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# TILINGS ASSOCIATED WITH BETA-NUMERATION AND SUBSTITUTIONS

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## Abstract

This paper surveys different constructions and properties of some multiple tilings (that is, finite-to-one coverings) of the space that can be associated with beta-numeration and substitutions. It is indeed possible, generalizing Rauzy's and Thurston's constructions, to associate in a natural way either with a Pisot number $\beta$ (of degree $d$) or with a Pisot substitution $\sigma$ (on $d$ letters) some compact basic tiles that are the closure of their interior, that have non-zero measure and a fractal boundary; they are attractors of some graph-directed Iterated Funtion System. We know that some translates of these prototiles under a Delone set $\Gamma$ (provided by $\beta$ or $\sigma$) cover $\mathbb{R}^{d-1}$; it is conjectured that this multiple tiling is indeed a tiling (which might be either periodic or self-replicating according to the translation set $\Gamma$). This conjecture is known as the *Pisot conjecture* and can also be reformulated in spectral terms: the associated dynamical systems have pure discrete spectrum. We detail here the known constructions for these tilings, their main properties, some applications, and focus on some equivalent formulations of the Pisot conjecture, in the theory of quasicrystals for instance. We state in particular for Pisot substitutions a finiteness property analogous to the well-known (F) property in beta-numeration, which is a sufficient condition to get a tiling.

## Introduction

Substitutions are elementary and natural combinatorial objects (one replaces a letter by a word) which produce sequences by iteration and generate simple symbolic dynamical systems with zero entropy. These systems, produced by this elementary algorithmic process, have a highly ordered self-similar structure. Substitutions prove to be useful in many

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mathematical fields (combinatorics on words [Lot02], ergodic theory and spectral theory [Que87, Fog02, Sol92, Sol97, DHS99], geometry of tilings [Ken96, Rob04], Diophantine approximation and transcendence [ABF04, Roy04, AS02]), as well as in theoretical computer science [BP97, Lot05] or physics [BT86, LGJJ93, VM00, VM01]. The connections with numeration systems are numerous (see for instance [Dur98a, Dur98b, Fab95]) and natural: one can define a numeration system based on finite factors of an infinite word generated by a primitive substitution $\sigma$, known as the Dumont-Thomas numeration [DT89, DT93, Rau90]; this numeration system provides generalized radix expansions of real numbers with digits in a finite subset of the number field $\mathbb{Q}(\beta)$, $\beta$ being the Perron-Frobenius eigenvalue of $\sigma$. The analogy between substitutions and beta-numeration is highlighted by the work of Thurston [Thu89], where tilings associated with beta-substitutions are introduced; a characteristic example is given by the Fibonacci substitution $1 \mapsto 12$, $2 \mapsto 1$ and by the Fibonacci numeration (where nonnegative integers are represented thanks to the usual Fibonacci recurrence with digits in $\{0, 1\}$ and no two one's in a row allowed); in the so-called Tribonacci substitution/numeration case (initially due to Rauzy [Rau82, Rau88]) one gets tilings of the plane by translates of three basic tiles (depicted Fig. 1) whose union is called the *central tile* or the *Rauzy fractal.*

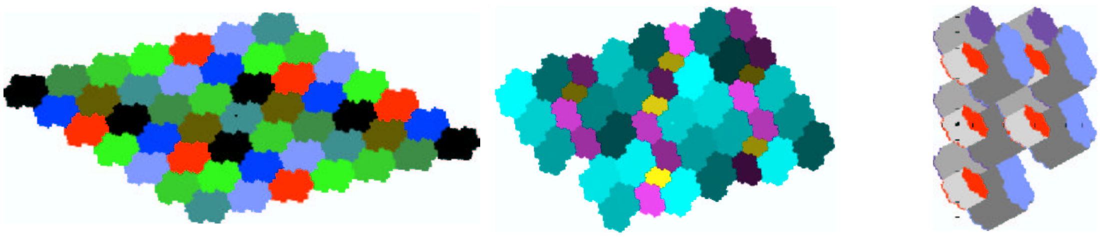


Figure 1: Periodic, aperiodic and 3-dimensional Rauzy tilings.

The Tribonacci situation has been widely studied and presents many interesting features. Indeed the symbolic dynamical system generated by the Tribonacci substitution (or equivalently the $\beta$-shift endowed with the odometer map, with $\beta > 1$ root of $X^3 - X^2 - X - 1$) is measure-theoretically isomorphic to a translation of the torus $\mathbb{T}^2$, the isomorphism being a continuous onto map [Rau82]: the symbolic dynamical system has thus pure discrete spectrum. Furthermore, the Tribonacci central tile has a "nice" topological behaviour (0 is an inner point and it is shown to be connected with simply connected interior [Rau82]), which leads to interesting applications in Diophantine approximation [CHM01]. Moreover it is also possible to construct a Markov partition for the toral automorphism of $\mathbb{T}^3$ given by the incidence matrix of the Tribonacci substitution, this construction being based on the Rauzy fractal and its associated 3-dimensional periodic tiling (depicted on Fig. 1).

More generally, it is possible to associate a central tile to any Pisot unimodular substitution [AI01, CS01a] or to beta-shifts with $\beta$ Pisot number [Aki98, Aki99, AS98, Aki00, Aki02, AN04b]. Let us note that not all central tiles associated with Pisot numbers need to satisfy the same topological properties as the Tribonacci tile does: they might be not connected or not simply connected, and 0 is not always an inner point of the central tile; see for instance, the examples given in [Aki02]).

There are mainly two methods of construction for central tiles. The first one is based on formal power series seen as digit expansions, and is inspired by the seminal paper [Rau82]; see e.g., [Mes98, Mes00], [CS01a, CS01b, Sie03], [Aki99, Aki98, Aki02]. A second approach via Iterated Function Systems and generalized substitutions has been developed following ideas from [Thu89, IK91], and [AI01, SAI01]. Indeed, central tiles can be described as attractors of some graph-directed Iterated Function System (IFS), as developed in [HZ98, Sir00a, Sir00b, SW02]. For more details on both approaches, see [Sie02].

Central tiles associated with Pisot beta-shifts and substitutive dynamical systems provide efficient geometric representations of the corresponding dynamical systems. One thus gets a combinatorial necessary and sufficient condition for a substitutive unimodular system of Pisot type to be measure-theoretically isomorphic to its maximal translation factor [Sie04]. This has also consequences for the effective construction of Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number [IO93, IO94, Pra99, Sie00]. Based on the approach of [KV98, VS93, SV98, Sid03a], an agebraic construction of symbolic representations of hyperbolic toral automorphimss as Markov partitions is similarly given in [Sch00, LS04], where homoclinic points are shown to play an essential role.

The central tile is divided into a finite number of basic tiles that are attractors for a graph-directed Iterated Function System based on an inflation-subdivision property, in the flavour of [LW96, MW88, Vin00]: each basic tile can be mapped onto a finite union of basic tiles, when multiplied by the paramater $\beta$ (the parameter of the beta-numeration or the dominant eigenvalue of the substitution assumed to be primitive). When $\beta$ is a Pisot number, one has a dilation, that is, all the eigenvalues of the subdivision matrix have modulus greater than 1. The maps in the IFS are thus contractive. Hence the compact non-empty sets satisfying this equation are uniquely determined (up to a scaling parameter) [MW88]; they have non-zero measure and are the closure of their interior [SW02].

By tiling, we mean here tilings by translation having finitely many tiles up to translation (a tile is assumed to be the closure of its interior), that is, tilings such that there exists a finite set of tiles $\mathcal{T}_i$ and a finite number of translation sets $\Gamma_i$ such that $\mathbb{R}^d = \cup_i \cup_{\gamma_i \in \Gamma_i} \mathcal{T}_i + \gamma_i$, and distinct translates of tiles have non-intersecting interiors; we assume furthermore, each compact set in $\mathbb{R}^d$ intersects a finite number of tiles; our sets of translation vectors $\Gamma_i$ are thus assumed to be Delaunay sets. See for instance [Ken90, Ken96, Ken99, Rad95, Rob96]. By *multiple tiling*, we mean according e.g. to [LW04], arrangements of tiles in $\mathbb{R}^d$ such that almost all points in $\mathbb{R}^d$ are covered exactly $p$ times for some positive integer $p$.

A central tile thus generates a self-replicating multiple tiling (in the sense of [KV98]) that is aperiodic and repetitive, i.e., any finite collection of tiles up to translation reoccurs in the tiling at a bounded distance from any point of the tiling. Repetitivity, also called quasiperiodicity, or uniform recurrence, is equivalent with the minimality of the tiling dynamical system [Sol97]. Furthermore the central tile does not only generate a self-replicating multiple tiling, but also a lattice multiple tiling of $\mathbb{R}^{d-1}$ in the case of a Pisot unimodular substitution $\sigma$ on $d$ letters, that can be lifted to $\mathbb{R}^d$: we thus get a second periodic multiple tiling, which, in the tiling case, produces a Markov partition. One recovers in a natural way from the Markov multiple tiling the lattice and the self-replicating ones by intersecting it with a suitable hyperplane (see Fig. 2). Consequently, as soon as one of those multiple tilings can be proved to be a tiling, then all the other multiple tilings are also indeed tilings [IR04]. Hence the Pisot conjecture states that as soon as $\beta$ is a Pisot number, then all these multiple tilings are tilings.

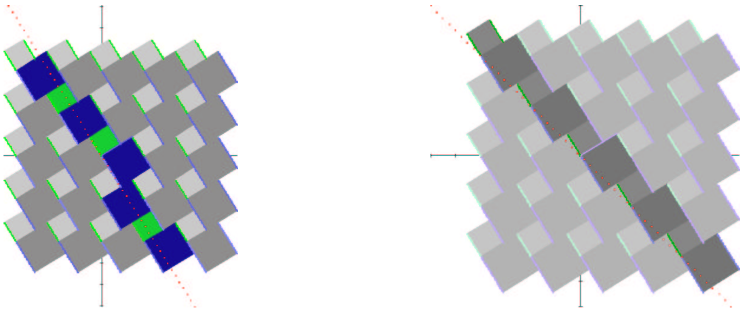


Figure 2: The Fibonacci Markov tiling: the self-replicating tiling lies in the intersection of the Markov tiling with the contracting direction; the lattice tiling is given by the projection on the contracting line of the pieces that cross the anti-diagonal line $x+y=0$.

We have chosen to handle both the beta-numeration and the substitutive cases for the following reasons. First, the literature on both subjects is often scattered among several series of papers, some of them dealing with beta-numeration (or analogously with Canonical Number systems, and Shift Radix Systems), other ones with symbolic dynamics and substitutions; a third group also deals with the interplay between the Pisot conjecture and spectral properties of tilings and quasicrystals. Second, the beta-shift is a rather natural framework for the introduction and motivation of the required algebraic formalism which is somehow heavier in the substitutive case. Lastly, the methods and motivations are very close and bring mutually insight on the subject.

Indeed, although the lattice multiple tiling seems more natural for substitutive dynamical systems, we have chosen here to focus on the self-replicating multiple tiling by following the same scheme and formalism as in the beta-numeration case, in order to unify both approaches. We thus introduce for that purpose a numeration system, the Dumont-Thomas numeration sytem [DT89, DT93, Rau90], based on the substitution, which allows us to expand real numbers. This point of view on the substitution self-replicating multiple tiling (that does not appear to our knowledge stated as such in the literature), allows us to introduce the substitutive counter-part to the well-known (F) property [FS92], which is a useful sufficient condition for tiling. We plan in a subsequent paper to develop this new point of view on substitution tilings.

Most of the resuls mentioned in the present paper are obtained under the assumption that $\beta$ is a Pisot unit. Nevertheless, it is one of the main challenges in the domain to try to relax the Pisot unit hypothesis, and to be able to prepare the playground for the non-Pisot case on the one hand, in the flavour of [KV98, LS04] and for the non-unit case, on the other hand, according to the $p$-adic approach developed in [Sie03, BS05]. Let us mention [AFHI05], where a simple example of an automorphism of the free group on 4 generators, whose associated matrix has 4 distinct complex eigenvalues, two of them of modulus larger than 1, and the other 2 of modulus smaller than 1 (non-Pisot case) is handled in details. Let us recall that substitutions are particular cases of free group morphisms, the main simplification being that we have no problem of cancellations.

This paper is organized as follows. We introduce in Section 1 the tiles and the multiple tiling associated with the beta-numeration when $\beta$ is assumed to be a unit Pisot number. We then show that both a self-replicating (Section 2) and a lattice multiple tiling (Section 3) with analogous prototiles can also be introduced for Pisot type substitution dynamical systems. We stronly use the formalism provided by the Dumont-Thomas numeration to develop the analogy between the numeration and the substitutive approaches. The lattice multiple tiling has an interesting dynamical and spectral interpretation that we develop in Section 3. Section 4 is devoted to the connections between the self-replicating multiple tiling and discrete geometry, more precisely, standard discrete planes. We then discuss the Pisot conjecture and formulate several equivalent statements as well as sufficient conditions in Section 5.

## 1. Beta-numeration

Let us define now the central tile associated with the beta-numeration; we recall all the required background on the beta-shift in Section 1.1; we then introduce the material required for the definition of the central tile and of its translation vectors set in Section 1.2 and 1.3; the multiple self-replicating tiling is defined in Section 1.4; we then work out the Tribonacci example in Section 1.5., and conclude this section by evoking some particular finiteness properties where the multiple tiling is known to be a tiling in Section 1.6. We mainly follow here [Thu89] and [Aki98, Aki99, Aki02].

### 1.1. Beta-shift

Let $\beta > 1$ be a real number. We assume throughout this paper that $\beta$ is an algebraic number. The (Renyi) *$\beta$-expansion* [Rén57, Par60] of a real number $x \in [0, 1]$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\mathcal{A}_\beta := \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ produced by the $\beta$-

transformation $T_\beta :\ x \mapsto \beta x \pmod 1$ as follows

$$\forall i \geq 1,\ u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \text{ and thus } x = \sum_{i \geq 1} u_i \beta^{-i}.$$

Let $d_\beta(1) = (t_i)_{i \geq 1}$ denote the $\beta$-expansion of 1. Numbers $\beta$ such that $d_\beta(1)$ is ultimately periodic are called *Parry numbers* and those such that $d_\beta(1)$ is finite are called *simple Parry numbers* (in this latter case, we omit the ending zero's when writing $d_\beta(1)$).

The algebraic integer $\beta$ is a said to be a *Pisot number* if all its algebraic conjugates have modulus less than 1: under this assumption, then $\beta$ is either a Parry number or a simple Parry number [BM86]; more generally, every element in $\mathbb{Q}(\beta)$ has an eventually periodic $\beta$-expansion according to [BM86, Sch80]. But conversely, if $\beta$ is Parry number we can only say that $\beta$ is a *Perron number*, that is, an algebraic integer greater than 1 all conjugates of which have absolute value less than that number [Lin84, DCK76]. Let us observe, as quoted in [Bla89], that there exist Parry numbers which are neither Pisot nor even Salem; consider e.g., $\beta^4 = 3\beta^3 + 2\beta^2 + 3$ with $d_\beta(1) = 0,323$; a *Salem number* is a Perron number all conjugates of which have absolute value less than or equal to 1, and among them one has modulus 1. It is conjectured in [Sch80] that every Salem number is a Parry number. This conjecture is sustained by the fact that if each rational in $[0,1)$ has a ultimately periodic $\beta$-expansion, then $\beta$ is either a Pisot or a Salem number. In particular, it is proved in [Boy89] that if $\beta$ is a Salem number of degree 4, then $\beta$ is a Parry number; see [Boy96] for the case of Salem numbers of degree 6. Note that the algebraic conjugates of a Parry number $\beta > 1$ are smallerr than $\frac{1+\sqrt{5}}{2}$ in modulus, this upper bound being sharp [FLP94, Sol94].

**Combinatorial caracterization of $\beta$-expansions.** We suppose that $\beta$ is a Parry number. Let $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and $d^*_\beta(1) = (t_1 \ldots t_{n-1}(t_n - 1))^\infty$, if $d_\beta(1) = t_1 \ldots t_{n-1} t_n$ is finite ($t_n \neq 0$). The set of $\beta$-expansions of real numbers in $[0,1)$ is exactly the set of sequences $(u_i)_{i \geq 1}$ in $\mathcal{A}_\beta^{\mathbb{N}^+}$ (where $\mathbb{N}^+$ denotes the set of positive inetegers) such that

$$\forall k \geq 1,\ (u_i)_{i \geq k} <_{\text{lex}} d^*_\beta(1). \tag{1}$$

For more details on the $\beta$-numeration, see for instance [Bla89, Fro00, Fro02].

**The $\beta$-shift.** It is natural to introduce the following symbolic dynamical system known as the (right one-sided) *$\beta$-shift* $(X^r_\beta, S)$ which is defined as the closure in $\mathcal{A}_\beta^{\mathbb{N}^+}$ of the set of $\beta$-expansions of real numbers in $[0,1)$, on which the shift map $S$ acts; let us recall that $S$ maps the sequence $(y_i)_{i \in \mathbb{N}^+}$ onto $(y_{i+1})_{i \in \mathbb{N}^+}$. Hence $X^r_\beta$ is equal to the set of sequences $(u_i)_{i \geq 1} \in \mathcal{A}_\beta^{\mathbb{N}^+}$ which satisfy

$$\forall k \geq 1,\ (u_i)_{i \geq k} \leq_{\text{lex}} d^*_\beta(1). \tag{2}$$

We can easily extend this admissibility condition to two-sided sequences and introduce the *two-sided symbolic $\beta$-shift* $(X_\beta, S)$ (the shift map $S$ maps now the sequence $(y_i)_{i \in \mathbb{Z}}$

onto $(y_{i+1})_{i\in\mathbb{Z}})$. The set $X_\beta$ is then defined as the set of two-sided sequences $(y_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}_\beta^{\mathbb{Z}}$ such that each left truncated sequence is less than or equal to $d_\beta^*(1)$, that is, $\forall k \in \mathbb{Z},\ (y_i)_{i\geq k} \leq_{\text{lex}} d_\beta^*(1)$.

We will use the following notation for the elements of $X_\beta$: if $y = (y_i)_{i\in\mathbb{Z}} \in X_\beta$, we set $u = (u_i)_{i\geq 1} = (y_i)_{i\geq 1}$ and $w = (w_i)_{i\geq 0} = (y_{-i})_{i\geq 0}$. One thus gets a two-sided sequence of the form

$$\ldots w_3 w_2 w_1 w_0, u_1 u_2 u_3 \ldots$$

and write it as $y = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) = (w, u)$. In other words, we will use the letters $(w_i)$ for denoting the "past" and $(u_i)$ for the "future" of the element $y = (w, u)$ of the two-sided shift $X_\beta$.

One similarly defines $X_\beta^l$ as the set of one-sided sequences $w = (w_i)_{i\geq 0}$ such that there exists $u = (u_i)_{i\geq 1}$ with $(w, u) \in X_\beta$. We call it the *left one-sided $\beta$-shift.*

**Sofic shift.** The finiteness condition in the definition of Parry numbers has a very powerful combinatorial interpretation: indeed, the $\beta$-shift $(X_\beta, S)$ is *sofic* (that is, the set of finite factors of the sequences in $X_\beta$ can be recognized by a finite automaton) if and only if $\beta$ is a Parry number (simple or not) [BM86]. The minimal automaton $\mathcal{M}_\beta$ recognizing the language $F_\beta$, defined as the set of finite factors of the sequences in $X_\beta$, can easily be constructed (see Figure 3). The number of states $n$ of this automaton is equal to the length of the period $n$ of $d_\beta^*(1)$ if $\beta$ is a simple Parry number with $d_\beta(1) = t_1 \ldots t_{n-1} t_n$, $t_n \neq 0$, and to the sum $n$ of its preperiod $m$ plus its period $p$, if $\beta$ is a non-simple Parry number with $d_\beta(1) = t_1 \ldots t_m (t_{m+1} \ldots t_{m+p})^\infty$ ($t_m \neq t_{m+p}$, $t_{m+1} \cdots t_{m+p} \neq 0^p$).

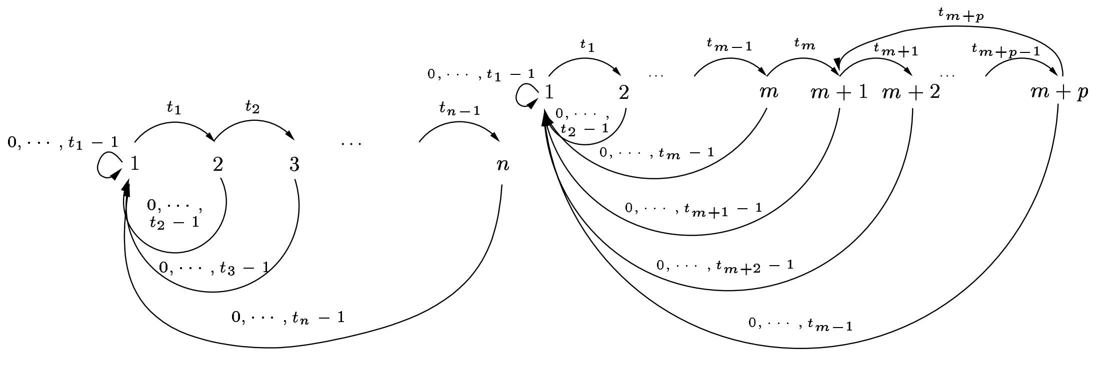



Figure 3: The automata $\mathcal{M}_\beta$ for $\beta$ simple Parry number ($d_\beta(1) = t_1 \ldots t_{n-1} t_n$) and for $\beta$ non-simple Parry number ($d_\beta(1) = t_1 \ldots t_m (t_{m+1} \ldots t_{m+p})^\infty$).

## 1.2. A tiling of the line

We can define the $\beta$-expansion of a real number $x$ greater than 1 as follows: let $k \in \mathbb{N}$ such $\beta^k \leq x < \beta^{k+1}$; one has $0 \leq \frac{x}{\beta^{k+1}} < 1$; let $u = (u_i)_{i\geq 1}$ denote the $\beta$-expansion of $\frac{x}{\beta^{k+1}}$;

the $\beta$-expansion of $x$ is then defined as the sequence $(\ldots 000u_1u_2\ldots u_{k+1}, u_{k+2}u_{k+3}\ldots)$. We thus can associate with any positive real number a two-sided sequence in $X_\beta$ which corresponds to its $\beta$-expansion (the converse being obviously untrue). We define the *$\beta$-fractional part* of a real number $x$ with $\beta$-expansion $(w,u) \in X_\beta$ as the one-sided sequence $u$.

**The sets Fin($\beta$) and $\mathbb{Z}_\beta^+$.** The *$\beta$-fractional part* of the positive real number $x$ with $\beta$-expansion $(w,u) \in X_\beta$ is defined as the sequence $u$; it is said to be *finite* if the sequence $u$ takes ultimately only zero values.

Let $\mathrm{Fin}(\beta)$ denote the set of positive real numbers having a finite $\beta$-fractional part, and $\mathbb{Z}_\beta^+ \subset \mathrm{Fin}(\beta)$ be the set of positive real numbers which have a zero fractional part in their $\beta$-expansion, that is,

$$\begin{aligned}\mathrm{Fin}(\beta) &= \{w_M\beta^M + \cdots + w_0 + u_1\beta^{-1} + \cdots + u_L\beta^{-L};\\ &\qquad\qquad M \in \mathbb{N},\ (w_M \cdots w_0u_1\ldots u_L) \in F_\beta\},\\ \mathbb{Z}_\beta^+ &= \{w_M\beta^M + \cdots + w_0;\ M \in \mathbb{N},\ (w_M\cdots w_0) \in F_\beta\} \subset \mathrm{Fin}(\beta).\end{aligned}$$

**A tiling of the line.** There is a tiling of the line that can be naturally associated with the $\beta$-numeration: let us place on the nonnegative half line the points of $\mathbb{Z}_\beta^+$; under the assumption that $\beta$ is a Parry number, one gets a tiling by intervals that take a finite number of lengths. Indeed we define the successor map $\mathrm{Succ} : \mathbb{Z}_\beta^+ \to \mathbb{Z}_\beta^+$ as the map which sends to an element $x$ of $\mathbb{Z}_\beta^+$ the smallest element of $\mathbb{Z}_\beta^+$ strictly larger than $x$. When $\beta$ is a Parry number, the set of values taken by $\mathrm{Succ}(x) - x$ on $\mathbb{Z}_\beta^+$ is finite and equal to $1, \beta - t_1, \beta^2 - t_1\beta - t_2, \ldots, \beta^{n-1} - t_1\beta^{n-2} - \cdots - t_{n-1}$, if $d_\beta(1) = t_1\ldots t_{n-1}t_n$ is finite ($t_n \neq 0$), and $1, \beta - t_1\beta, \beta^2 - t_1\beta - t_2, \cdots, \beta^{m+p-1} - t_1\beta^{m+p-2} - \cdots - t_{m+p-1}$ if $d_\beta(1) = t_1\ldots t_m(t_{m+1}\ldots t_{m+p})^\infty$, $t_m \neq t_{m+p}$, $t_{m+1}\ldots t_{m+p} \neq 0$, according to [Thu89]. We indeed divide $\mathbb{Z}_\beta^+$ according to the state *reached* in the automaton $\mathcal{M}_\beta$ when feeding the automaton by the digits of the elements of $\mathbb{Z}_\beta^+$ read from left to right, that is, most significant digit first. This tiling of the line has many interesting features when $\beta$ is furthermore assumed to be Pisot. Let us first recall a few definitions issued from the mathematical theory of quasicrystals.

**Definition 1** *A set $X \subset \mathbb{R}^n$ is said* uniformly discrete *if there exists a positive real number $r$ such that for any $x \in X$, the open ball located at $x$ of radius $r$ contains* at most *one point of $X$; a set $X \subset \mathbb{R}^n$ is said* relatively dense *it there exists a positive real number $R$ such that for any $x$ in $\mathbb{R}^n$, the open ball located at $x$ of radius $R$ contains* at least *one point of $X$.*

*A subset of $\mathbb{R}^n$ is a* Delaunay set *if it is uniformly discrete and relatively dense. A Delaunay set is a* Meyer set *if $X - X$ is also a Delaunay set.*

We deduce from the above results that if $\beta$ is a Parry number, then $\mathbb{Z}_\beta^+$ is a Delaunay set. More can be said when $\beta$ is a Pisot number.

**Proposition 1 ([BFGK98, VGG04])** *When $\beta$ is a Pisot number, then $\mathbb{Z}_\beta^+$ is a Meyer set.*

*Proof.* Since $\mathbb{Z}_\beta^+$ is relatively dense, we first deduce that $\mathbb{Z}_\beta^+ - \mathbb{Z}_\beta^+$ is also relatively dense. Now if $S$ is a finite subset of $\mathbb{Z}$, then $\{P(\beta);\ P \in S[X]\}$ is easily seen to be a discrete set [Sol97], Lemma 6.6: indeed $P(\beta)$ is an algebraic integer for any polynomial $P$ with coefficients in $\mathbb{Z}$; furthermore, since $\beta$ is assumed to be Pisot, there exists $C$ such that $|P_1(\beta^{(i)}) - P_2(\beta^{(i)})| \leq C$ for any algebraic conjugate $\beta^{(i)}$ (distinct from $\beta$), and for $P_1, P_2 \in S[X]$; since $\prod_i (P_1 - P_2)(\beta^{(i)})(P_1 - P_2)(\beta) \in \mathbb{Z}$ and is non-zero for $P_1, P_2 \in S[X]$ with $P_1(\beta) \neq P_2(\beta)$, we deduce a positive uniform lower bound for $P_1(\beta) - P_2(\beta)$ with $P_1(\beta) \neq P_2(\beta)$, which is sufficient to conclude. ∎

A Meyer set [Mey92, Mey95] is a mathematical model for quasicrystals [Moo97, BM00]; a Meyer set is also equivalently defined as a Delaunay set for which there exists a finite set $F$ such that $X - X \subset X + F$; this endows a Meyer set with a structure of "quasi-lattice": Meyer sets play indeed the role of the lattices in the theory of crystalline structure. For some families of $\beta$ (mainly Pisot quadratic units), an internal law can even be produced formalizing this quasi-stability under subtraction and even multiplication [BFGK98]. Beta-numeration reveals itself as a very efficient and promising tool for the modelisation of families of quasicrystals thanks to beta-grids [BFGK98, BFGK00, VGG04]. Note that Proposition 1 is also proved in [VGG04] by exhibiting a cut-and-project scheme; see also Section 3.4 where we detail connections between such a generation process for quasicrystals and lattice tilings.

An important issue is to characterize those $\beta$ for which $\beta$ is $\mathbb{Z}_\beta^+$ is uniformly discrete or even a Meyer set. Observe that $\mathbb{Z}_\beta^+$ is at least always a discrete set. It is conjectured that $\mathbb{Z}_\beta^+$ is uniformly discrete as soon as $\beta$ is a Perron number. It can easily be seen that $\mathbb{Z}_\beta^+$ is uniformly discrete if and only if the $\beta$-shift $X_\beta$ is *specified*, that is, if $d_\beta(1)$ contains bounded strings of zeros; for more details, see for instance [Bla89] and the discussion in [VGG04]. Let us note that if $\mathbb{Z}_\beta^+$ is a Meyer set, then $\beta$ is a Pisot or a Salem number [Mey95].

### 1.3. Geometric representation

The right one-sided shift $X_\beta^r$ admits as a natural geometric representation the interval $[0, 1]$; namely, one associates with a sequence $(u_i)_{i\geq 1} \in X_\beta^r$ its real value $\sum_{i\geq 1} u_i \beta^{-i}$. We even have a measure-theoretical isomorphism between $X_\beta^r$ endowed with the shift, and $[0, 1]$ endowed with the map $T_\beta$. We want now to give a similar geometric interpretation of the set $X_\beta^l$ as the central tile (defined as an explicit compact set in the product of Euclidean spaces) of a self-replicating multiple tiling. We first need to introduce some algebraic formalism in order to embed $\mathbb{Z}_\beta^+$ in a hyperplane provided by the algebraic conjugates of $\beta$; the closure of the "projected" points will be defined as the central tile. Let us note that we shall give a geometric interpretation to this projection process in

Section 3, and a geometric representation of the whole two-sided shift $X_\beta$ in Section 5.1.

**Canonical embedding.** Let $\beta^{(2)}, \ldots, \beta^{(r)}$ denote the real conjugates of $\beta$, and let $\beta^{(r+1)}, \overline{\beta^{(r+1)}}, \ldots, \beta^{(r+s)}, \overline{\beta^{(r+s)}}$ be its complex conjugates. If $d$ denotes the degree of $\beta$, then $d = r + 2s$. We set $\beta^{(1)} = \beta$. Let $\mathbb{K}^{(k)}$ be the complete field containing $\beta^{(k)}$, that is to say $\mathbb{R}$ if $1 \le k \le r$, and $\mathbb{C}$, if $k > r$. We furthermore denote by $\mathbb{K}_\beta$ the *representation space*

$$\mathbb{K}_\beta := \mathbb{R}^{r-1} \times \mathbb{C}^s \simeq \mathbb{R}^{d-1}.$$

For $x \in \mathbb{Q}(\beta)$ and $1 \le i \le r$, let $x^{(i)}$ denote the conjugate of $x$ in $\mathbb{K}^{(i)}$. Let us consider now the following algebraic embeddings:

- The *canonical embedding* on $\mathbb{Q}(\beta)$ maps a polynomial to all its conjugates
  $\Phi_\beta : \mathbb{Q}(\beta) \to \mathbb{K}_\beta, \ P(\beta) \mapsto (P(\beta^{(2)}), \ldots, P(\beta^{(r)}), P(\beta^{(r+1)}), \ldots, P(\beta^{r+s})).$
- The series $\lim_{n\to\infty} \Phi_\beta(\sum_{i=0}^n w_i\beta^i) = \sum_{i\ge 0} w_i \Phi_\beta(\beta^i)$ are convergent in $\mathbb{K}_\beta$ for every $(w_i)_{i\ge 0} \in X_\beta^l$. The *representation map of* $X_\beta^l$ is then defined as
  $\varphi_\beta : \ X_\beta^l \to \mathbb{K}_\beta, \ (w_i)_{i\ge 0} \mapsto \lim_{n\to\infty} \Phi_\beta(\sum_{i=0}^n w_i\beta^i).$

Note that the map $\varphi_\beta$ is continuous, hence the image of a closed set in $X_\beta^l$, which thus is compact, is again a compact set.

**Definition 2** *We define the* central tile $\mathcal{T}_\beta$ *as*

$$\mathcal{T}_\beta = \overline{\Phi_\beta(\mathbb{Z}_\beta^+)} = \varphi_\beta(X_\beta^l).$$

There is a natural decomposition of $\mathbb{Z}_\beta^+$ according to the value taken by the function $x \mapsto \mathrm{Succ}(x) - x$. Let $n$ denote the number of states in the minimal automaton $\mathcal{M}_\beta$. For any $1 \le i \le n$, then $\mathrm{Succ}(x) - x = T_\beta^{i-1}(1)$ if and only if the last state read is the state $i$ as labeled on the graphs depicted Figure 3. Hence the central tile can be naturally divided into $n$ pieces, called *basic tiles*, as follows for $1 \le i \le n$:

$$\begin{aligned}\mathcal{T}_\beta(i) &= \overline{\Phi_\beta\left(\{x \in \mathbb{Z}_\beta^+;\ \mathrm{Succ}(x) - x = T^{i-1}(\beta)\}\right)} \\ &= \varphi_\beta(\{w \in X_\beta^l;\ w \text{ is a path in the reversed image of the automaton } \mathcal{M}_\beta \\ &\qquad\qquad \text{starting from state } i\}),\end{aligned}$$

where the reversed image of $\mathcal{M}_\sigma$ is obtained by reversing the direction of all the arrows in $\mathcal{M}_\beta$.

### 1.4. The self-replicating multiple tiling

We assume in the remaining of this section that $\beta$ is a Pisot number. In order to be able to cover $\mathbb{K}_\beta$ by translates of the basic tiles according to a Delaunay translation set, we would like to consider a set whose image by $\Phi_\beta$ is dense in $\mathbb{K}_\beta$ without being too large: a good candidate is the set $\mathbb{Z}[\beta]_{\geq 0}$ of nonnegative real numbers in $\mathbb{Z}[\beta]$. Indeed it is proved in [Aki99] (Proposition 1) that $\Phi_\beta(\mathbb{Z}[\beta]_{\geq 0})$ is dense in $\mathbb{K}_\beta$, the proof being based on Kronecker's approximation theorem. According to [Aki02], we introduce the (countable) set $\mathrm{Frac}(\beta) \subset X_\beta^r$ defined as the set of $\beta$-expansions of real numbers in $\mathbb{Z}[\beta] \cap [0,1)$.

$$\mathrm{Frac}(\beta) = \{d_\beta(x),\, x \in \mathbb{Z}[\beta] \cap [0,1)\} \subset X_\beta^r.$$

Let $u = (u_i)_{i\geq 1} \in \mathrm{Frac}(\beta)$. By definition of $\mathrm{Frac}(\beta)$, then $\sum_{i\geq 1} u_i\beta^{-i} \in \mathbb{Q}(\beta)$; hence we can apply $\Phi_\beta$ to $\sum_{i\geq 1} u_i\beta^{-i}$. We define the tile $\mathcal{T}_u$ as

$$\mathcal{T}_u : \ = \overline{\Phi_\beta(\{W_M\beta^M + \cdots + W_0 + u_1\beta^{-1} + \cdots + u_l\beta^{-L} + \cdots ;\ (\cdots 000W_M \cdots W_0, u) \in X_\beta\})}.$$

An immediate consequence is

$$\mathcal{T}_u \ = \phi_\beta(\textstyle\sum_{i\geq 1} u_i\beta^{-i}) + \varphi_\beta(\{w \in X_\beta^l;\, (w,u) \in X_\beta\}).$$

Hence the tiles $\mathcal{T}_u$ are finite unions of translates of the basic tiles $\mathcal{T}(i)$ for $1 \leq i \leq n$ by considering the minimal automaton $\mathcal{M}_\beta$; furthermore, it is proved in [Aki02] that there are at most $n$ such tiles up to translation.

**Theorem 1 ([Aki02, Aki99])** *We assume that $\beta$ is a* Pisot unit. *The set*

$$\Gamma_\beta := \Phi_\beta\left(\{\textstyle\sum_{i\geq 1} u_i\beta^{-i};\ u \in \mathit{Frac}(\beta)\}\right)$$

*is a Delaunay set. The (finite up to translation) set of tiles $\mathcal{T}_u$, for $u \in \mathit{Frac}(\beta)$, covers $\mathbb{K}_\beta$, that is,*

$$\mathbb{K}_\beta = \bigcup_{u\in \mathit{Frac}(\beta)} \mathcal{T}_u = \bigcup_{1\leq i\leq n} \ \bigcup_{\substack{u \in \mathit{Frac}(\beta),\\ \gamma = \Phi_\beta(\sum_{j\geq 1} u_j\beta^{-j}),\\ (u_j) \text{ being the label of}\\ \text{a path read from state } i}} \mathcal{T}(i) + \gamma. \tag{3}$$

*For each $u$, the tile $\mathcal{T}_u$ has non-empty interior, hence it has non-zero measure.*

*Proof.* Let us prove that the set of translation vectors $\Gamma_\beta$ is a Delaunay set. Let $u$ be the $\beta$-fractional part of the nonegative number $R(\beta) \in \mathbb{Z}[\beta]$ that we expand in base $\beta$; there thus exist $P_1, P_2 \in \mathcal{A}_\beta[X]$ such that $\sum_{i\geq 1} u_i\beta^{-i} = (P_1 - P_2)(\beta) = P_3(1/\beta)$, with $P_3$ with coefficients in a finite set that only depends on the unit $\beta$. The uniform discreteness is thus a direct consequence of the fact that if $\mathcal{S} \subset \mathbb{Q}(\beta)$ is a finite set, then $\{\Phi_\beta(P(1/\beta));\ P \in \mathcal{S}[X]\}$ is a discrete set in $\mathbb{K}_\beta$ ([SW02], Lemma 2.3). The proof of this result is based on the fact that for a given norm $|| \ ||$ in $\mathbb{K}_\beta$, for any constant $C > 0$, there are only finitely many algebraic integers $x$ in $\mathbb{Q}(\beta)$ such that $|x| < C$ and $||\Phi_\beta(x)|| < C$. Since $\beta$ is assumed to be a unit, then $1/\beta$ is an algebraic integer as well as $P(1/\beta)$, for any $P \in S[X]$. Furthermore, there exists $C' > 0$ such that for any polynomial $P \in S[X]$, $P(1/\beta) < C'$. Hence, for a fixed $C > 0$, the set of elements $\Phi_\beta(P(1/\beta))$, $P \in \mathcal{S}[X]$, whose norm is smaller than $C$, is finite.

We now use the fact that $\Phi_\beta(\mathbb{Z}[\beta]_{\geq 0})$ is dense in $\mathbb{K}_\beta$ ([Aki99], Proposition 1). We first deduce that $\Gamma_\beta$ is relatively dense. Let us now prove that one has the covering (3). Let $x \in \mathbb{K}_\beta$. There exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials in $\mathbb{Z}[X]$ with $P_n(\beta) \geq 0$, for all $n$, such that $(\Phi_\beta(P_n(\beta)))_n$ tends towards $x$. For all $n$, $P_n(\beta) \in \mathcal{T}_{u^{(n)}}$, with $u^{(n)}$ the $\beta$-fractional part of $P_n(\beta)$. By uniform discreteness of $\Gamma_\beta$, there exist infinitely many $n$ such that $u^{(n)}$ take the same value, say, $u$. Since the tiles are closed, $x \in \mathcal{T}_u$. We now deduce form Baire's theorem that the tiles have non-empty interior. ■

**IFS structure.** Let us prove now that our basic tiles are *graph-directed attractors* for a graph-directed self-affine Iterated Function System, according to the formalism of [MW88, LW96].

We denote by $\mathcal{S}_\beta := \{1, \ldots, n\}$ the set of states of the minimal automaton $\mathcal{M}_\beta$. The notation $a \mapsto_i b$ stands for the fact that there exists an arrow labeled by $i$ (in $\mathcal{A}_\beta$) from $a$ to $b$ in the minimal automaton $\mathcal{M}_\beta$. We denote by $h_\beta : \mathbb{K}_\beta \to \mathbb{K}_\beta$ the $\beta$-multiplication map that multiplies the coordinate of index $i$ by $\beta^{(i)}$, for $2 \leq i \leq d$.

**Theorem 2 ([Aki02, SW02, IR04])** *Let $\beta$ be a Parry number. The basic tiles of the central tile $\mathcal{T}_\beta$ are solutions of the following graph-directed self-affine Iterated Function System:*

$$\forall a \in \mathcal{S}_\beta,\ \mathcal{T}_\beta(a) = \bigcup_{b\in\mathcal{S}_\beta,\ b\mapsto_i a} h_\beta(\mathcal{T}(b)) + \Phi_\beta(i). \tag{4}$$

*If $\beta$ is assumed to be a Pisot unit, then the basic tiles have disjoint interiors and they are the closure of their interior. Furthermore, there exists an integer $k \geq 1$ such that the covering (3) is almost everywhere $k$-to-one. This multiple tiling is repetitive: any finite collection of tiles up to translation reoccurs in the tiling at a bounded distance from any point of the tiling.*

Let us observe that the subdivision matrix of the IFS is the adjacency matrix of the reversed image of $\mathcal{M}_\beta$. Let us note that this matrix is *primitive* (it admits a power with only positive entries) since the graph is strongly connected and aperiodic (the lengths of its cycles are relatively prime since there exist cycles of length 1).

*Proof.* Let $a \in \mathcal{S}_\beta$ be given. Let $w = (w_k)_{k\geq 0} \in X_\beta^l$ such that $w$ is a path in the automaton $\mathcal{M}_\beta$ starting from state $a$. One has:

$$\begin{aligned}\Phi_\beta(w) &= \Phi_\beta\left(\textstyle\sum_{k\geq 1} w_k\beta^k\right) + \Phi_\beta(w_0) = h_\beta \circ \Phi_\beta\left(\textstyle\sum_{k\geq 1} w_k\beta^{k-1}\right) + \Phi_\beta(w_0)\\ &= h_\beta \circ \Phi_\beta((w_k)_{k\geq 1}) + \Phi_\beta(w_0).\end{aligned}$$

One deduces (4) by noticing that $(w_k)_{k\geq 1}$ is a path in the reversed image of $\mathcal{M}_\beta$ starting from state $b$ with $b \mapsto_{w_0} a$ in $\mathcal{M}_\beta$.

We assume that $\beta$ is a Pisot unit. The present proof is an adaption of [SW02] concerned with Pisot substitution dynamical systems. We use the following property of [Sie03] describing the action of the multiplication map $h_\beta$ on the Lebesgue measure $\mu_{\mathbb{K}_\beta}$ of $\mathbb{K}_\beta$: for every Borelian set $B$ of $\mathbb{K}_\beta$, $\mu_{\mathbb{K}_\beta}(h_\beta(B)) = \frac{1}{\beta}\mu_{\mathbb{K}_\beta}(B)$. Let us prove that the union in (4) is a disjoint union up to sets of zero measure. One has for a given $a \in \mathcal{S}_\beta$ according to (4) )

$$\begin{aligned} \mu_{\mathbb{K}_\beta}(\mathcal{T}_\beta(a)) &\leq \textstyle\sum_{b, b \not\mapsto_i a} \mu_{\mathbb{K}_\beta}(h_\beta(\mathcal{T}_\beta(b))) \\ &\leq 1/\beta \textstyle\sum_{b, b\not\mapsto_i a} \mu_{\mathbb{K}_\beta}(\mathcal{T}_\beta(b)). \end{aligned} \tag{5}$$

Let $\mathbf{m} = (\mu_{\mathbb{K}_\beta}(\mathcal{T}_\beta(a)))_{a\in\mathcal{S}_\beta}$ denote the vector in $\mathbb{R}^n$ of measures in $\mathbb{K}_\beta$ of the basic tiles; we know from Theorem 1 that $\mathbf{m}$ is a non-zero vector with nonnegative entries. According to Perron-Frobenius theorem, the previous equality implies that $\mathbf{m}$ is an eigenvector of the adjacency matrix of the reversed image of $\mathcal{M}_\beta$ which is primitive. We thus have equality in (5) which implies that the unions are disjoint up to sets of zero measure. One similarly proves that this equality in measure still holds by replacing $\beta$ by $\beta^n$, for every $n$.

Take two distinct pieces, $\mathcal{T}_\beta(b)$ and $\mathcal{T}_\beta(c)$ say, with $b \neq c$. By primitivity and from the shape of $\mathcal{M}_\beta$, there exist two paths of the same length $n$ starting respectively from state $b$ and $c$ in $\mathcal{M}_\beta$ and arriving at state $a$. Hence both basic tiles occur in (4), when iterated $n$ times and for $i = 0$, with the same translation term (which is indeed equal to 0) and they are thus distinct. We have proved that the $n$ basic tiles are disjoint up to sets of zero measure.

We deduce that the basic tiles are the closure of their interior from the unicity of the solution of the IFS [MW88], since the closure of the interior of the pieces satisfy the same IFS equation [SW02].

It remains to deduce from Lemma 1 below that there exists an integer $k$ such that this covering is almost everywhere $k$-to-one.

**Lemma 1** *Let $(\Omega_i)_{i\in I}$ be a collection of open sets in $\mathbb{R}^k$ such that i) $\cup_{i\in I}\overline{\Omega_i} = \mathbb{R}^k$, ii) for any compact set $K$, $I_k := \{i \in I;\ \overline{\Omega_i} \cap K \neq \emptyset\}$ is finite. For $x \in \mathbb{R}^k$, let $f(x) := Card\{i \in I;\ x \in \overline{\Omega_i}\}$. Let $\Omega = \mathbb{R}^k \setminus \cup_{i\in I}\delta(\omega_i)$, where $\delta(\Omega_i)$ denotes the boundary of $\Omega_i$. Then $f$ is locally constant on $\Omega$.*

The repetitivity is a consequence of the primitivity of the matrix $\mathbf{M}_\sigma$. Assume now that there exist $B(x,r)$ and $B(y,r)$ with $f(x) = k \leq f(y)$. We deduce from (4) that the covering $h_\beta^{-n}(B(x,r))$ has the same property. We then use repetitivity. For $N$ large enough, the ball $h_\beta^{-N}(B(x,r))$ contains $B(y,r)$, which ends the proof of Theorem 2. ∎

*Proof of Lemma 1.* According to ii), one deduces that $\Omega$ is an open set in $\mathbb{R}^k$. By i), one has for all $x$ $f(x) \geq 1$, and by ii), $f$ takes bounded values.

For any fixed $\ell \in \mathbb{N}$, the set $J_\ell := \{x \in \mathbb{R}^k; f(x) \geq \ell\}$ is a closed set of $\mathbb{R}^k$. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of elements in $J_\ell$ and let $x$ denote its limit. There exist $(i_1^{(n)})_{n\in\mathbb{N}}, \ldots, (i_\ell^{(n)})_{n\in\mathbb{N}}$ such that for all $n$, $x_n \in \overline{\Omega}_{i_j^{(n)}}$, for $1 \leq j \leq \ell$. By using ii), there exist $i_1, \ldots, i_\ell$ such that for infinitely many $n$, $x_n \in \overline{\Omega_{i_j}}$, for $1 \leq j \leq \ell$, hence $x \in \cap_{1\leq j\leq \ell}\overline{\Omega_{i_j}}$ and $f(x) \geq \ell$.

Let us introduce now for $x \in \mathbb{R}^k$, $g(x) := \mathrm{Card}\{i \in I;\ x \in \Omega_i\}$. Let us note that $f$ and $g$ do coincide over $\Omega$. We similarly prove that any fixed $\ell \in \mathbb{N}$, $\{x \in \mathbb{R}^k; g(x) \geq \ell\}$ is an open set of $\mathbb{R}^k$.

Now let $x \in \Omega$; let $r > 0$ such that the open ball $B(x,r)$ of center $x$ and radius $r$ is included in $\Omega$; such a ball exists since $\Omega$ is an open set. For all $\ell$, $B_\ell := \{y \in B(x,r);\ f(y) \geq \ell\}$ is both an open and a closed set of $B(x,r)$, from what precedes. Hence it is either equal to the empty set or to $B(x,r)$, by connectedness of $B(x,r)$. We have

$$\ldots B_\ell \subset B_{\ell-1} \subset \cdots \subset B_1 \subset B_0 = B(x,r).$$

¿From ii), one cannot get for all $\ell$, $B_\ell = B(x,r)$. Let $\ell_0 = \max\{\ell;\ B_\ell = B(x,r)\}$. For all $y \in B(x,r)$, one has $f(y) \geq \ell_0$, but $B_{\ell_0+1} = \emptyset$, hence $f(y) < \ell_0 + 1$; this thus implies that $\forall y \in B(x,r)$ $f(y) = \ell_0$. ■

### 1.5. An example: The Tribonacci number.

Let $\beta$ be the *Tribonacci number*, that is, the Pisot root of the polynomial $X^3-X^2-X-1$. One has $d_\beta(1) = 111$ ($\beta$ is a simple Parry number) and $d_\beta^*(1) = (110)^\infty$. Hence $X_\beta$ is the set of sequences in $\{0,1\}^{\mathbb{Z}}$ in which there are no three consecutive 1's. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is reduced to the $\mathbb{Q}$-isomorphism $\tau_\alpha$ which maps $\beta$ on $\alpha$, where $\alpha$ is one of the complex roots of $X^3 - X^2 - X - 1$. The set $\mathcal{T}_\beta$ which satisfies

$$\mathcal{T}_\beta = \{\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_iw_{i+1}w_{i+2} \neq 0\}$$

is a compact subset of $\mathbb{C}$ called the *Rauzy fractal*. This set was introduced in [Rau82], see also [IK91, Mes98, Mes00]. It is shown in Fig. 4 with its division into the three basic tiles $\mathcal{T}(i)$, $i = 1,2,3$, indicated by different shades. They correspond respectively to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_10 = 11$; this is easily seen thanks to the automaton $\mathcal{M}_\beta$ shown in Fig. 4. One has

$$\begin{cases} \mathcal{T}_\beta(0) = \alpha(\mathcal{T}_\beta(0) \cup \mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2)) \\ \mathcal{T}_\beta(1) = \alpha(\mathcal{T}_\beta(0)) + 1 \\ \mathcal{T}_\beta(2) = \alpha(\mathcal{T}_\beta(1)) + 1. \end{cases}$$

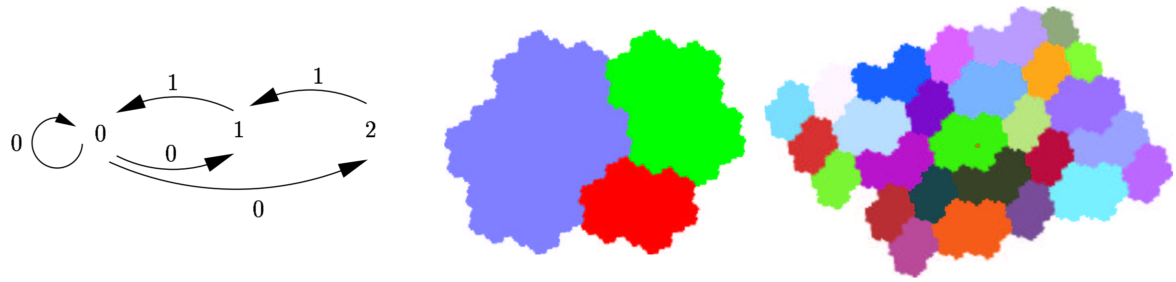



Figure 4: Tribonacci number: the minimal automaton $\mathcal{M}_\beta$ and the central tile divided into its basic tiles; The self-replicating multi-tiling

One interesting property in this numeration is that the $\beta$-fractional parts of the elements of $\mathbb{Z}[\beta]_{\geq 0}$ are all finite [FS92], as detailed in Section 1.6. This implies that the multiple tiling is a tiling.

If $U$ is a finite word which is a $\beta$-fractional part, and which begins with the letter 0, then $\mathcal{T}_U = \mathcal{T} = \mathcal{T}_\beta(0) \cup \mathcal{T}_\beta(1) \cup \mathcal{T}_\beta(2)$; if $U$ begins with the factor 10, then $\mathcal{T}_U = \mathcal{T}_\beta(0) \cup \mathcal{T}_\beta(1)$; if $U$ begins with the factor 11, then $\mathcal{T}_U = \mathcal{T}\beta(0)$. The corresponding self-replicating tiling is shown Fig. 1.

### 1.6. Finiteness conditions

Let us recall that the tiles in the covering (3) are labeled by the fractional parts of elements in $\mathbb{Z}[\beta]_{\geq 0}$. When the elements of $\mathbb{Z}[\beta]_{\geq 0}$ all have a finite fractional part, as in the Tribonacci case, then much more can be said. This finiteness condition is called the (F) property, *finiteness property*, and has been introduced by C. Frougny and B. Solomyak [FS92]: an algebraic integer $\beta > 1$ is said to have property (F) if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta]_{\geq 0}. \qquad (F)$$

Property (F) implies that $\beta$ is a Pisot number which is a simple Parry number; hence not all Pisot numbers have property (F). A sufficient condition for $\beta$ to satisfy the (F) property is the following: if $\beta > 1$ is the dominant root (that is, if it has the maximal modulus along all the roots) of the polynomial $X^d - t_1 X^{d-1} - \cdots t_d$, with $t_i \in \mathbb{N}$, $t_1 \geq t_2 \geq \cdots \geq t_d \geq 1$, then $\beta$ satisfies (F) [FS92]; the same conclusion holds if more generally $t_1 > t_2 + \cdots + t_d$ [Hol96]. A complete characterization of some families of Pisot numbers with property (F) exists: the quadratic case is studied in [FS92], the case of cubic units has been considered in [Aki00].

The (W) condition, called *weak finiteness condition*, has first been introduced by M. Hollander in [Hol96]: he has proved that the (W) property implies the pure discreteness of the spectrum of the beta-shift. The (W) condition can be stated as follows for $\beta$ Pisot number:

$$\forall z \in \mathbb{Z}[\beta^{-1}] \cap [0,1),\ \forall \varepsilon > 0,\ \exists x, y \in \mathrm{Fin}(\beta) \text{ such that } z = x - y \text{ and } y < \varepsilon. \quad (W)$$

Note that if $\beta$ has property (W), then it must be a Pisot or a Salem number [ARS04]. All the quadratic units [FS92] and all the cubic units [ARS04] are known to satisfy (W). The (W) property has been proved in [Aki02] to be equivalent with the fact that the multiple tiling (3) is in fact a tiling. Hence the Pisot conjecture is equivalenty reformulated as:

**Conjecture** [Akiyama [Aki02], Sidorov [Sid03b]] *For every Pisot number $\beta$, then (W) holds.*

An algorithm is described in [ARS04] which can tell whether a given Pisot $\beta$ has property (F) or (W). It is proved in [ARS04] that (W) holds for all the cubic units and in higher degree, for each dominant root of a polynomial $X^d - t_1 X^{d-1} - \cdots t_d$,with $t_i \in \mathbb{N}$, $t_1 > |t_2| + \cdots + |t_d|$, and $(t_1, t_2) \neq (2, -1)$.

The (F) and (W) properties can be reformulated in topological terms: (F) is equivalent with the fact that the origin is an inner *exclusive point* of the central tile [Aki99, Aki02]; an inner point in a tile $\mathcal{T}_u$ is said *exclusive* if it is contained in no other tile $\mathcal{T}_v$ with $u \neq v$; (W) is equivalent to the fact that there exists an exclusive inner point in the central tile [Aki02].

More generally, the study of the topological properties of the central tile is one important issue of the domain. The connectedness of the central tile in the Pisot unit case is studied in [AN04b]: it is proved that when $\beta$ is a Pisot number of degree 3, then each central tile is arcwise connected, but examples of Pisot numbers of degree 4 are produced with a disconnected central tile. In particular it is proved in [AN04a] that each tile corresponding to a Pisot unit is arcwise connected if $d_\beta(1)$ is finite and terminates with 1. A complete description of the $\beta$-expansion of 1 is also given for cubic and quartic Pisot units; for the general case of cubic Pisot numbers, see [Bas02].

These results are inspired by techniques used for some particular generalized radix representations, the so-called Canonical Number System case. Note that there has been a recent and very promising attempt through the notion of Radix Number Systems [ABB$^+$05] to embrace both approaches, that is, beta-numeration and Canonical Number Systems: in both cases tiling properties can be ensured by similar finiteness properties, that can be expressed in terms of the orbit of a certain dynamical system.

## 2. Substitution numeration systems and Rauzy fractals

The first example of a central tile associated with a substitution is the Tribonacci tile (also known as *Rauzy fractal*), which is due to Rauzy [Rau82]. (Let us note that the central tile is usually called Rauzy fractal when associated with a substitution, but for the sake of consistency, we call it here again central tile,and introduce the term Rauzy fractal in Section 3.1 for its geometric representation.) The associated tiling is a lattice tiling having some deep dynamical interpretation. We discuss it in Section 3. One natural question is to figure out what structure can play the role of $\mathbb{Z}[\beta]_{\geq 0}$. We thus introduce for that purpose a numeration system, the Dumont-Thomas numeration sytem, based on the substitution, that allows one to expand real numbers: the only difference is that the digits will not only belong to $\mathbb{Z}$ but to some finite subset of $\mathbb{Z}[\beta]$.

We recall in Section 2.1 basic definitions on substitutive dynamical systems; we work out the notion of desubstitution in Section 2.2; a family of substitutions, the so-called $\beta$-substitutions, which allows us to recover the beta-numeration developed in Section 1, is described in Section 2.3, in order to introduce in Section 2.4 the Dumont-Thomas numeration; we then consider the self-replicating multiple tiling in Section 2.5. Let us note that we need an extra combinatorial assumption, the so-called strong coincidence condition, so that the basic tiles have distinct interiors, which always holds for $\beta$-substitutions.

The main assumption made in this section is that then substitution $\sigma$ is supposed to be Pisot with a dominant eigenvalue $\beta$ Pisot unit: the characteristic polynomial of the substitution may thus be reducible.

## 2.1. Substitutions

A *substitution* $\sigma$ is an endomorphism of the free-monoïd $\mathcal{A}^*$ such that the image of any letter of $\mathcal{A}$ never equals the empty word $\varepsilon$, and for at least one letter $a$, we have $|\sigma^n(a)| \to +\infty$. A substitution naturally extends to the set of bi-infinite words $\mathcal{A}^{\mathbb{Z}}$:

$$\sigma(\dots w_{-2}w_{-1}.w_0w_1\dots) = \dots\sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\dots$$

The two assumptions above guarantee the existence of bi-infinite words generated by iterating the substitution. To be more precise, a *periodic point* of $\sigma$ is a bi-infinite word $u = (u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ that satifies $\sigma^n(u) = u$ for some $n > 0$ and for which there exists a letter $a$ such that every factor $W$ of $u$ is also a factor of some iteration $\sigma^{n(W)}(a)$; if $\sigma(u) = u$, then $u$ is a *fixed point* of $\sigma$. Every substitution has at least one periodic point [Que87]. The substitution is said *shift-periodic* when there exists a bi-infinite word that is periodic for both the shift map $S$ and the substitution $\sigma$.

A substitution $\sigma$ is said *primitive* if there exists an integer $n$ (independent of the letters) such that $\sigma^n(a)$ contains at least one occurrence of the letter $b$ for every pair $(a, b) \in \mathcal{A}^2$. In that case, if $u$ is a periodic point for $\sigma$, then the closure $X_u$ in $\mathcal{A}^{\mathbb{Z}}$ of the shift orbit of $u$ does not depend on $u$ and we denote by $(X_\sigma, S)$ the *symbolic dynamical system generated by* $\sigma$. The system $(X_\sigma, S)$ is *minimal* (every non-empty closed shift-invariant subset equals the whole set) and *uniquely ergodic* (there exists a unique shift-invariant probability measure $\mu_{X_\sigma}$ on $X_\sigma$ [Que87]); it is made of all the bi-infinite words whose set of factors coincides with the set of factors $F_\sigma$ of $u$ (which does not depend of the choice of $u$ by primitivity).

**Incidence matrix.** Let $\mathbf{l} : \mathcal{A}^* \mapsto \mathbb{N}^n$ be the natural homomorphism obtained by abelianization of the free monoid; if $\mathcal{A} = \{1, \ldots, n\}$ and $|W|_a$ denotes the number of occurrences of the letter $a \in \mathcal{A}$ in a finite word $W$, then we have $\mathbf{l}(W) = (|W|_k)_{k=1,\ldots,n} \in \mathbb{N}^n$. To each substitution $\sigma$ on $\mathcal{A}$ is canonically associated its *abelianization linear map* whose matrix $\mathbf{M}_\sigma = (m_{i,j})_{1\le i,j\le n}$ (called *incidence matrix of* $\sigma$) is defined by $m_{i,j} = |\sigma(j)|_i$, so that we have $\mathbf{l}(\sigma(W)) = \mathbf{M}_\sigma \mathbf{l}(W)$ for every $W \in \mathcal{A}^*$. If $\sigma$ is primitive, the *Perron-Frobenius theorem* says that the incidence matrix $\mathbf{M}_\sigma$ has a simple real positive dominant eigenvalue $\beta$.

A substitution $\sigma$ is *of Pisot type* if its incidence matrix $\mathbf{M}_\sigma$ has a dominant eigenvalue $\beta$ such that for every other eigenvalue $\lambda$, one gets: $0 < \lambda < 1 < \beta$. The characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$. We deduce [Fog02] that the dominant eigenvalue $\beta$ is a Pisot number, substitutions of Pisot type are primitive, and that substitutions of Pisot type are not shift-periodic. For this last point, it is indeed easy to recognize whether a substitution is not shift-periodic following [HZ98]: if $\sigma$ is a primitive substitution the matrix of which has a non-zero eigenvalue of modulus less that 1, then no fixed point of $\sigma$ is shift-periodic. Hence, if a substitution is of Pisot type then its characteristic polynomial is irreducible, whereas when the dominant eigenvalue of a primitive substitution is assummed to be a Pisot number, it may be reducible. We do not need any irreducibility assumption in all this section. This assumption will be required during Section 3 and in Section 4.

A substitution $\sigma$ is said *unimodular* if det $\mathbf{M}_\sigma = \pm 1$. The assumption corresponding to $\beta$ Pisot unit made in Section 1 becomes here that $\sigma$ is assumed to be a Pisot unimodular substitution.

### 2.2. Combinatorial numeration system: desubstitution

We need to be able to desubstitute, that is, to define a notion of inverse map for the action of the substitution $\sigma$ on $X_\sigma$. For that purpose, we decompose any $w \in X_\sigma$ as a combinatorial power series. Hence, a combinatorial expansion is defined on $X_\sigma$,which plays the role of an exotic numeration system on bi-infinite sequence. The action of $\sigma$ can be compared with the action of the shift in $X_\beta$ (but not with the action of the shift on $X_\sigma$).

**Desubstitution: a combinatorial division by $\sigma$.** Every bi-infinite word $w \in X_\sigma$ has a unique decomposition $w = S^\nu(\sigma(v))$, with $v \in X_\sigma$ and $0 \le \nu < |\sigma(v_0)|$, where $v_0$ is the $0^{th}$ coordinate of $v$ [Mos92]. This means that any word of the dynamical system can be uniquely written in the following form, with $\ldots v_{-n} \ldots v_{-1}.v_0 v_1 \ldots v_n \cdots \in X_\sigma$:

$$w = \ldots \mid \underbrace{\ldots}_{\sigma(v_{-1})} \mid \underbrace{w_{-\nu} \ldots w_{-1}.w_0 \ldots w_{\nu'}}_{\sigma(v_0)} \mid \underbrace{\ldots}_{\sigma(v_1)} \mid \underbrace{\ldots}_{\sigma(v_2)} \mid \cdots$$

Here, the doubly infinite word $v$ appears to be the "quotient" of $w$ in the "division" by $\sigma$. The "rest" of this division consists in the the three-tuple $(p, w_0, s)$, that is, the decomposition of $\sigma(v_0)$ of the form $pw_0s$, where $p = w_{-\nu} \dots w_{-1}$ (prefix) and $s = w_1 \dots w_{\nu'}$ (suffix). The word $w$ is completely determined by the quotient $v$ and the rest $(p, w_0, s)$.

Let $\mathcal{P}$ be the finite set of all rests or *digits*, called *prefix-suffix set* associated with $\sigma$:

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pas\}\,.$$

The *desubstitution map* $\theta : X_\sigma \to X_\sigma$ maps a bi-infinite word $w$ to its quotient $v$. The decomposition of $\sigma(v_0)$ of the form $pw_0s$ is denoted $\gamma : X_\sigma \to \mathcal{P}$ (mapping $w$ to $(p, w_0, s)$).

**Prefix-suffix expansion.** The *prefix-suffix expansion* is the map $E_{\mathcal{P}} : X_\sigma \to \mathcal{P}^{\mathbb{N}}$ which maps a word $w \in X_\sigma$ to the sequence $(\gamma\,(\,\theta^i w)\,)_{i \geq 0} \in \mathcal{P}^{\mathbb{N}}$, that is, the orbits of $w$ through the desubstitution according to the partition defined by $\gamma$. For example, the prefix-suffix expansion of periodic points for $\sigma$ has only empty prefixes.

Let $w \in X_\sigma$ and $E_{\mathcal{P}}(w) = (p_i, a_i, s_i)_{i \geq 0}$ be its prefix-suffix expansion. If there are infinitely many prefixes and suffixes that are non-empty, then $w$ and $E_{\mathcal{P}}(w)$ satisfy:

$$w = \lim_{n \to \infty} \sigma^n(p_n) \dots \sigma(p_1) p_0 . a_0 s_0 \sigma(s_1) \dots \sigma^n(s_n).$$

Hence, the prefix-suffix expansion can be considered as an expansion of the points of $X_\sigma$ in a "combinatorial" power series. The three-tuples $(p_i, a_i, s_i)$ play the role of digits in this combinatorial expansion.

**Prefix-suffix automaton.** Any prefix-suffix expansion is the label of an infinite path in the so-called *prefix-suffix automaton* $\mathcal{M}_\sigma$ of $\sigma$, whose set of vertices is the alphabet $\mathcal{A}$ and whose edges satisfy the following: there exists an edge labeled by $(p, a, s) \in \mathcal{P}$ from $b$ toward $a$ if $pas = \sigma(b)$; we set $b \mapsto_{(p,a,s)} a$. The automaton for the Tribonacci substitution $1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$ is given in Figure 5.

Let us note that one can associate a prefix-suffix automaton with any substitution (even if it is neither primitive nor non-shift-periodic). The adjacency matrix of the prefix-suffix automaton (one indentifies the automaton with the underlying graph) is exactly the transpose of the incidence matrix of the substitution. Hence the substitution $\sigma$ is primitive if and only if its prefix-sufix automaton is strongly connected and aperiodic (the lengths of its cycles are relatively prime).

Denote by $X_{\mathcal{P}}$ the set of path labels in the prefix-suffix automaton; it is the support of a subshift of finite type. Any such path is the expansion of a bi-infinite word, since the map $E_{\mathcal{P}}$ is continuous and onto $X_{\mathcal{P}}$. A countable number of bi-infinite words are not characterized by their prefix-suffix expansion: $E_{\mathcal{P}}$ is one-to-one except on the orbit denoted $X_\sigma^{per}$ of periodic points of $\sigma$, where it is finite-to-one (see the proofs in [CS01a, HZ01]).

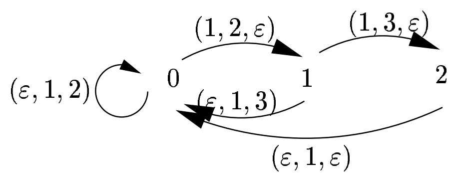



Figure 5: Prefix-sufix automaton for the Tribonacci substitution.

## 2.3. A specific case: $\beta$-substitution

Let $\beta > 1$ be a Parry number such as defined in Section 1. As introduced for instance in [Thu89] and in [Fab95], one can associate in a natural way a substitution $\sigma_\beta$, called *$\beta$-substitution* with $(X_\beta, S)$ over the alphabet $\{1, \cdots, n\}$, where $n$ denotes the number of states of the automaton $\mathcal{M}_\beta$: $j$ is the $k$-th letter occuring in $\sigma_\beta(i)$ (that is, $\sigma_\beta(i) = pjs$, where $p, s \in \{1, \cdots, n\}^*$ and $|p| = k - 1$) if and only if there is an arrow in $\mathcal{M}_\beta$ from the state $i$ to the state $j$ labeled by $k - 1$. One easily checks that this definition is consistent.

An explicit formula for $\sigma_\beta$ can be computed by considering the two different cases, $\beta$ simple and $\beta$ non-simple Parry number.

- Assume $d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite, with $t_n \neq 0$. Thus $d^*_\beta(1) = (t_1 \dots t_{n-1}(t_n - 1))^\infty$. One defines $\sigma_\beta$ over the alphabet $\{1, 2, \dots, n\}$ as shown in (6).
- Assume $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 [Fro02]). Hence $d_\beta(1) = d^*_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+p})^\infty$, with $m \geq 1$, $t_m \neq t_{m+p}$ and $t_{m+1} \dots t_{m+p} \neq 0^p$. One defines $\sigma_\beta$ over the alphabet $\{1, 2, \dots, m+p\}$ as shown in (6).

As a consequence of the definition, the incidence matrix of $\sigma_\beta$ coincides with the transpose of the adjacency matrix of the automaton $\mathcal{M}_\beta$.

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1} 2 \\ 2 & \mapsto 1^{t_2} 3 \\ \vdots & \vdots \\ n-1 & \mapsto 1^{t_{n-1}} n \\ n & \mapsto 1^{t_n}. \end{cases} \qquad \sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1} 2 \\ 2 & \mapsto 1^{t_2} 3 \\ \vdots & \vdots \\ m+p-1 & \mapsto 1^{t_{m+p-1}} (m+p) \\ m+p & \mapsto 1^{t_{m+p}} (m+1). \end{cases} \tag{6}$$

Substitution associated with a simple Parry number — Substitution associated with a non-simple Parry number

If the number of letters $n$ equals the degree of $\beta$, then $\sigma_\beta$ is a Pisot substitution. Otherwise the characteristic polynomial of the incidence matrix of $\sigma_\beta$ may be reducible.

Hence we cannot apply here directly the substitutive formalism to the substitution $\sigma_\beta$ since it is generally not Pisot. The dominant eigenvalue of $\sigma_\beta$ is still a Pisot number but other eigenvalues may occur, as in the smallest Pisot case: let $\beta$ be the Pisot root of $X^3 - X - 1$; one has $d_\beta(1) = 10001$ ($\beta$ is a simple Parry number) and $d^*_\beta(1) = (10000)^\infty$; $\sigma_\beta:\ 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$; the characteristic polynomial of its incidence matrix is $(X^3 - X - 1)(X^2 - X + 1)$, hence $\sigma_\beta$ is not a substitution of Pisot type. Furthermore, the extra roots that occur need not lie inside the unit circle: consider for instance, as quoted in [Boy89, Boy96], $\beta$ the dominant root of $P(X) = X^7 - 2X^5 - 2X^4 - X - 1$; then the complementary factor $Q(X)$ (such that $P(X)Q(X)$ is the characteristic polynomial of the incidence matrix of $\sigma_\beta$) is non-reciprocal, there thus exist roots outside the unit circle.

The prefix-suffix automaton of the substitution $\sigma_\beta$ is strongly connected with the finite automaton $\mathcal{M}_\beta$ recognizing the set $F_\beta$ of finite factors of the $\beta$-shift $X_\beta$. Let us first note that the proper prefixes of the images of letters contain only the letter 1; it is thus natural to code a proper prefix by its length: if $(p, a, s) \in \mathcal{P}$, then $p = 1^{|p|}$. Hence it is easily seen that one recovers the automaton $\mathcal{M}_\beta$ by replacing in the prefix-suffix automaton the set of labelled edges

$$\mathcal{P} = \{(p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pas\}$$

by the following set of labelled edges

$$\{|p|,\ \text{with } (p, a, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pas\}.$$

Hence, the following relation holds between the set $F_\beta$ of finite factors of the $\beta$-shift $X_\beta$ and the set $F_\mathcal{P}$ of factors of $X_\mathcal{P}$, that is, the set of finite words recognized by the prefix-suffix automaton:

$$w_M \dots w_0 \in F_\beta \text{ iff } \exists\, a_0, \dots a_M \in \mathcal{A},\ s_0, \dots s_M \in \mathcal{A}^\star \quad (1^{w_M}, a_M, s_M) \dots (1^{w_0}, a_0, s_0) \in F_\mathcal{P}. \tag{7}$$

From this relation, one can interpret any point $w_M\beta^M + \cdots + w_0$ in $\mathbb{Z}^+_\beta$ as the real value of the combinatorial expansion $(1^{w_M}, a_M, s_M) \dots (1^{w_0}, a_0, s_0)$ associated with the $\beta$-substitution. We formalize this interpretation in the next section.

### 2.4. Dumont-Thomas numeration

The Dumont-Thomas numeration system [DT89, DT93, Rau90] generalizes the approach given above to any primitive substitution whose dominant eigenvalue is a Pisot number; hence it is a generalized numeration system associated with a given substitution $\sigma$, which is not only defined on the set of nonnegative integers (this provides a very convenient way to expand uniquely prefixes of fixed points of $\sigma$), but also on real numbers.

Let $v$ be a one-sided fixed point of $\sigma$ having $a_0$ as a first letter. Let us first define the numeration on $\mathbb{N}$. This numeration depends on this particular choice of a fixed point, that is, on the letter $a_0$. One checks ([DT89] Theorem 1.5) that every finite prefix of $v$ can be uniquely expanded as $\sigma^n(p_n)\sigma^{n-1}(p_{n-1})\cdots p_0$, where $p_n \neq \varepsilon$, and $(p_n, a_n, s_n)\cdots(p_0, a_0, s_0)$ is the sequence of labels of a path in the prefix-suffix automaton $\mathcal{M}_\sigma$ arriving at the state $a_0$; one has for all $i$, $\sigma(p_i) = p_{i-1}a_{i-1}s_{i-1}$, that is, $a_n \mapsto_{(p_{n-1}a_{n-1}s_{n-1})} a_{n-1} \cdots \mapsto_{(p_0,a_0,s_0)} a_0$. Conversely, any path in $\mathcal{M}_\sigma$ arriving at $a_0$ generates a finite prefix of $v$. This numeration works a priori on finite words but we can expand the nonnegative natural integer $N$ as $N = |\sigma^n(p_n)| + \cdots + |p_0|$, where $N$ denotes the length of the expanded prefix of $v$. The expansions of prefixes of $v$ play here the role of $\mathbb{Z}_\beta^+$ in the beta-numeration case.

Let us now expand real numbers. Let $\beta$ denote the dominant eigenvalue of the incidence matrix of the primitive substitution $\sigma$. We assume that $\beta$ is a Pisot number. We want to expand real numbers in base $\beta$, with digits which may not belong to $\mathbb{Z}$ anymore, but to a finite subset of $\mathbb{Q}(\beta)$. Let $F_{\mathcal{P}}$ denote the set of finite words recognized by the prefix-suffix automaton $\mathcal{M}_\sigma$. We want to define a map $\delta_\sigma : \mathcal{A}^* \to \mathbb{Q}(\beta)$ such that one could associate with a combinatorial expansion $(p_n, a_n, s_n)\ldots(p_0, a_0, s_0) \in F_{\mathcal{P}}$ the real value $\delta_\sigma(p_n)\beta^n + \cdots + \delta_\sigma(p_0) \in \mathbb{Z}[\beta^{-1}]$. Hence the map $\delta_\sigma$ need to satisfy $\delta_\sigma(\sigma(P)) = \beta\delta_\sigma(P)$, so that a natural and suitable choice is given by

$$\delta_\sigma : \ \mathcal{A}^* \to \mathbb{Q}(\beta), \ \delta_\sigma(P) = <\mathbf{l}(P), \mathbf{v}_\beta>,$$

where $\mathbf{v}_\beta$ is a (simple) dominant eigenvector for the transpose of the matrix $\mathbf{M}_\sigma$, i.e., $\mathbf{v}_\beta$ is a left eigenvector associated with $\beta$. To recover the $\beta$-expansion in the case of a $\beta$-substitution, $\mathbf{v}_\beta$ has to be normalized so that its first coordinate is equal to 1. In the substitutive case, we just normalize $\mathbf{v}_\beta$ so that its coordinates belong to $\mathbb{Q}(\beta)$. The map $\delta_\sigma$ sends the letter $a$ on the corresponding coordinate of the left eigenvector. We now get the following representation:

**Theorem 3 ([DT89])** *Let $\sigma$ be a primitive substitution on the alphabet $\mathcal{A}$ whose dominant eigenvalue is a Psiot number. Let us fix $a_0 \in \mathcal{A}$. Every real number $x \in [0, \delta_\sigma(a_0))$ can be uniquely expanded as*

$$x = \sum_{i \geq 1} \delta_\sigma(p_i)\beta^{-i},$$

*where the sequence of digits $(p_i)_{i\geq 1}$ is the projection on the first component of an infinite path $(p_i, a_i, s_i)_{i\geq 1}$ in $\mathcal{M}_\sigma$ issued from the letter $a_0$, that is, for all $i \geq 0$, $\sigma(a_i) = p_{i+1}a_{i+1}s_{i+1}$ with the extra condition that there exists infinitely non-empty suffixes in the sequence $(s_i)_{i\geq 1}$. We call this expansion $(\sigma, a_0)$-expansion of $x$ and denote it $d_{(\sigma,a_0)}(x)$.*

This theorem provides an analogue of the Parry condition (1), the proof being also based on the greedy algorithm. The underlying dynamics depends of each interval $[0, \delta_\sigma(a_0))$, and is defined as follows:

$$\begin{array}{rcl} T_\sigma : \bigcup_{a\in\{1\ldots d\}}[0, \delta_\sigma(a)) \times \{a\} & \mapsto & \bigcup_{a\in\{1\ldots d\}}[0, \delta_\sigma(a)) \times \{a\} \\ (x, a) & \to & (\beta x - \delta_\sigma(p), b) \text{ with } \left\{ \begin{array}{l} \sigma(a) = pbs \\ \beta x - \delta_\sigma(p) \in [0, \delta_\sigma(b)). \end{array} \right. \end{array}$$

Theorem 3 states that this map is well defined, meaning that for every $(x, a)$ there exists a unique $(y, b)$ satisfying the above conditions.

Let us note that one may obtain a different type of numeration for each letter. Nevertheless, one easily checks that for a $\beta$-substitution, then all the associated numerations are consistent with the $\beta$-numeration: in particular, $\mathbf{v}_\beta$ is normalized so that $\delta_\sigma(1) = 1$, and the numeration associated with the letter 1 is exactly the $\beta$-numeration.

This numeration shares many properties with the $\beta$-numeration. In particular, every element of $\mathbb{Q}(\beta) \cap [0, \delta_\sigma(a_0))$ admits an eventually periodic expansion, when $\beta$ is a Pisot number, and conversely. The proof can be conducted exactly in the same way as in [Sch80].

## 2.5. The self-replicating substitution multiple tiling

We now have gathered all the required tools to be able to define the central tile defined as the image under a suitable representation map of the one-dimensional prefix-suffix expansions.

Let $\sigma$ be a primitive unimodular substitution whose dominant eigenvalue $\beta$ is a Pisot unit. The cardinality of the alphabet on which $\sigma$ is defined is denoted $n$ whereas $d$ stands for the algebraic degree of $\sigma$. We use the same notation as in Section 1.5 concerning the *canonical embedding* that we denote here $\Phi_\sigma$; we also denote $\mathbb{K}_\sigma$ the *representation space* (one has $\mathbb{K}_\sigma \simeq \mathbb{R}^{d-1}$). We define the *representation map* as $\varphi_\sigma : X_\sigma \to \mathbb{K}_\beta, \; w \mapsto \lim_{n\to+\infty} \Phi_\sigma\left(\sum_{i\geq 0}^{n} \delta_\sigma(p_i)\beta^i\right)$, with $E_\mathcal{P}(w) = (p_i, a_i, s_i)_{i\geq 0}$. We similarly as in the beta-numeration case define:

$$\begin{aligned}\mathrm{Fin}(\sigma) &= \{\delta_\sigma(q_M)\beta^M + \cdots + \delta_\sigma(p_0) + \delta_\sigma(p_1)\beta^{-1} + + \cdots + \delta_\sigma(p_L)\beta^{-L}; \\ &\qquad M, L \in \mathbb{N},\ (q_M, b_M, r_M)\ldots(p_0, a_0, s_0)(p_1, a_1, s_1)\ldots(p_L, a_L, s_L) \in F_\mathcal{P}\},\end{aligned}$$

$$\mathbb{Z}_\sigma^+ = \{\delta_\sigma(q_M)\beta^M + \cdots + \delta_\sigma(q_0);\ M \in \mathbb{N},\ (q_M, b_M, s_M)\ldots(q_0, b_0, s_0) \in F_\mathcal{P}\} \subset \mathrm{Fin}(\sigma).$$

**Definition 3** *We define the* central tile $\mathcal{T}_\sigma$ *as*

$$\mathcal{T}_\sigma = \overline{\Phi_\sigma(\mathbb{Z}_\sigma^+)} = \varphi_\sigma(X_\sigma).$$

Recall that $n$ denotes the number of letters in the alphabet $\mathcal{A}$ on which $\sigma$ is defined. The central tile is here again divided into $n$ pieces, called *basic tiles*, as follows: for $a \in \mathcal{A}$,

$$\begin{aligned}\mathcal{T}_\sigma(a) &= \varphi_\sigma(\{w \in X_\sigma; E_\mathcal{P}(w) \text{ is a path} \\ &\qquad \text{in the reversed image of the automaton } \mathcal{M}_\sigma \text{ starting from state } a\}).\end{aligned}$$

To fit with the formalism and the proofs developed for the $\beta$-numeration, we intend, for each $a_0 \in \mathcal{A}$, to introduce a set $\mathrm{Frac}(\sigma, a_0)$ defined as the set of fractional $(\sigma, a_0)$-expansions of a suitable set (analogous to $\mathbb{Z}[\beta]_{\geq 0}$) whose image under $\Phi_{sigma}$ has to be relatively dense and which has to provide a Delaunay set when embedding $\mathrm{Frac}(\sigma, a_0)$. A first idea is thus to consider the set $\mathbb{Z}[\beta] \cap [0, \delta_\sigma(a_0))$. However, such a set is not stable anymore through the action of the $\sigma$-expansion $T_\sigma : x \to \beta x - \delta_\sigma(p)$, since $\delta_\sigma(p)$ does not always belong to $\mathbb{Z}$ (nor to $\mathbb{Z}[\beta]$). As a solution to this problem, we introduce the following countable set that contains all finite expansions with digits $\delta_\sigma(p)$:

$$D = \min\{k \in \mathbb{N};\ \forall\, a, b, p, s, \text{ such that } \sigma(a) = pbs,\ s \neq \varepsilon, \text{ then } k\delta_\sigma(p) \in \mathbb{Z}[\beta]\}$$

$$\mathrm{Frac}(\sigma, a_0) = d_{(\sigma,a_0)}\left(\frac{\mathbb{Z}[\beta]}{D} \cap [0, \delta_\sigma(a_0))\right).$$

$$\mathrm{Frac}(\sigma) = \bigcup_{a \in \{1\dots,n\}} \mathrm{Frac}(\sigma, a).$$

Notice that for a $\beta$-substitution with the normalization $\delta_\sigma(1) = 1$, then $D = 1$ and $\mathrm{Frac}(\sigma) = \mathrm{Frac}(\beta)$ as introduced in Section 1.

Let $u = (p_i, a_i, s_i)_{i \geq 1} \in \mathrm{Frac}(\sigma, a_0)$; then $\sum_{i\geq 1} \delta_\sigma(p_i)(\beta^{-i}) \in \mathbb{Q}(\beta)$. We define the tile $\mathcal{T}_u$ as

$$\begin{aligned}\mathcal{T}_u \;&= \phi_\sigma(\textstyle\sum_{i\geq 1} \delta_\sigma(p_i)\beta^{-i}) + \varphi_\sigma(\{(p_{-i}, a_{-i}, t_{-i}) \in E_{\mathcal{P}}(X_\sigma);\\ &\qquad\qquad (p_i, a_i, s_i) \text{ is a two-sided path in } \mathcal{M}_\sigma)\}.\end{aligned}$$

**Coincidence.** In order to get basic tiles with disjoint interiors we need here an extra condition, called the strong coincidence condition, that is satisfied by beta-substitutions in particular. The condition of coincidence was introduced in [Dek78] for substitutions of constant length. It was generalized to non-constant length substitutions by Host in unpublished manuscripts. A formal and precise definition appears in [AI01]: a substitution is said to satisfy the *strong coincidence condition* if for any pair of letters $(i, j)$, there exist two integers $k, n$ such that $\sigma^n(i)$ and $\sigma^n(i)$ have the same $k$-th letter, and the prefixes of length $k-1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under the abelianization map. It is conjectured that every substitution of Pisot type satisfies the condition of coincidence: no counter-example is known; and this conjecture holds for two-letter substitutions [BD02].

The following theorem can be proved similarly as Theorem 1 and 2, thanks to the suitable choice of $\mathrm{Frac}(\sigma)$.

**Theorem 4** *We assume that $\sigma$ is a primitive substitution whose dominant eigenvalue $\beta$ is a* Pisot unit. *The set*

$$\Gamma_\sigma := \Phi_\sigma\left(\textstyle\sum_{i\geq 0} \delta_\sigma(p_i)\beta^{-1};\ (p_i, q_i, s_i)_{i\geq 1} \in \mathit{Frac}(\sigma)\right)$$

*is a Delaunay set. The finite up to translation set of tiles $\mathcal{T}_u$, for $u \in \mathit{Frac}(\sigma)$, covers $\mathbb{K}_\sigma$.*

$$\mathbb{K}_\beta = \bigcup_{u \in \mathit{Frac}(\sigma)} \mathcal{T}_u \tag{8}$$

*For each $u$, the tile $\mathcal{T}_u$ has non-empty interior, hence it has non-zero measure. The basic tiles of the central tile $\mathcal{T}_\sigma$ are solutions of the following graph-directed self-affine Iterated Function System:*

$$\forall a \in \mathcal{A},\ \mathcal{T}_\sigma(a) = \bigcup_{\substack{b \in \mathcal{A}, \\ b \mapsto_{(p_i,a_i,s_i)} a}} h_\beta(\mathcal{T}(b)) + \Phi_\sigma(\delta_\sigma(p_i)).$$

*The basic tiles are the closure of their interior. We assume furthermore that $\sigma$ satisfies the strong coincidence condition. Then the basic tiles have disjoint interiors.*

## 3. The lattice multiple tiling: a dynamical point of view

In this section, we give a geometric and dynamical interpretation of the central tile; for that purpose, we introduce a lattice multiple tiling that provides a geometric representation of the substitutive dynamical system; the shift is thus proved to be measure-theoretically isomorphic to an exchange of domains acting on the basic tiles.

### 3.1. Geometric construction of the Rauzy fractal

In all that follows $\sigma$ denotes a *primitive substitution whose dominant eigenvalue is a unit Pisot number.* Let $n$ denote the cardinality of the alphabet $\mathcal{A}$ on which $\sigma$ is defined. Let $u$ be a two-sided periodic point of $\sigma$. This bi-infinite word $u$ is embedded as a broken line in $\mathbb{R}^n$ by replacing each letter in the periodic point by the corresponding vector in the canonical basis $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ in $\mathbb{R}^n$. More precisely, the broken line is defined as $\{\mathbf{l}(u_0 \cdots u_k);\ k \in \mathbb{N}\}$ (see Fig. 6).

**Algebraic normalized eigenbasis.** We need now to introduce a suitable choice of basis of the representation space with respect to eigenspaces associated with our substitution $\sigma$ and its (simple) dominant eigenvalue $\beta$. Let $d$ denote the algebraic degree of $\beta$ and $n$ the cardinality of the alphabet on which $\sigma$ is defined; one has $d \leq n$, we may be in a reducible acse. Let us recall that $\mathbf{v}_\beta \in \mathbb{Q}(\beta)^n$ denote an expanding left eigenvector of the incidence matrix $\mathbf{M}_\sigma$. Let $\mathbf{u}_\beta \in \mathbb{Q}(\beta)^d$ be the unique right-eigenvector of $\mathbf{M}_\sigma$ associated with $\beta$, normalized so that $\langle \mathbf{u}_\beta, \mathbf{v}_\beta \rangle = 1$, where $\langle\ \rangle$ denotes the usual Hermitian scalar product. An eigenvector $\mathbf{u}_{\beta_k}$ for each eigenvalue $\beta_k$ ($1 \leq k \leq d$) is obtained by replacing $\beta$ with $\beta_k$ in $\mathbf{u}_\beta$. We set $\beta_1 = \beta$. We complete this free family into a basis of $\mathbb{R}^n$ (to be more precise, we should make a distinction between real and complex eigenvalues, but for the sake of clarity, we work here in $\mathbb{R}^n$) by selecting eigenvectors, say, $\mathbf{u}_\mathbf{i}^{'}$, for $1 \leq i \leq n-d$, associated with the eigenvalues that are not conjugate to $\beta$. We similarly complete the family $(\mathbf{v}_{\beta_1}, \ldots, \mathbf{v}_{\beta_d})$ into a seond basis that we choose dual to the first one, that is, $\langle \mathbf{u}_i{}', \mathbf{v}_i{}' \rangle = 1$, for every $1 \leq i \leq n-d$.

Let $\mathbb{H}_c = \{ \mathbf{x} \in \mathbb{R}^n, \ \langle \mathbf{x}, \mathbf{v}_\beta \rangle = 0\}$ denote the subspace of $\mathbb{R}^n$ generated by the vectors $\mathbf{u}_{\beta_2}, \ldots, \mathbf{u}_{\beta_d}$; we call it *$\beta$-contracting plane.* Let $\mathbb{H}_e$ denote the expanding line generated by $\mathbf{u}_\beta$, similarly called *$\beta$-expanding line*, and let $\mathbb{H}_r$ denote the space generated by the eigenvectors $\mathbf{u}_i^{'}$, for $1 \leq i \leq n-d$. Let $\pi : \mathbb{R}^n \to \mathbb{H}_c$ be the projection onto $\mathbb{H}_c$ along $\mathbb{H}_e \oplus \mathbb{H}_r$, according to the natural decomposition $\mathbb{R}^n = \mathbb{H}_c \oplus \mathbb{H}_e \oplus \mathbb{H}_r$, and $\pi'$ the projection onto the expanding line $\mathbb{H}_e$ along $\mathbb{H}_c \oplus \mathbb{H}_r$. Then $\pi$ and $\pi'$ can easily be expressed with respect to our dual basis: for any $\mathbf{x} \in \mathbb{R}^n$, one has

$$\pi(\mathbf{x}) = \sum_{2 \leq k \leq d} \langle \mathbf{x}, \mathbf{v}_{\beta_k} \rangle \mathbf{u}_{\beta_k} \text{ and } \pi'(\mathbf{x}) = \langle \mathbf{x}, \mathbf{v}_\beta \rangle \mathbf{u}_\beta.$$

Indeed $\langle \mathbf{u}_{\beta_1}, \mathbf{u}_{\beta_k} \rangle = 0$, for every $2 \leq k \leq d$, and $\langle \mathbf{u}_i^{'}, \mathbf{u}_{\beta_k} \rangle = 0$, for every $1 \leq k \leq d$ and $1 \leq i \leq n-d$. As a consequence, the projections $\pi(\mathbf{x})$ and $\pi'(\mathbf{x})$ of a rational vector $\mathbf{x} \in \mathbb{Q}^n$ are completely determined by the algebraic conjuguates of $\langle \mathbf{x}, \mathbf{v}_\beta \rangle$.

**Rauzy fractal and projection of the broken line.** An interesting property of the broken line is that after projection by $\pi$, then one obtains a bounded set in $\mathbb{H}_c$. It appears that the closure of this set is exactly the central tile, after identification of $\mathbb{H}_c$ and $\mathbb{K}_\sigma$ (Fig. 6). Let us denote $\Psi_\sigma$ the identification map from $\mathbb{K}_\sigma$ to $\mathbb{H}_c$ that gives a geometric representation in $\mathbb{H}_c$ of points whose coordinates in the right eigenvector basis of $\mathbb{H}_c$ belong to $\mathbb{K}_\sigma$:

$$\begin{aligned}&\Psi : (x_2, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}) \in \mathbb{K}_\sigma \to \\ &\quad x_2 \mathbf{u}_{\beta_2} + \ldots x_r \mathbf{u}_{\beta_r} + x^{r+1} \mathbf{u}_{\beta_{r+1}} + \overline{x^{r+1}} \overline{\mathbf{u}_{\beta_{r+1}}} + \ldots x^{r+s} \mathbf{u}_{\beta_{r+s}} + \overline{x^{r+s}} \overline{\mathbf{u}_{\beta_{r+s}}} \in \mathbb{H}_c.\end{aligned}$$

**Theorem 5** *Let $\sigma$ be a primitive substitution whose dominant eigenvalue is a Pisot unit. Let $u = (u_i)_{i \in \mathbb{Z}}$ be a periodic point of $\sigma$. Then the central tile satisfies*

$$\Psi \mathcal{T}_\sigma = \overline{\pi\left(\{\mathbf{l}(u_0 \cdots u_k); \ k \in \mathbb{N}\}\right)} := \mathcal{R}_\sigma \tag{9}$$

*The basic tiles satisfy for $a \in \mathcal{A}$*

$$\Psi \mathcal{T}_\sigma(a) = \overline{\pi\left(\{\mathbf{l}(u_0 \cdots u_{k-1}); \ k \in \mathbb{N}, \ u_k = a\}\right)} := \mathcal{R}_\sigma(a).$$

*The embedding of the central tile in the contracting hyperplane is called the Rauzy fractal and is denoted $\mathcal{R}_\sigma$.*

*Proof.* Let $u$ be a periodic point of $\sigma$. We assume w.l.o.g. that $u$ is a fixed point of $\sigma$ (otherwise we consider a suitable power of $\sigma$). Let $k$ be fixed. Let us expand according to the Dumont-Thomas numeration system $u_0 \ldots u_{k-1}$ as $\sigma^n(p_n) \ldots p_0$, with $(p_n, a_n, s_n) \ldots (p_0, a_0, s_0) \in F_{\mathcal{P}}$. One has

$$\begin{aligned}\pi(\mathbf{l}(u_0 \ldots u_{k-1})) &= \sum_{2 \leq j \leq d} \langle \mathbf{l}(\sigma^n(p_n) \ldots \sigma(p_1) p_0), \mathbf{v}_{\beta_j} \rangle \mathbf{u}_{\beta_j} \\ &= \sum_{2 \leq j \leq d} \left(\beta_j^n \langle \mathbf{l}(p_n), \mathbf{v}_{\beta_j} \rangle + \cdots + \langle \mathbf{l}(p_0), \mathbf{v}_{\beta_j} \rangle\right) \mathbf{u}_{\beta_j} \\ &= \Psi_\sigma \left[\beta_j^n \langle \mathbf{l}(p_n), \mathbf{v}_{\beta_j} \rangle + \cdots + \langle \mathbf{l}(p_0), \mathbf{v}_{\beta_j} \rangle\right]_{2 \leq j \leq r+s} \\ &= \Psi_\sigma \circ \Phi_\sigma(\delta_\sigma(p_n)\beta^n + \cdots + \delta_\sigma(p_0)) \in \Psi_\sigma \circ \Phi_\sigma(\mathbb{Z}_\sigma^+).\end{aligned}$$

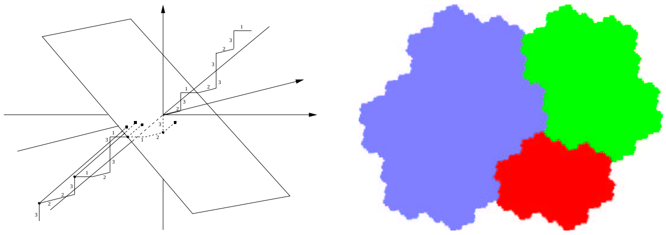


Figure 6: The projection method to get the Rauzy fractal for the Tribonacci substitution.

Note that $a_n$ is the first letter of $u$. Conversely, we deduce from the Dumont-Thomas numeration system, that every word $\sigma^n(p_n)\dots p_0$ with $(p_n, a_n, s_n)\dots(p_0, a_0, s_0) \in F_{\mathcal{P}}$ is the prefix of a the sided-fixed point of $\sigma$ with first letter $a_n$. By density and primitivity, one checks that we do not need to consider all the broken lines associated with all the fixed points to generate the central tile. Any fixed point is sufficient and all te fixed points provide the same result.

The basic tile $\mathcal{T}_\sigma(a_0)$ similarly corresponds to the finite factors in $F_{\mathcal{P}}$ which end in $a_0$. ■

### 3.2. Domain exchange dynamical system

We have given a geometric interpretation of our combinatorial expansions of finite prefixes as projections on the contracting space $\mathbb{H}_c$ of the vertices of the broken line. This allows us to introduce a dynamics on the central tile as a domain exchange. Indeed, from (9) one deduces that one can translate any point of a basic tile $\mathcal{T}_\sigma(a)$ by the projection of the $a$-th canonical vector $\mathbf{e}_a$ without exiting from the Rauzy fractal:

$$\begin{aligned}\mathcal{R}_\sigma(a) + \pi(\mathbf{e}_a) &= \overline{\pi\left(\{\mathbf{l}(u_0\cdots u_{k-1}a);\ k\in\mathbb{N},\ u_k = a\}\right)}\\ &\subset \overline{\pi\left(\{\mathbf{l}(u_0\cdots u_{k-1}u_k);\ k\in\mathbb{N}\}\right)} = \mathcal{R}_\sigma\end{aligned}$$

By projecting this relation in $\mathbb{K}_\sigma$, since $\pi(\mathbf{e}_a) = \Psi_\sigma(\Phi_\sigma\delta_\sigma(a))$, one gets

$$\mathcal{T}_\sigma(a) + \Phi_\sigma\delta_\sigma(a) \subset \mathcal{T}_\sigma. \tag{10}$$

If furthermore, the substitution satisfies the condition of strong coincidence, the basic tiles are almost everywhere disjoint, so that (10) defines a domain exchange on the central tile: $x \in \mathcal{T}_{\sigma(a)} \mapsto x + \Phi_\sigma \circ \delta_\sigma(a) \in \mathcal{T}$.

The following theorem means that the coding of the orbits of the points in the central tile under the action of this domain exchange are described by the substitutive dynamical system. It is indeed natural to code, up to the partition provided by the $n$ basic tiles, the action of the domain exchange over the central tile $\mathcal{T}_\sigma$. Theorem 6 states that the coding

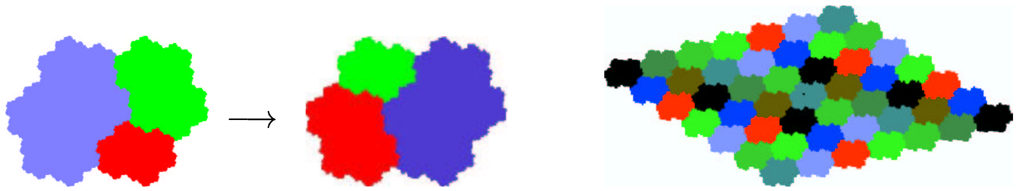


Figure 7: Domain exchange over the Rauzy fractal; lattice substitution multiple tiling

map, from $\mathcal{T}_\sigma$ into the three-letter alphabet full shift $\{1,\ldots,n\}^{\mathbb{Z}}$ is almost everywhere one-to-one, and onto the substitutive dynamical system $(X_\sigma, S)$. We thus have given an interpretation of the shift map as an exchange of domains.

The Dumont-Thomas expansion appears as a realization of the isomorphism between the set of trajectories and the central tile. Indeed, recall that the representation map of the substitutive dynamical system $(X_\sigma, S)$ is defined as the representation in the central tile of the combinatorial-suffix expansions. More precisely, one has $\varphi_\sigma(w) = \Phi_\sigma(\sum_{i\geq 1}\langle \mathbf{l}(p_i), \mathbf{v}_\beta\rangle \beta^i) \in \mathcal{T}_\sigma$ for every $w \in X_\sigma$, where $(p_i, a_i, s_i)$ is the combinatorial prefix-suffix expansion of $w$, that is, $w = \lim_{n\to\infty} \sigma^n(p_n)\ldots\sigma(p_1)p_0.a_0 s_0 \sigma(s_1)\ldots\sigma^n(s_n)$.

**Theorem 6 ([AI01, CS01b])** *Let $\sigma$ be a primitive substitution that satisfies the strong coincidence condition, and whose dominant eigenvalue is a Pisot unit. Then the domain exchange $x \in \mathcal{T}_\sigma(a) \mapsto x + \Phi_\sigma \circ \delta_\sigma(a) \in \mathcal{T}_\sigma$ is defined almost everywhere on the central tile.*

*The substitutive dynamical system $(X_\sigma, S)$ is measure-theoretically isomorphic to the central tile endowed with this exchange of domains.*

*The representation map of the substitutive dynamical system $(X_\sigma, S)$ is a continuous and almost everywhere one-to-one realization of the isomorphism, that is*

$$\varphi_\sigma \circ S = \varphi_\sigma + \pi(\mathbf{e}_a) \ \textit{on}\ \mathcal{T}_\sigma(a).$$

### 3.3. The lattice substitution multiple tiling

We want now to associate with our substitution $\sigma$ a lattice covering of $\mathbb{K}_\sigma$. This will also allow us to associate with a beta-numeration a lattice tiling by consdering the associated $\beta$-substitution. We assume that $\beta$ is a Pisot unit.

The domain exchange is defined only almost everywhere, which prevents us from defining a continuous dynamics on the full central tile. A solution to this problem consists in factorizing the central tile $\mathcal{T}$ by the smallest possible lattice so that the translation vectors $\Phi_\sigma \circ \delta_\sigma(a)$ do coincide: we thus consider the group $\sum_{i=1}^{n-1} \mathbb{Z}\Phi_\sigma(\delta_\sigma(i) - \delta_\sigma(n))$. In the case where this group is discrete, the quotient is a compact group and the domain exchange factorizes into a minimal translation on a compact group. This group is discrete if and only if $(\delta_\sigma(i) - \delta_\sigma(n))$ are rationnally independent with 1 in $\mathbb{Z}[\beta]$. A sufficient condition is thus that $n$ equals the degree $d$ of $\beta$, that is, that every eigenvalue of the substitution is a conjugate of the dominant eigenvalue $\beta$. We thus assume that we are in the irreducible case, that is, that $\sigma$ is a Pisot type substitution (every eigenvalue is a conjugate of the dominant eigenvalue $\beta$ that is a Pisot number). This implies severe restrictions for the beta-numeration: the cardinality of the alphabet of the $\beta$-substitution is equal to the degree of $\beta$.

**Theorem 7 ([AI01, CS01b])** *Let $\sigma$ be a unimodular substitution of Pisot type. Then the central tile generates a lattice multiple tiling of $\mathbb{K}_\sigma$:*

$$\mathcal{T}_\sigma + \sum_{i=1}^{d-1} \mathbb{Z}\Phi_\sigma(\delta_\sigma(i) - \delta_\sigma(d)) = \mathbb{K}_\sigma. \tag{11}$$

This multiple tiling is classical for substitutions but it is rarely associated with a beta-numeration, although it has a nice spectral interpretation: when the multiple tiling is a tiling, then the beta-shift has pure discrete spectrum ([BK04, Sie04]).

*Proof.* An intuitive approach of the proof is given by its interpretation in the full space $\mathbb{R}^d$. Indeed, the Pisot type hypothesis implies that the set $\mathcal{L}_0 = \sum_1^{d-1} \mathbb{Z}(\mathbf{e}_i - \mathbf{e}_d)$ is a lattice of $\mathbb{R}^d$. The translates along the lattice $\mathcal{L}_0$ of the vertices of the broken line $\mathbf{l}(u_0u_1 \dots u_{N-1})$, $N \in \mathbb{N}$, cover the following upper half space:

$$\{\mathbf{l}(u_0u_1 \dots u_{N-1}) + \gamma;\ N \in \mathbb{N},\ \gamma \in \mathcal{L}_0\} = \{x \in \mathbb{Z}^d;\ \langle x, \mathbf{v}_\beta \rangle \geq 0\}.$$

As an application of the Kronecker theorem, the projection of the upper half space is dense in the contracting plane $\mathbb{H}_c$. The translation vector set $\sum_{i=1}^{d-1} \mathbb{Z}\Phi_\sigma(\delta_\sigma(i) - \delta_\sigma(d))$ is a Delaunay set by definition, since we have chosen a lattice. Consequently, given any point $P$ of $\mathbb{H}_c$, there exists a sequence of points $(\pi(\mathbf{l}(u_0u_1 \dots u_{N_k-1})) + \gamma_k)_k$ with $\gamma_k$ in the lattice $\mathcal{L}_0$ which converges to $P$ in $\mathbb{H}_c$. Since the embedding $\mathcal{R}_\sigma = \Psi_\sigma(\mathcal{T}_\sigma)$ of the central tile in $\mathbb{H}_c$ is bounded, there are infinitely many $k$ for which the points $\gamma_k$ of the lattice $\mathcal{L}_0$ take the same value, say $\gamma$; we thus get $P \in \mathcal{R}_\sigma + \gamma$, which implies (11). ∎

Even in some specific cases in the reducible case $d < n$, some of the numbers $\delta_\sigma(i)$ are equal so that a factorization can be performed even if the substitution is not of Pisot type. See for instance [EI02] for a detailed study of the $\beta$-substitution associated with the smallest Pisot number.

### 3.4. Model sets

The aim of this section is to reformulate the previous results in terms of "cut and project scheme": Theorem 8 below states that the vertices of the broken line are exactly the points of $\mathbb{Z}^d$ selected by shifting the central tile $\mathcal{T}_\sigma$ (considered as an "acceptance window"), along the expanding eigendirection $\mathbf{u}_\beta$. We assume here that $\sigma$ is a substitution of Pisot type.

A *cut and project scheme* consists of a direct product $\mathbb{R}^k \times H$, $k \geq 1$, where $H$ is a locally compact abelian group, and a lattice $D$ in $\mathbb{R}^k \times H$, such that with respect to the natural projections $p_0 : \mathbb{R}^k \times H \to H$ and $p_1 : \mathbb{R}^k \times H \to \mathbb{R}^k$:

1. $p_0(D)$ is dense in $H$;
2. $p_1$ restricted to $D$ is one-to-one onto its image $p_1(D)$.

This cut and project scheme is denoted $(\mathbb{R}^k \times H, D)$.

A subset $\Gamma$ of $\mathbb{R}^k$ is a *model set* if there exists a cut and project scheme $(\mathbb{R}^k \times H, D)$ and a relatively compact set (i.e., a set such that its closure is compact) $\Omega$ of $H$ with non-empty interior such that

$$\Gamma = \{p_1(P);\ P \in D,\ p_0(P) \in \Omega\}.$$

The set $\Gamma$ is called the *acceptance window* of the cut and project sheme. Meyer sets (introduced in Section 1.2) appear naturally in this context: a *Meyer set* $S$ is a subset of some model set of $\mathbb{R}^k$, for some $k \geq 1$, which is *relatively dense*. For more details, see for instance [BM00, Moo97, Mey92, Mey95, LW04, Sen95]. Notice that the locally abelian compact group which usually occur in the previous definition are either Euclidean or $p$-adic spaces. For more details,

We recall that $\pi'$ denote the projection in $\mathbb{R}^d$ on the expanding line generated by $\mathbf{u}_\beta$ along the plane $\mathbb{H}_c$, whereas $\pi$ denotes the projection on the plane $\mathbb{H}_c$ along the expanding line.

**Theorem 8** *Let $\sigma$ be a unimodular Pisot type substitution such that the lattice multiple tiling is indeed a tiling. We assume furthermore that the projections by $\pi$ of the vertices of the broken line belong to the interior of the Rauzy fractal. The subset $\pi'(\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\})$ of the expanding eigenline obtained by projecting by $\pi'$ the vertices of the broken line associated with a periodic point of the substitution $\sigma$ is a Meyer set associated with the cut and project scheme $(\mathbb{R} \times \mathbb{R}^{d-1}, \mathbb{Z}^d)$, with acceptance window the interior of the Rauzy fractal $\mathcal{R}_\sigma = \Psi_\sigma(\mathcal{T}_\sigma)$. In other words,*

$$\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\} = \{P = (x_1, \cdots, x_d)) \in \mathbb{Z}^d;\ \sum_{1 \leq i \leq d} x_i \geq 0; \pi(P) \in Int(\mathcal{R}_\sigma)\}. \tag{12}$$

*Proof.* Let $H = \mathbb{R}^{d-1}$, $D = \mathbb{Z}^d$, $k = 1$. The set $H = \mathbb{R}^{d-1}$ is in one-to-one correspondence with the plane $\mathbb{H}_c$, whereas $\mathbb{R}$ is in one-to-one correspondence with the expanding eigenline. Up to these two bijections, the natural projections become respectively $\pi'$ and $\pi$ and are easily seen to satisfy the required conditions It remains to prove (12) to conclude.

For every $N$, $\pi(\mathbf{l}(u_0\cdots u_{N-1}))\in \mathrm{Int}(\mathcal{R}_\sigma)$ . Conversely, let $P=(x_1,\ldots,x_d)\in\mathbb{Z}^d$ with $\sum x_i\geq 0$ such that $\pi(P)\in Int(\Psi\mathcal{T})$. Let $N=x_1+\ldots x_d$. According to Theorem 7, there exists $\gamma\in\mathcal{L}_0$ such that $P=f(u_0\cdots u_{N-1})+\gamma$. Since $\pi(P)=\pi(\mathbf{l}(u_0\cdots u_{N-1}))+\pi(\gamma)=\pi(\mathbf{l}(u_0\cdots u_{N-1}))+\gamma$, one gets $P=\mathbf{l}(u_0\cdots u_{N-1})$. ■

## 4. Discrete planes and $\mathbb{Z}^2$-actions

The aim of this section is to present an alternative construction and interpretation of the self-replicating multiple tiling associated with a primitive Pisot unimodular substitution. This construction is based on the notion of geometric generalized substitution due to [AI01], see also [IR04]. We respectively introduce the notion of discrete plane in Section 4.1, of substitution acting on it in Section 4.2, and end this section by giving a substitutive version for the finiteness condition (F) introduecd in Section 1.6.

### 4.1 Discrete plane and discrete multiple tiling

We assume that $\sigma$ is a unimodular Pisot substitution.

Let us first consider a discretisation of the contracting hyperplane $\mathbb{H}_c=\{\,\mathbf{x}\in\mathbb{R}^n,\ \langle\mathbf{x},\mathbf{v}_\beta\rangle=0\}$ corresponding to the notion of arithmetic plane introduced in [Rev91] which consists in approximating the plane $\mathbb{H}_c$ by selecting points with integral coordinates above and within a bounded distance of the plane: more generally, given $\mathbf{v}\in\mathbb{R}^n$, $\mu,\omega\in\mathbb{R}$, the *lower* (resp. *upper*) *discrete hyperplane* $\mathfrak{P}(\mathbf{v},\mu,\omega)$ is the set of points $\mathbf{x}\in\mathbb{Z}^d$ satisfying $0\leq\langle\mathbf{x},\mathbf{v}\rangle+\mu<\omega$ (resp. $0<\langle\mathbf{x},\mathbf{v}\rangle+\mu\leq\omega$). The parameter $\mu$ is called the *translation parameter* whereas $\omega$ is called the *thickness.* Moreover, if $\omega=\max\{v_i\}=|\mathbf{v}|_\infty$ (resp. $\omega=\sum v_i=|\mathbf{v}|_1$) then $\mathfrak{P}(\mathbf{v},\mu,\omega)$ is said to be *naive* (resp. *standard*). In the standard case, one approximates a plane with irrational normal vector $\mathbf{v}\in\mathbb{R}^d$ by square faces oriented along the three coordinates planes. For more details, see the surveys [BCK04, BFJ05].

We now associate with the standard arithmetic discrete plane with parameter $\mu=0$ associated with $\mathbf{v}_\beta$ the *stepped surface* $\mathcal{S}_\sigma$ defined as the union of faces of integral cubes that connect these points, as depicted Fig. 8. (We consider only standard discrete planes in this paper, hence we call them discrete planes, for the sake of simplicity.) More precisely, an *integral cube* denotes any translate with integral vertices of the fundamental unit cube $\mathcal{C}=\{\sum_{1\leq i\leq n}\lambda_i\mathbf{e}_i;\ \lambda_i\in[0,1],\ \text{ for all } i\}$. The vertices (that is, the points with integer coordinates) of the boundary of the set of integral cubes that intersect the lower open half-space $\{\mathbf{x};\ \langle\mathbf{x},\mathbf{v}\rangle<0\}$ (we work here with lower arithmetic planes) are exactly the points of the arithmetic discrete plane $\mathfrak{P}(\mathbf{v},\mu,\omega)$, that we denote for short $\mathfrak{P}_\sigma$ [BV00, ABI02, ABS04, AI01].

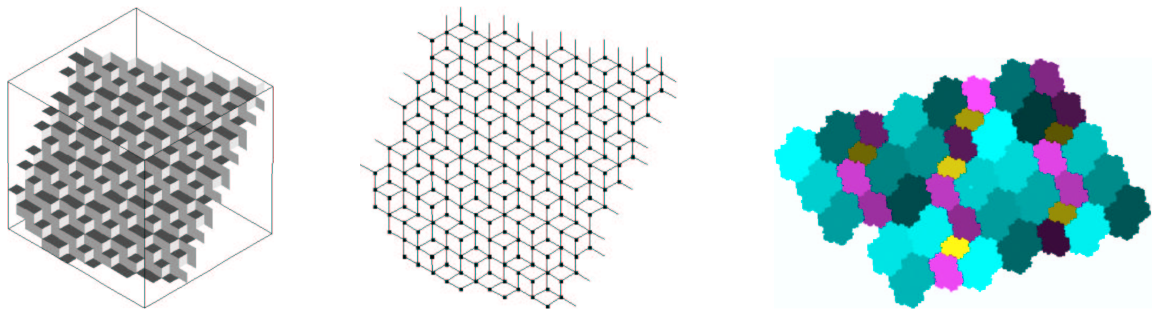


Figure 8: Stepped surface; pointed faces in the stepped surface; discrete multiple tiling

Now we consider the following set of faces for $1 \leq i \leq n$:

$$(\mathbf{x}, i) := \{\mathbf{x} + \sum_{j \neq i} \lambda_j \mathbf{e}_j;\ 0 \leq \lambda_j \leq 1, \quad \text{for } 1 \leq j \leq n,\ j \neq i\}.$$

One checks that such faces provide a partition (up to the boundaries of the faces) of the stepped surface.

**Theorem 9 ([AI01, ABI02, ABS04])** *Let $\sigma$ be a unimodular substitution of Pisot type. The stepped surface $\mathcal{S}_\sigma$ associated the contracting space is spanned by:*

$$\mathcal{S}(\mathbf{v}_\beta) = \bigcup_{0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} (\mathbf{x}, i). \tag{13}$$

Let us project now the discrete plane $\mathfrak{P}_\sigma$ on the contracting space $\mathbb{H}_c$ and replace each face $(\mathbf{x}, i)$ by the corresponding basic piece of the Rauzy fractal $\mathcal{R}_\sigma(i)$. The covering (13) becomes

$$\bigcup_{0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i).$$

It is shown in [AI01, ABI02, IR02] that this covering provides a multiple tiling of the contracting hyperplane, the translation vector set $\pi(\{\mathbf{x};\ \exists i,\ (\mathbf{x}, i) \in \mathcal{S}_\sigma)\}$, being a Delaunay set. The proof becomes easy in our context if one notices that the projections, when embedded in $\mathbb{K}_\sigma$, of the discrete plane correspond to

$$\bigcup_{0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \Phi_\sigma \delta_\sigma(< \mathbf{x}, \mathbf{v}_\beta >) + \mathcal{T}_\sigma(i).$$

Indeed we recover it directly from the following lemma:

**Lemma 2** *For every $\mathbf{x} \in \mathbb{Z}^d$,*

$$0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle \text{ if and only } \pi(\mathbf{x}) \in \Psi_\sigma \circ \Phi_\sigma \left( \frac{\mathbb{Z}[\beta]}{D} \cap [0, \delta_\sigma(i)) \right).$$

*Proof.* Let $\mathbf{x} \in \mathbb{Z}^d$ such that $0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle$. By definition $z := \langle \mathbf{x}, \mathbf{v}_\beta \rangle \in \frac{\mathbb{Z}[\beta]}{D}$. Since $\mathbf{x}$ has rational coordinates, its coordinates in the normalized eigenbasis are conjugate, so that $\mathbf{x} = z\mathbf{u}_\beta + \Psi(\Phi_\sigma(z))$ and $\pi(\mathbf{x}) \in \Psi_\sigma\left(\Phi_\sigma \frac{\mathbb{Z}[\beta]}{D} \cap [0, \delta_\sigma(i)\right)$.

Conversely, let $\pi(\mathbf{x}) = \Psi_\sigma \circ \Phi_\sigma(z)$ with $z \in \frac{\mathbb{Z}[\beta]}{D} \cap [0, \delta_\sigma(i))$. Let $z' = \langle \mathbf{x}, \mathbf{v}_\beta \rangle$. Since $\mathbf{x}$ is rational, $\Phi_\sigma(z)$ has for coordinates the conjugates of $z'$. Since $z \in \mathbb{Q}[\beta]$, one gets $z' = z$. ■

In other words, $\Psi_\sigma \Gamma_\sigma = \pi(\{\mathbf{x};\ \exists i,\ (\mathbf{x}, i) \in \mathcal{S}_\sigma)\}$, that is, $\cup_{\mathbf{x} \in \mathbb{Z}^d, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i)$ is the embedding in the contracting space of the self-replicating multiple tiling. One interest of this approach is that it establishes a correspondence bewteen two multiple tilings which a priori have nothing in common, a multiple tiling provided one the one hand by a discrete approximation of the contracting plane, and on the other hand, a multiple tiling obtained via the Dumont-Thomas numeration as a generalization of the $\beta$-numeration. We thus reformulate is as follows:

**Theorem 10** *Let $\sigma$ be a unimodular substitution of Pisot type. The projections by $\pi$ of the discrete plane $\mathfrak{P}_\sigma$ associated with the contracting space $\mathbb{H}_c$ generates a covering of the contracting hyperplane, called the* discrete multiple-covering *of the substitution:*

$$\mathbb{R}^{d-1} = \bigcup_{\mathbf{x} \in \mathbb{Z}^d,\, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i).$$

*The discrete multiple-tiling is the embedding of the self-replicating multiple tiling, that is,*

$$\mathbb{K}_\sigma = \bigcup_{\mathbf{x} \in \mathbb{Z}^d,\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \Phi_\sigma \circ \delta_\sigma(< \mathbf{x}, \mathbf{v}_\beta >) + \mathcal{T}_\sigma(i).$$

### 4.2. Substitutive construction of the discrete plane

**Substitution rule** We define $\mathcal{F}^*$ as the $\mathbb{R}$- vector space generated by $\{(\mathbf{x}, i);\ \mathbf{x} \in \mathbb{Z}^d,\ i \in \{1, 2, \cdots, d\}\}$. We define the map the following generation process which can be considered as a geometric realization of the substitution $\sigma$ on the geometric set $\mathcal{F}^*$ made of finite sums of faces:

$$E_1^*(\sigma)(\mathbf{x}, a) = \bigcup_{\sigma(b) = pas} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{l}(p)), b).$$

**Proposition 2** *[AI01] The discrete surface $\mathfrak{S}_\sigma$ is stable under the action of $E_1(\sigma)^*$ and contains the faces $(\mathbf{0}, 1)$, ..., $(\mathbf{0}, d)$.*

**Lemma 3**
- *If $(\mathbf{x}, a) \in \mathcal{S}(\mathbf{v}_\beta)$ and $b \to_{(p,a,s)} a$ in the prefix-suffix automaton $\mathcal{M}_\sigma$, then $(M^{-1}\mathbf{x} + M^{-1}\mathbf{l}(p), b) \in \mathcal{S}(\mathbf{v}_\beta)$.*

- *If* $a_{-k-1} \mapsto_{(p_{-k-1},a_{-k-1},s_{-k-1})} a_{-k} \dots \mapsto_{(p_{-1},a_{-1},s_{-1})} a_0$ *is a walk in the prefix-suffix automaton, then* $(M^k\mathbf{l}(p_{-1}) + \dots + p_{-(k+1)}, a_0)$ *is a face of the discrete surface* $\mathcal{S}\sigma$.

*Proof.* By defintion, $(\mathbf{y}, b) \in E_1^*(\sigma)(\mathbf{x}, a)$ iff $\mathbf{y} = M^{-1}\mathbf{x} + M^{-1}\mathbf{l}(p)$ with $\sigma(b) = pas$, that is, if there exists an arrow $b \to_{(p,a,s)} a$ in the prefix-suffix automaton such that $\mathbf{y} = M^{-1}\mathbf{x} + M^{-1}\mathbf{l}(p)$. If $(\mathbf{x}, a) \in \mathcal{S}_\sigma$, then $(\mathbf{y}, b) \in E_1(\sigma)^*\mathcal{S}_\sigma = \mathcal{S}_{sigma}$. The second point is a direct consequence of the first one and of the fact that the discrete surface contains the faces $(\mathbf{0}, i)$, for $1 \leq i \leq d$. ■

As a consequence, a point in $\frac{\mathbb{Z}[\beta]}{D}$ has a finite $(\sigma, a)$-expansion if and only if the corresponding face of the discrete surface belongs to an iteration of the geometric substitution $E_1^*(\sigma)$ on the central tile $(\mathbf{0}, a)$.

**Theorem 11** *Let* $\sigma$ *be a unimodular Pisot type substitution. Let* $(F)$ *be the following extended* $(F)$*-property:*

$$(F) \qquad \forall a \in \{1 \dots d\}, \forall z \in \frac{\mathbb{Z}[\beta]}{D} \cap [0, \delta_\sigma(a)), \qquad d_{(\sigma,a)}(z) \text{ is finite .}$$

*The extended* $(F)$*-property is satisfied if and only if the images of the unit cube located at the origin* $\mathcal{C} = \cup_{1\leq i\leq d}(\mathbf{0}, i)$ *under the iterated action of* $E_1^*(\sigma)$ *cover the full discrete surface, that is,*

$$\mathcal{S}_\sigma = \bigcup_{n\in\mathbb{N}} E_1^*(\sigma)^n((\mathbf{0}, \mathbf{1}) \cup \dots (\mathbf{0}, \mathbf{d})).$$

*If the extended (F)-property is satisfied and if the substitution satsifies the strong coincidence proptery, the self-replicating substitution multiple tiling, which coincides with the discrete multiple tiling, is indeed a tiling.*

*Proof* Let $(\mathbf{x}, a) \in \mathcal{S}_\sigma$. From Lemma 2, $z =< \mathbf{x}, \mathbf{v}_\beta > \in \frac{\mathbb{Z}[\beta]}{D} \cap [0, \delta_\sigma(a))$. Let $a_{-k-1} \mapsto_{(p_{-k-1},a_{-k-1},s_{-k-1})} a_{-k} \dots \mapsto_{(p_{-1},a_{-1},s_{-1})} a_0$ be its finite $(\sigma, a)$ expansion, according to the extended $(F)$-property. Let $y = M^k\mathbf{l}(p_{-1}) + \dots + p_{-(k+1)}$. Then, by construction, $\langle \mathbf{y}, \mathbf{v}_\beta \rangle = z =< \mathbf{x}, \mathbf{v}_\beta >$ so that $\mathbf{y} = \mathbf{x}$ (both are integral points with the same projection on the discrete plane). From Lemma 3 $(\mathbf{x}, a) \in E_1^*(\sigma)^n(\mathbf{0}, \mathbf{a})$. The converse follows similarly.

The strong coincidence condition implies that for every pair of faces $(\mathbf{x}, i), (\mathbf{y}, j) \in E_1^*(\sigma)^n((\mathbf{0}, \mathbf{1}) \cup \dots (\mathbf{0}, \mathbf{d}))$, the tiles $\pi(\mathbf{x}) + \mathcal{R}_\sigma(i)$ and $\pi(\mathbf{y}) + \mathcal{R}_\sigma(j)$ are measurably disjoint. Since every point in the self-replicating substitution multiple tiling belongs to such an iterate, all the tiles are disjoint and the multiple tiling is a tiling. ■

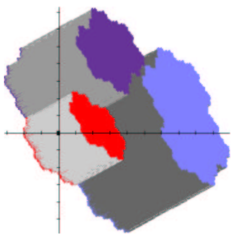
## 5. Equivalent tilings

Different multiple tilings where introduced in the previous sections: we have introduced a multiple tiling related to a substitution numeration system, also defined for every substitution whose dominant eigenvalue is a unit Pisot number, a lattice multiple tiling and a discrete multiple tiling ??? pas fait tress clair, mettre def quelque part????, both defined when the substitution satisfies the extra hypothesis to be of Pisot type, that is, to have an irreducible characteristic polynomial. When existing, the discrete and the substitution multitilings appear to coincide.

Ito and Rao prove in [IR04] that the lattice multiple tiling and the self-replicating multiple tiling are simultaneously tilings. The proof is based on the following construction: from the Rauzy fractal associated to a substitution, that is, a compact set in a $(d-1)$-dimensional space, one builds a compact set with nonempty interior in $\mathbb{R}^d$as the union of $d$ cylinders with a transverse component along the expanding direction, based on each piece of the Rauzy fractal (living in the contracting space $\mathbb{H}_c$) with height the size of the corresponding interval in $\mathbb{R}$:

$$\tilde{\mathcal{R}}_\sigma = \bigcup_{a=1}^{d} \mathcal{R}_\sigma(a) + [0, \delta_\sigma(a))\mathbf{u}_\beta.$$

This set is called the *Markov Rauzy fractal* (depicted Fig. 1 and 2). If the substitution satisfies the strong coincidence condition, the pieces $\mathcal{R}_\sigma(a)$ are disjoint in measure in $\mathbb{H}_c$ so that the cylinder are almost-everywhere disjoint in $\mathbb{R}^d$.



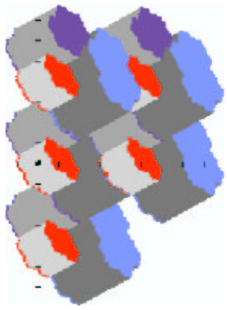


Figure 9: Markov Rauzy fractal; Markov multiple tiling

In [IR04] are proved further essential facts about this Markov Rauzy fractal:

- The Markov Rauzy fractal provides a lattice multiple tiling of $\mathbb{R}^d$:

$$\bigcup_{\mathbf{z}\in\mathbb{Z}^d} \tilde{\mathcal{R}}_\sigma + \mathbf{z} = \mathbb{R}^d.$$

- If one intersects the contracting hyperplane with the space multiple tiling, one recovers the discrete multiple tiling:

$$\bigcup_{\mathbf{z}\in\mathbb{Z}^d} \pi(\tilde{\mathcal{R}}_\sigma + +\mathbf{z}) \cap \mathbb{H}_c = \bigcup_{a\in\{1,\ldots d\},\, \mathbf{z}\in\mathbb{Z}^d,\, \langle \mathbf{z},\mathbf{v}_\beta\rangle\in[0,\delta_\sigma(a))} \mathcal{R}_\sigma + \pi(\mathbf{z}).$$

- The projections on $\mathbb{H}$ of the pieces that cross the anti-diagonal hyperlane provide the lattice multiple tiling:

$$\bigcup_{\mathbf{z}\in\mathbb{Z}^d,\, \langle \mathbf{z},(1,\ldots 1)\rangle=0} \pi(\tilde{\mathcal{R}}_\sigma + \mathbf{z}) = \mathcal{R}_\sigma + \sum_{1}^{d-1} \mathbb{Z}\pi(\mathbf{e}_i - \mathbf{e}_d).$$

**Theorem 12 ([IR04])** *Let $\sigma$ be a unimodula Pisot substitution. The self-replicating multiple tiling, the lattice multiple tiling, the discrete multiple tiling and the Markov multiple tiling are simulnateously tilings or not.*

For each of those multiple tilings, some tiling conditions expressed according to the context in which the multiple tiling is defined (numeration, substitution, discrete geometry, symbolic dyunamics...) Some are necessary and sufficient, others are only sufficient but effective. We do not have space enough in the present paper to introduce all the formalisms that are necessary to detail each of those conditions. Hence, the reader will find as a conclusion a small description of each condition including the corresponding references.

- Combinatorics [IR04, BK04]: the discrete multiple tiling is a tiling if and only if the substitution satisfies a combinatorial condition called the super-coincidence condition, which can be seen as a geometric generalization of the strong coincidence property. Roughly speaking, two stairs corresponding to a periodic point and starting from the discrete surface have to share a common vector.
- Combinatorics [Hui04]: the super-coincidence is satisfied if and only if the balanced pair algorithm terminates.
- Geometry [IR04]: the discrete multiple tiling is a tiling if and only if the measure of at least a basic tile $\mathcal{R}_\sigma(i)$ is equal to the measure of the rhombus $\pi((0,i))$.
- Discrete geometry [IR04]: the discrete multiple tiling is a tiling if and only if the boundaries of the polygonals $X_n(i) := \pi(E_1^*(\sigma)^(n)(\mathbf{0},i))$ tend to the boundary of the $i$-th basic tile $\mathcal{R}_\sigma(i)$ for the Haussdorf metric.
- Spectral theory [BK04, Sie04]: the lattice multiple tiling is a tiling if and only if the substitutive dynamical system $(X_\sigma, S)$ has a purely discrete spectrum.

- Numeration [Sie04, Thu04]: the lattice multiple tiling is a tiling if and only if a finite graph describing the intersection of the pieces in the tiling is "small enough", that is, if it does not recognize the same langage as the prefix-suffix automaton. Similarly, the discrete multiple tiling is a tiling if and only if the contact graph is small enough. Both conditions provide algorithms that are rather long but allow one to test whether a multiple tiling is or not a tiling (which is not the case for the balanced pair algorithm).

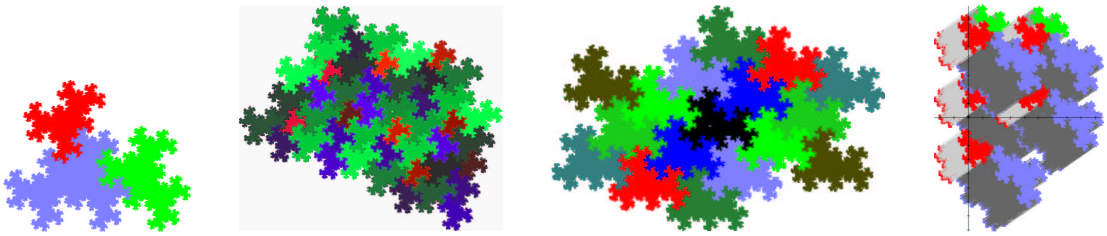


Figure 10: Central tile for the unimodular substitution of Pisot type $1 \to 12$, $2 \to 3$, $3 \to 1$. Self-replicating tiling, lattice tiling and Markov tiling. The associated Pisot number satisfies $\beta^3 = \beta^2 + 1$. This substitution satisfies the numeration condition in [Sie04, Thu04] so that all multiple tilings are tilings.

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