative degree. Since $\mathbb{L}$ is a compact Abelian group, there exists a unique normalized Haar measure m. We can define the Jacobi-Perron algorithm on $\mathbb{L}^d$ for any $d \ge 1$. In this case, m^d is invariant under this algorithm. Suppose that $\left(\frac{P_1^{(n)}}{Q^{(n)}},\ldots,\frac{P_d^{(n)}}{Q^{(n)}}\right)$ is the n-th convergent of $(f_1,\ldots,f_d) \in \mathbb{L}^d$. For any $R \in \mathbb{F}_q[X]$, it is possible to prove the following : for any $A_1,\ldots,A_d,A_{d+1} \in \mathbb{F}_q[X]$ such that $A_1,\ldots,A_d,A_{d+1},R$ are relatively prime,

$$\begin{aligned} \lim_{N\to\infty} \frac{\sharp\{1 \le n \le N : (P_1^{(n)},\ldots,P_d^{(n)},Q^{(n)}) \equiv (A_1,\ldots,A_d,A_{d+1}) \text{ (mod. } R)\}}{N} \\ = c_R \quad \text{for } m^d\text{-a.e. } (f_1,\ldots,f_d) \in \mathbb{L}^d, \end{aligned}$$

where c_R is a constant depending only on d and R. The proof is essentially the same as that of Theorem 1 of this paper. We refer to K. Inoue and H. Nakada [5] for the study of the rates of convergence for Jacobi-Perron algorithm over $\mathbb{L}^d$ and to R. Natsui [8] for the $\mathbb{L}$-version of Jager-Liardet's result in the case of continued fractions.

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