Tiling an Interval of the Discrete Line

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Abstract. We consider the problem of tiling a segment $\{0, \ldots, n\}$ of the discrete line. More precisely, we ought to characterize the structure of the patterns that tile a segment and their number. A pattern is a subset of $\mathbb{N}$. A tiling pattern or tile for $\{0, \ldots, n\}$ is a subset $A \in \mathcal{P}(\mathbb{N})$ such that it exists $B \in \mathcal{P}(\mathbb{N})$ and such that the direct sum of A and B equals $\{0, \ldots, n\}$. This problem is related to the difficult question of the decomposition in direct sums of the torus $\mathbb{Z}/n\mathbb{Z}$ (proposed by Minkowski). Using combinatorial and algebraic techniques, we obtain for any $n \in \mathbb{N}$ an explicit characterization of all tiles of $\{0, \ldots, n\}$. We prove that the tiles are direct sums of some arithmetic sequences of specific lengths. Besides, we show there are as many tiles whose smallest tilable segment is $\{0, \ldots, n\}$ as tiles whose smallest tilable segment is $\{0, \ldots, d\}$ for all strict divisors d of n. This enables us to exhibit a polynomial time algorithm to compute for a given pattern the smallest segment that it tiles if any, as well as a recurrence formula for counting the tiles of a segment.

1 Introduction

Tilings are intriguing in many regards. Their structure, *i.e.*, the way in which the tiles are assembled, may be remarkably complex. As a matter of fact, a theorem from Berger [3] states that, given a set of patterns, determining whether this set tiles the plane is undecidable. This result let us think there exist sets of tiles that tile the plane only in complex ways. Indeed, Penrose and others [11, 4] demonstrated there exist aperiodic sets of tiles (aperiodic means that it tiles the plane, but that none of its tilings admits an invariant by translation). However, some related questions remain open. The smallest known aperiodic set of tiles contains 13 tiles and it is unknown whether there is one with only one non necessarily connected tile. Over and above that, it is undetermined whether the tiling of the plane with one non connected pattern is decidable. Nevertheless, an interesting result from Beauquier-Nivat [1] states that if the pattern is connected the problem is decidable, and if it exists a tiling, there is also a (doubly)-periodic tiling (*i.e.*, one that is invariant by two non-collinear vectors).

Even when restricted to bounded regions of the plane, tiling problems remain difficult combinatoric questions on which little is known. Numerous articles report on specific cases. Among others, the problem of tiling a connected region (respectively, a simply connected region) by dominos is polynomial [13] (resp., linear [14]). But, if generalized to a region that is non necessarily simply connected, the problem of tiling by rectangles of size 1×3 and 3×1 becomes NP-complete [2].

Regarding these difficulties, it is natural to focus on tiling problems for the discrete line $\mathbb{Z}$. These problems are related to additive number theory, which studies the decompositions of sets of numbers in sums of sets of numbers. A major theorem in this field is the decomposition of integers in sums of 4 squares (Lagrange's theorem), which is written $4C = \mathbb{N}$ where $C := \{n^2 : n \in N\}$. Let us also mention Golbach's conjecture (in a letter to L. Euler, 1742), which asks whether any even integer is the sum of two primes.

Indeed in additive number theory, tiling the discrete line with a tile is equivalent, given a set A (representing the tile), to finding a set B (representing the positions of the tile's copies) such that the function $f(a, b) := a + b$ is one-to-one from $A \times B$ into $\mathbb{Z}$. In this case, we denote it $A \oplus B = \mathbb{Z}$. A classical result [9] states that in this case, it always exists a positions set B that is periodic (*i.e.*, for which there is an integer k such that $B + k = B$). As an immediate corollary, one obtains the decidability of the tiling of the discrete line by a single pattern. Despite that, its algorithmic complexity remains open although a lot of efforts have been dedicated to study bases for the integers [5, 15]. Moreover, the periodicity of the positions set B raises the question of the characterization of sets A and B such that $A \oplus B = \mathbb{Z}/n\mathbb{Z}$. This problem formulated by Minkowski more than hundred years ago is still mainly unsolved despite the last progress made by Hajòs [7, 8].

In this work, we focus on the characterization of sets A and B satisfying $A \oplus B = [\![n]\!]$, where $[\![n]\!]$ denotes the interval $\{0, ..., n-1\}$. To our knowledge, this question has not been addressed in the literature. In a first part (Sections 2 and 3), we demonstrate using techniques from word theory that if $A \oplus B = [\![n]\!]$ then either A or B tiles $[\![d]\!]$, for d a proper divisor of n. For any $n \in \mathbb{N}$, let us say a tile is *n-specific* if its smallest tilable segment is $[\![n]\!]$. More precisely, we exhibit a bijection between n-specific tiles and d specific tiles for all strict divisors d of n. This result allows to count the tilings of $[\![n]\!]$. The obtained sequence is described in the Encyclopedia of Integer Sequences [12] by Zumkeller without relationship to the theory of tilings.

Besides, we prove a theorem on the size of the smallest tilable segment in function of the tile's diameter. This solves in a specific case a conjecture of Nivat stating that the smallest torus $\mathbb{Z}/n\mathbb{Z}$ that can be tiled by a pattern of diameter d satisfies $n \leq 2d$. Moreover, this enables us to exhibit a polynomial time algorithm to decide whether a pattern tiles at least one interval of $\mathbb{Z}$ (Section 4).

In a second part (Section 4), using more algebraic techniques, we demonstrate that any tile of $[\![n]\!]$ can be decomposed in irreducible tiles (*i.e.*, tiles that are not sums of smaller tiles), which we characterize explicitly. Furthermore, we know for any n how many irreducible tiles there are.

1.1 Definitions and notation.

Subsets of $\mathbb{N}$ and polynomials. Let $\mathbb{N}$, resp. $\mathbb{Z}$, be the set of non-negative integers, resp. of integers, and $\mathcal{P}(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$. We denote the set of polynomials with coefficients in $\{0,1\}$ by $\{0,1\}[X]$. We define the mapping ρ that to a finite subset of $\mathbb{N}$ associates a polynomial of $\{0,1\}[X]$ by:

$$\begin{array}{rl} \rho : \mathcal{P}(\mathbb{N}) \to & \{0,1\}[X] \\ A \quad \longrightarrow & P_A(X) := \sum_{a \in A} X^a \end{array}$$

Clearly, ρ is one-to-one. For all $A \in \mathcal{P}(\mathbb{N})$, we denote by $c(A)$ its minimal element, by $d(A)$ its maximal element, and by $\#(A)$ its *cardinality*. $d(A)$ is also the degree of P_A.

Let $A, B \in \mathcal{P}(\mathbb{N})$ and $k \in \mathbb{N}$. The following operations on sets have correspondents for polynomials:

union : $P_{A \cup B} = P_A + P_B$ if and only if $A \cap B = \emptyset$,
difference : $P_{A \setminus B} = P_A - P_B$ if and only if $B \subset A$,
translation : if one denotes $A + k := \{a + k : a \in A\}$, then $P_{A+k}(X) = P_A X^k$.

We introduce a notation for the *direct sum*. Let us denote by $A \uplus B$ the union with repetition for all $b \in B$ of the translates $A + b$. In general, this union is a multi-set on $\mathbb{N}$, *i.e.*, $P_{A \uplus B} := P_{\biguplus_{b \in B} A+b} = \sum_{b \in B} P_{A+b}$ is a polynomial with integral coefficients that are eventually strictly greater than 1. If it exists $C \in \mathcal{P}(\mathbb{N})$ such that $C = A \uplus B$, then we denote it by $C = A \oplus B$. In this case, $P_{A \uplus B} := P_{A \oplus B} = P_A P_B$ and it belongs to $\{0,1\}[X]$. In other words, we investigate the case where the sum is stable in $\mathcal{P}(\mathbb{N})$, or where the product of polynomials is stable in $\{0,1\}[X]$. One says that a polynomial is *irreducible* in $\{0,1\}[X]$ if it does not admit a non trivial factorization in $\{0,1\}[X]$. When transposed to subsets of $\mathbb{N}$, A is *irreducible* means it is impossible to decompose A in a non trivial direct sum (*i.e.*, other than $\{0\} \oplus A$).

Besides, we say A is a *prefix* of B if and only if $A \subset B$ and $\forall i \in B$ such that $i \leq d(A)$, $i \in A$. By convention, one admits that $\emptyset$ is prefix of any other subset of $\mathbb{N}$. We denote by $[\![k]\!]$ the finite interval of $\mathbb{N}$ of length k whose minimal element is 0, *i.e.*, the interval $[0, k-1]$. We use the word segment as an alternate for interval.

In the sequel, for any finite subset A of $\mathbb{N}$, we assume that $c(A) = 0$ (this is always true up to a translation). We call A a *pattern* or *motif.* For a pattern A, $d(A)$ is also termed *diameter.*

2 Properties of the direct sum

In this section, we investigate the properties of the direct sum that are useful to study the tilings of an interval. Note that the propositions hereunder are true for subsets of $\mathbb{N}$, but not necessarily for multi-sets on $\mathbb{N}$.

Proposition 1 (Sums of prefixes). *Let A, B, B', C, C' be subsets of $\mathbb{N}$ such that $A \neq \emptyset$ and C is prefix of C'. Then, together $A \oplus B = C$ and $A \oplus B' = C'$ imply that B is prefix of B'.*

Proof. [Prop. 1] We proceed by induction on $\#(C)$. If $C = \emptyset$ then $B = \emptyset$, and B is prefix of B'. Now, assume $\#(C) > 0$. By hypothesis, there exists $(a, b) \in A \times B$ and $(a', b') \in A \times B'$ such that $a+b = \min C$ and $a' + b' = \min C$. By definition of $\oplus$, $(a + b = \min C)$ implies that $a = \min A$ and $b = \min B$, while $(a' + b' = \min C)$ implies that $a' = \min A = a$ and $b' = \min B'$. We get $b = \min B = b' = \min B'$. Hence, one has:

$$A \oplus (B \setminus \{b\}) = C \setminus (\{b\} \oplus A), \qquad A \oplus (B' \setminus \{b\}) = C' \setminus (\{b\} \oplus A),$$

and $C \setminus (\{b\} \oplus A)$ is prefix of $C' \setminus (\{b\} \oplus A)$. By induction, we obtain that $B \setminus \{b\}$ is prefix of $B' \setminus \{b\}$, and thus, that B is prefix of B'. □

Proposition 2 (Sum of a partition). *Let A, B, D be subsets of $\mathbb{N}$ such that $D \subseteq A$ and $A \oplus B$ be a subset of $\mathbb{N}$. Let us denote by $\complement_A D$ the complement of D in A. Then $(D \oplus B)$ and $\left(\complement_A D \oplus B\right)$ partition $A \oplus B$.*

Proof. [Prop. 2] Clearly, one has $P_{A \oplus B} = P_A P_B = P_D P_B + (P_A - P_D) P_B$. □

This proposition is not verified when $A \oplus B$ is multi-set on $\mathbb{N}$ that is not a subset of $\mathbb{N}$. For multi-sets, we have the following property: Let C, D be such that $A = C \uplus D$, then $(C \oplus B) \uplus (D \oplus B) = A \oplus B$. In general it is not true that $(C \oplus B) \cap (D \oplus B) = \emptyset$, even if $C \cap D = \emptyset$.

We state two propositions of simplification.

Proposition 3 (Difference of intervals). *Let A, B, C be subsets of $\mathbb{N}$ and $m, n \in \mathbb{N}$. If $A \oplus B = [\![m]\!]$ and $A \oplus C = [\![n]\!]$ with $n \geq m$, then it exists $D \subset \mathbb{N}$ such that $A \oplus D = [\![n-m]\!]$ and $D := \complement_C B - m$.*

Proof. [Prop. 3] As $[\![m]\!]$ is a prefix of $[\![n]\!]$, we know by Proposition 1 that B is a prefix of C. Hence, $A \oplus \complement_C B = [\![n-m]\!] \oplus \{m\} = [\![n-m]\!] + m$, which yields

$$A \oplus \left(\complement_C B - m\right) = [\![n-m]\!],$$

since the addition is associative. □

Example 1 *Set $A := \{0, 2\}$, $B := \{0, 1, 4, 5\}$ and $B' := \{0, 1, 4, 5, 8, 9\}$. One has $A \oplus B = [\![8]\!]$ and $A \oplus B' = [\![12]\!]$,* i.e., *$m := 8$ and $n := 12$. Let $D := \complement_C B - m = \{8, 9\} - 8 = \{0, 1\}$, one obtains $A \oplus D = [\![4]\!] = [\![n-m]\!]$.*

Proposition 4 (gcd of intervals). *Let A, B be subsets of $\mathbb{N}$ and $m, n \in \mathbb{N}$. If $M \oplus A = [\![n]\!]$ and $M \oplus B = [\![m]\!]$, then it exists $C \in \mathbb{N}$ such that $M \oplus C = [\![\gcd(n, m)]\!]$.*

Proof. [Prop. 4] It follows from Proposition 3 by application of Euclid's algorithm. □

Proposition 5 (Multiple of an interval). *Let A, B be subsets of $\mathbb{N}$ and $n \in \mathbb{N}$ such that $A \oplus B = [\![n]\!]$. Then, for all $l \in \mathbb{N}$,*

$$A \oplus \left(\oplus_{i=0}^{l-1}(B + in)\right) = [\![ln]\!] .$$

Proof. [Prop. 5] We have that $A \oplus B = [\![n]\!]$. Then for all $i \in [0, l-1]$, one has

$$A \ \oplus \ \left(in + B\right) = B \ \oplus \ \left(in + A\right) = (in + [\![n]\!])$$

and hence,

$$A \oplus \cup_{i=0}^{l-1}\left(in + B\right) \ = \ \cup_{i=0}^{l-1}\left(in + \left(A \oplus B\right)\right) \ = \ \cup_{i=0}^{l-1}(in + [\![n]\!]) \ = \ [\![ln]\!] .$$

□

Note that if $\#(A)$ is prime, then A can be decomposed only in the direct sum of the neutral element and itself. We close with an elementary property.

Proposition 6. *For any $A \in \mathcal{P}(\mathbb{N})$, one has $\#(A) \leq d(A) + 1$ and both members are equal if and only if $A = [\![d(A)]\!]$.*

3 Tiling an interval of the discrete line

In this section, let $n \in \mathbb{N}$ be an integer and f be a finite subset of $\mathbb{N}$ such that $d(f) < n$. We use the following notation:

- for any $x < y$, we denote $f \cap [x, y]$ by $f[x, y]$, and $f \cap [x, y[)$ by $f[x, y[$;
- for any $0 \leq x \leq d(f)$, let us denote by $f[x]$ the subset $\{i \in f : i < x\}$.

Definition 1 (Tiling, dual). *Let $n \geq 0$ and f be a pattern such that $d(f) < n$. We say that f* tiles *$[\![n]\!]$ if and only if there exists $\hat{f}_n$, a subset of $\mathbb{N}$, such that $f \oplus \hat{f}_n = [\![n]\!]$. We call $\hat{f}_n$ the* dual *of f for n.*

For a given n, the dual is unique. The notion of dual is idempotent: the dual of the dual of f is f itself, and $\hat{f}_n$ also tiles $[\![n]\!]$. We say that a pattern f that tiles $[\![n]\!]$ is *trivial* if $f := [0, n-1] = [\![n]\!]$ or $f := \{0\}$. We define a notion of *self-period* for a pattern.

Definition 2 (Self-period a pattern). *Let $n \in \mathbb{N}$, f be a pattern such that $d(f) < n$ and p be an integer such that $0 \leq p < d(f)$. We say that p is* self-period of f for length n *if and only if for any $i \in [0, n-p[$ one has*

$$i \in f \Leftrightarrow (i + p) \in f \ .$$

For length n, we denote by $\Pi_n(f)$ the set of self-periods of f, and by $\pi_n(f)$ its smallest non null self-period.

Definition 3 (Completely self-periodic). *We say that a pattern is* completely self-periodic *for length n if and only if it is an arithmetic sequence.* I.e., *if and only if one has $j \in f \Leftrightarrow (\exists i \in [0, \lfloor n/c \rfloor] : j = ic)$, where c denotes the common difference.*

Note that if a pattern f is completely self-periodic then its common difference is its smallest non-null period, $\pi_n(f)$. We choose the word "self-period" to avoid confusion with the notion of a tiling's period mentioned in the introduction. However, for the sake of simplicity, we use the word period instead of self-period in the sequel, since the context prevents ambiguity. Furthermore, let us point out the connection between the notions of a pattern self-periodicity and of word periodicity.

Example 2 *Consider $n := 12$. The pattern $f := \{0, 1, 4, 5, 8, 9\}$ has periods 0, 4 and 8. So, $\pi_{12}(f) = 4$ and $\Pi_{12}(f) = \{0, 4, 8\}$. It can be decomposed in $\{0, 1, 4, 5, 8, 9\} = \{0, 1\} \oplus \{0, 4, 8\}$. These patterns, $\{0, 1\}$ and $\{0, 4, 8\}$ are completely periodic for lengths 2 and 12 resp., with smallest period 1 and 4 resp. Pattern f tiles $[\![12]\!]$; its dual for $n := 12$ is $\hat{f}_{12} := \{0, 2\}$, it tiles $[\![4]\!]$, $[\![8]\!]$ and $[\![12]\!]$. It is true that $\#(f) \times \#\left(\hat{f}_{12}\right) = 6 \times 2 = 12$.*

3.1 Properties of patterns that tile an interval.

Let f be a pattern. In the sequel, we assume that f tiles $[\![n]\!]$. First, we list some elementary properties of f.

Proposition 7. *Let f be a pattern that tiles $[\![n]\!]$. One has*

1. *$\#(f) \times \#\left(\hat{f}_n\right) = n$,*
2. *$d(f) + d\left(\hat{f}_n\right) = n - 1$,*
3. *Either $d(f) \geq n/2$ and $d\left(\hat{f}_n\right) < n/2$, or the opposite is true. Thus, we have either $d(f) > d\left(\hat{f}_n\right)$, or $d(f) < d\left(\hat{f}_n\right)$.*

Proof. [Prop. 7] The first two points are immediate consequences of the tiling's definition in terms of direct sum. For the second one, note that the sum of the largest elements of f and $\hat{f}_n$ must equal $n - 1$. For the third point, remark that a pattern and its dual cannot share more than the element 0. Thus, the direct sum between f and $\hat{f}_n$ implies that $d(f) \neq d\left(\hat{f}_n\right)$, hence the proposition. □

Now, let us state a simple and useful property. It follows from the positivity of the pattern's elements and from the properties of the direct sum.

Proposition 8. *For any $x \in [\![n]\!]$, one has $[0,x] \subseteq f[0,x] \oplus \hat{f}_n$.*

Proof. [Prop. 8.] That f tiles $[\![n]\!]$ implies $[0,x] \subseteq f \oplus \hat{f}_n$. On the other hand, we know by Property 2 that

$$(f[0,x] \oplus \hat{f}_n) \cap (f[x+1,n[\oplus \hat{f}_n) = \emptyset,$$

but any element of $f[x+1,n[\oplus \hat{f}_n$ is strictly greater than x. Hence, $[0,x] \subseteq f[0,x] \oplus \hat{f}_n$. □

Let us show first that a non trivial pattern that tiles $[\![n]\!]$ admits a smallest non null period smaller than $\lfloor n/2 \rfloor$. Next Property demonstrates that this period divides n.

Proposition 9. *Let f be a pattern that tiles $[\![n]\!]$ and such that $d(f) > d\left(\hat{f}_n\right)$. Then: $\pi_n(f) \leq \lfloor n/2 \rfloor$.*

Proof. [Prop. 9.] If f is trivial, *i.e.*, $f := [\![n]\!]$ then $\pi_n(f) = 1$ and the property is satisfied. Now, we consider the case of a non trivial pattern f.

By contradiction, assume that $\pi_n(f) > \lfloor n/2 \rfloor$. Thus, we have $n - \pi_n(f) < \pi_n(f)$. Let us first show that $[\![n-\pi_n(f)]\!] = f[n-\pi_n(f)] \oplus \hat{f}_n$. By Property 8, the inclusion $[\![n-\pi_n(f)]\!] \subseteq f[n-\pi_n(f)] \oplus \hat{f}_n$ is true. By contradiction, suppose there is i an element of $f[n-\pi_n(f)] \oplus \hat{f}_n$ such that $i \geq n - \pi_n(f)$. In other words, there exist $j \in f[n-\pi_n(f)]$ and $k \in \hat{f}_n$ such that $i = j+k$ and $i \geq n-\pi_n(f)$. Since $\pi_n(f)$ is a period of f, one gets $f[\pi_n(f), n[= f[n-\pi_n(f)] + \pi_n(f)$ and so, $(j+k+\pi_n(f)) \in f[\pi_n(f), n[\oplus \hat{f}_n$. But

$$j+k+\pi_n(f) \quad \geq \quad n-\pi_n(f)+\pi_n(f) \quad = \quad n,$$

which contradicts $f \oplus \hat{f}_n = [\![n]\!]$. We have shown that $[\![n-\pi_n(f)]\!] = f[n-\pi_n(f)] \oplus \hat{f}_n$.

Thus, we have that $\hat{f}_n$ tiles both $[\![n-\pi_n(f)]\!]$ and $[\![n]\!]$, and so, one obtains by Property 4 that $\hat{f}_n$ tiles $[\![\gcd(n,\pi_n(f))]\!]$. Since by hypothesis $\pi_n(f) > \lfloor n/2 \rfloor$, one has $\gcd(n,\pi_n(f)) < \lfloor n/2 \rfloor$ and $f[\gcd(n,\pi_n(f))] \oplus \hat{f}_n = [\![\gcd(n,\pi_n(f))]\!]$. By Proposition 5, it implies that $\gcd(n,\pi_n(f))$ is a period of f for length n, which contradicts the minimality of $\pi_n(f)$. □

Lemma 1 *Let f be a pattern that tiles $[\![n]\!]$ and satisfies $d(f) > d\left(\hat{f}_n\right)$. Thus, $\pi_n(f)$ divides n and*

$$f[\pi_n(f)] \ \oplus \ \hat{f}_n = [\![\pi_n(f)]\!] \ .$$

Proof. First, consider the case where f is trivial. One has $f = [\![n]\!]$, $\hat{f}_n = \{0\}$, and so, $\pi_n(f) = 1$ and divides n. Thus, $f[\pi_n(f)] = [\![\pi_n(f)]\!]$ and $f[\pi_n(f)] \ \oplus \ \hat{f}_n = [\![\pi_n(f)]\!]$. Now assume that f is not trivial. One knows by Property 9 that $\pi_n(f) \leq \lfloor n/2 \rfloor$. Set $d := n/\pi_n(f)$ and set $r := n \bmod \pi_n(f)$; we know that $d \geq 2$. Let us first show that $f[\pi_n(f)] \ \oplus \ \hat{f}_n = [\![\pi_n(f)]\!]$.

By Property 8, one gets the inclusions $[\![\pi_n(f)]\!] \subseteq f[\pi_n(f)] \oplus \hat{f}_n$ and $[\![r]\!] \subseteq f[r] \oplus \hat{f}_n$. It implies that:

$$\left(\cup_{i=1}^{d} [\![\pi_n(f)]\!] + i\pi_n(f)\right) \bigcup \left([\![r]\!] + d\pi_n(f)\right) \quad \subseteq \quad f \oplus \hat{f}_n \quad = \quad [\![n]\!].$$

But, each translate of $[\![\pi_n(f)]\!]$ has cardinality $\pi_n(f)$ and $[\![r]\!]$ has cardinality r. So the left hand side has cardinality $d\pi_n(f) + r$ which equals n the cardinality of the right hand side. It follows that both sides are equal, and that $[\![\pi_n(f)]\!] = f[\pi_n(f)] \oplus \hat{f}_n$, what we wanted, as well as $f[r] \oplus \hat{f}_n = [\![r]\!]$.

Now, let us prove by contradiction that $\pi_n(f) \mid n$, and assume that $r \neq 0$. By Proposition 5, $f[r] \oplus \hat{f}_n = [\![r]\!]$ implies that r is a period of f for length n, which contradicts the minimality of $\pi_n(f)$. □

Next corollary follows from the patterns' properties and from Lemma 1.

Corollary 1. *If f tiles $[\![n]\!]$ and $d(f) > d\left(\hat{f}_n\right)$ then $d\left(\hat{f}_n\right) < \pi_n(f)$.*

By Property 5, we have that any tile of $[\![n]\!]$ also tiles $[\![ln]\!]$ for any integer $l > 0$. We deduce next corollary from Lemma 1 and Property 5.

Corollary 2. *Let f be a pattern and d be the smallest integer such that f tiles $[\![d]\!]$. If $d > 0$, then the $[\![ld]\!]$, for $l \in \mathbb{N}$, are all the intervals f can tile.*

Theorem 3. *Let n be an integer. Among the patterns f that tile $[\![n]\!]$, it exists a one-to-one mapping that, to any pattern f such that $d(f) \leq n/2$, associates a pattern that tiles $[\![d]\!]$ for d a divisor of n. This bijection associates to such a pattern f its dual $\hat{f}_n$.*

Proof. [Theorem 3.] Theorem follows from Lemma 1 and Property 5. □

One obtains a canonical decomposition of patterns tiling $[\![n]\!]$ in irreducible patterns. Indeed, Theorem 3 allows us to write any tile f of $[\![n]\!]$ as the direct sum of i/ a completely periodic pattern for length n (with period a divisor strict of n) and ii/ one or more patterns that tiles $[\![d]\!]$, with d a strict divisor of n, and are completely periodic for length d. This decomposition result also is a corollary of Theorem 6 (section 4).

3.2 Numbers of tiles of an interval

Let $n \in \mathbb{N}$ such that $n > 0$. We denote by Ξ_n the set of tiles of $[\![n]\!]$. Let Δ_n be the subset of patterns in Ξ_n whose diameter is smaller than or equal to $\lfloor n/2 \rfloor$ (*i.e.*, those who tile $[\![d]\!]$ for d a strict divisor of n), and let Ψ_n be the complement of Δ_n in Ξ_n (*i.e.*, those patterns with diameter strictly greater than $\lfloor n/2 \rfloor$). By definition, one has $\Xi_n = \Delta_n \cup \Psi_n$. We denote the cardinalities of these sets by ξ_n, δ_n, and ψ_n, respectively.

Theorem 4. *Let $n \in \mathbb{N}$ be an integer such that $n > 1$. One has $\xi_1 = 1$ and*

$$\boxed{\xi_n \quad = \quad 1 + \sum_{d \in \mathbb{N}\ :\ d|n,\ d \neq n} \xi_d\ .} \tag{1}$$

Proof. [Theorem 4.] Direct investigation of the case $n := 1$ establishes that $\Xi_1 = \{\{0\}\}$ and $\xi_1 = 1$, since $\Delta_1 = \emptyset$, $\Psi_1 = \{\{0\}\}$, $\delta_1 = 0$, and $\psi_1 = 1$. By definition, for any $n > 0$, one has $\Xi_n = \Delta_n \cup \Psi_n$, and by Lemma 1 we obtain that

$$\Delta_n \quad = \bigcup_{d \in \mathbb{N}: d|n, d \neq n} \Psi_d\ , \qquad\qquad \delta_n \quad = \sum_{d \in \mathbb{N}: d|n, d \neq n} \psi_d\ . \tag{2}$$

As by definition for any $n > 1$, $\Psi_n = \{\hat{f}_n : f \in \Delta_n\}$, we obtain that $\delta_n = \psi_n$, and therefore that $\xi_n = 2\delta_n$. Rewritting the latter using 2 yields

$$\xi_n \quad = \quad 2 \sum_{d \in \mathbb{N}: d|n, d \neq n} \psi_d\ . \tag{3}$$

Finally, we show that 1 is equivalent to 3. For any $n > 1$, one has

$$\begin{aligned} \xi_n \quad &= \quad 2 \textstyle\sum_{d \in \mathbb{N}: d|n, d \neq n} \psi_d \quad &&= 1 + \psi_1 + 2 \textstyle\sum_{d \in \mathbb{N}: d|n, d>1, d \neq n} \psi_d \\ &= 1 + \xi_1 + \textstyle\sum_{d \in \mathbb{N}: d|n, d>1, d \neq n} \xi_d \quad &&= \quad 1 + \textstyle\sum_{d \in \mathbb{N}: d|n, d \neq n} \xi_d\ . \end{aligned}$$

□

The values of ξ_n for $n > 0$ are those of Sequence entry A067824 in [12], and (1) corresponds to the recurrence relation given for this sequence par Zumkeller.

Corollary 5. *If $n > 1$ is prime then $\Delta_n = \Psi_1$, $\Psi_n = \{[\![n]\!]\}$, $\Xi_n = \{\{0\}, [\![n]\!]\}$, $\delta_n = \psi_n = 1$ and $\xi_n = 2$.*

4 Algebraic approach

4.1 Polynomials decomposition

Let us denote by $\mathcal{C}$ the set of *super-composite* integers, *i.e.*, all integers whose prime factorization contains at least two different primes. It is known that $X^n - 1$ admits a unique decomposition (up to the order of its factors) in irreducible elements of $\mathbb{Z}[X]$. Indeed, $\mathbb{Z}[X]$ is an Euclidean ring, and as such, is factorial. This decomposition is

$$X^n - 1 = \prod_{d|n} \Phi_d. \tag{4}$$

where Φ_d is the *d-th cyclotomic polynomial* [10]. We use the following properties of cyclotomic polynomials.

Proposition 10.
- *The degree of* Φ_d *is* $\varphi(d)$*, where* φ *is Euler's function.*
- $\Phi_d(1) = p$ *if* d *is a power of a prime* p *and* $\Phi_d(1) = 1$ *otherwise.*
- *The polynomial* Φ_d *belongs to* $\{0,1\}[X]$ *if and only if* $d \notin \mathcal{C}$.

Proof. • First statement is a classical result [10].
• First, using (4), we obtain by induction that $\Phi_{p^n}(X) = \sum_{i=0}^{p-1} X^{ip^{(n-1)}}$, and thus, that $\Phi_d(1) = p$ if $d = p^n$. Moreover, we have

$$1 + \cdots + X^{n-1} \quad = \prod_{d|n,\ d\neq 1} \Phi_d(X) \quad = \prod_{d|n,\ d\neq 1,\ d\in\mathcal{C}} \Phi_d(X) \prod_{d|n,\ d\neq 1,\ d\notin\mathcal{C}} \Phi_d(X). \tag{5}$$

Evaluating (5) in 1 yields

$$n \quad = \prod_{d|n,\ d\neq 1} \Phi_d(1) \quad = \prod_{d|n,\ d\neq 1,\ d\in\mathcal{C}} \Phi_d(1) \prod_{d|n,\ d\neq 1,\ d\notin\mathcal{C}} \Phi_d(1).$$

However, it is clear that $n = \prod_{d|n, d\neq 1, d\notin\mathcal{C}} \Phi_d(1)$ (since for each $p^i|n$, we have $\Phi_{p^i}(1) = p$) and so, $1 = \prod_{d|n,d\neq 1,d\in\mathcal{C}} \Phi_d(1)$. It means that if $d \notin \mathcal{C}$ then $\Phi_d(1) = 1$.
• Third statement is a consequence of the second statement. We have already shown that if $d \notin \mathcal{C}$ then $\Phi_d = \Phi_{p^n}(X) = \sum_{i=0}^{p-1} X^{ip^{(n-1)}}$ belongs to $\{0,1\}[X]$. Now, we know from first statement that if $n \in \mathcal{C}$ then $\Phi_n(1) = 1$. In $\{0,1\}[X]$, for any polynomial P, its number of monomials equals $P(1)$. As $\Phi_n(X)$ has more than one monomial, it cannot be in $\{0,1\}[X]$. □

As ρ is a bijection, it induces a one-to-one correspondence between the pairs $(A,B) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ such that $A \oplus B = [\![n]\!]$, and the pairs $(P,Q) \in (\{0,1\}[X] \times \{0,1\}[X])$ such that $P(X)Q(X) = 1 + \cdots + X^{n-1}$. Moreover, as $1 + \cdots + X^{n-1}$ is factorizable in $\prod_{d|n,d\neq 1} \Phi_d(X)$, there exists a partition of $\{d|n, d \neq 1\}$ in D_1 and D_2 such that $P(X) = \prod_{d\in D_1} \Phi_d(X)$ and $Q(X) = \prod_{d\in D_2} \Phi_d(X)$.

4.2 Results

Lemma 2 *Let* $P_1, \ldots, P_k$ *be polynomials of* $\{0,1\}[X]$ *such that* $\prod_{i=1}^k P_i$ *belongs to* $\{0,1\}[X]$*. For each extracted sequence* $P_{s_1}, \ldots, P_{s_t}$*, with* $1 \leq s_1, \ldots, s_t \leq k$*, one has* $\prod_{i=1}^t P_{s_i} \in \{0,1\}[X]$.

Proof. Classically, we define the infinite norm $\|.\|_\infty$ by $\|a_0 + \cdots + a_n X^n\|_\infty := \max(|a_i|)$. The infinite norm satisfies the following property: for each $P,Q \in \mathbb{N}[X]$, $\|P\|_\infty \|Q\|_\infty \leq \|PQ\|_\infty$. Moreover, if $\prod_{i=1}^t P_{s_i} \notin \{0,1\}[X]$, we have $\|\prod_{i=1}^t P_{s_i}\|_\infty > 1$, which contradicts our hypothesis. □

For all $n \in \mathbb{N}$, we call *total valuation* of n, denoted by ν_n, the sum of the powers in the prime factorization of n. We call *factorial sequence* of n, a sequence $u_0, u_1, \ldots, u_s$ such that $u_0 := 1$, $u_s := n$, and u_{i+1}/u_i is a prime number. Observe that all factorial sequences of n have ν_n+1 terms. From a factorial sequence of n, we can build a *sequence of decomposition* $(D_{u_{i-1},u_i})_{1\leq i\leq s}$ with $D_{u_{i-1},u_i} := \{d|u_i : d \nmid u_{i-1}\}$. For conciseness, for all $D \in \mathcal{P}(\mathbb{N})$ we write $\Phi_D := \prod_{d\in D} \Phi_d$.

Lemma 3 *Let* $n, p \in \mathbb{N}$ *with* p *prime.* $\Phi_{D_{n,pn}}$ *belongs to* $\{0,1\}[X]$ *and is irreducible in* $\{0,1\}[X]$.

Proof. As $\Phi_{D_{n,np}} \Phi_{\{d|n:d\neq 1\}} = 1 + \cdots + X^{np-1}$ and $\Phi_{\{d|n:d\neq 1\}} = 1 + \cdots + X^{n-1}$, by division, we obtain that $\Phi_{D_{n,np}} \in \{0,1\}[X]$. As $\Phi_{D_{n,np}}(1) = np/n$ is prime, it follows that $\Phi_{D_{n,np}}$ is necessarily irreducible in $\{0,1\}[X]$. □

Theorem 6. *Each factorization of* $1 + \cdots + X^{n-1}$ *in irreducible elements in* $\{0,1\}[X]$ *has the following form* $\prod_{1\leq i\leq s} \Phi_{D_{u_{i-1},u_i}}$ *where* $(D_{u_{i-1},u_i})_{1\leq i\leq s}$ *is a sequence of decomposition of* n*, and reciprocally. Moreover, for all* $1 \leq i \leq s$*,* $\Phi_{D_{u_{i-1},u_i}}(1)$ *is a prime factor of* n.

Note that the factorization may not be unique. Indeed, $\{0,1\}[X]$ is not a factorial ring.
Proof. We proceed by induction on the value of ν_n. If $\nu_n = 1$ then n is prime and $1 + \cdots + X^{n-1} = \Phi_n$. Now suppose the theorem holds for all n such as $\nu_n < k$. Let $n \in \mathbb{N}$ such that $\nu_n = k$ and let $P_1 \ldots P_t$ be a factorization of $1 + \cdots + X^{n-1}$ in irreducible polynomials of $\{0,1\}[X]$. By Lemma 2, we have that $\prod_{i=1}^{t-1} P_i \in \{0,1\}[X]$. Two cases arise:

1. There is j such that $\deg(P_j) > n/2$.
   Up to a renumbering of the polynomials, one can always suppose that $j = t$. By Theorem 3, we get that there exist a polynomial $Q \in \{0,1\}[X]$ and a divisor d of n such that $Q\prod_{i=1}^{t-1} P_i = 1 + \cdots + X^{d-1}$. Thus, there is $T \in \{0,1\}[X]$ such that $TQ\prod_{i=1}^{t-1} P_i = 1 + \cdots + X^{n-1}$, which implies $P_t = TQ$. But, P_t is supposed to be irreducible, which gives that $Q = 1$. As $\nu_d < k$, we can apply the induction hypothesis to the factorization $\prod_{i=1}^{t-1} P_i$ of $1 + \cdots + X^{d-1}$ and deduce that it arises from a sequence of decomposition of d.
   Now, let us show by contradiction that n/d is prime. We suppose that there exists d_1 such that $d|d_1$ and $d_1|n$. We can find two polynomials $Q_1, Q_2 \in \{0,1\}[X]$ such that $Q_1, Q_2 \neq 1$, $Q_1\prod_{i=1}^{t-1} P_i = 1 + \cdots + X^{d_1-1}$, and $Q_2Q_1\prod_{i=1}^{t-1} P_i = 1 + \cdots + X^{n-1}$. We obtain that $P_t = Q_1Q_2$, which contradicts the irreducibility of P_t. We have shown that n/d is prime. Finally, it can be seen that $P_t = \prod_{\substack{m|n \\ m\nmid d}} \Phi_m$.

2. For all j, $\deg(P_j) \leq n/2$.
   In this case, for all j, there exist $Q_j \in \{0,1\}[X]$ and d_j a strict divisor of n such that $P_jQ_j = 1 + \cdots + X^{d_j-1}$. By induction hypothesis, P_j is an irreducible factor of Φ_{D_j} where D_j belongs to a sequence of decomposition of d_j. However, $\bigcup_{1\leq j\leq t} D_i$ partitions the divisors of n. So, there is i such that $n \in D_i$, which contradicts that D_i belongs to a sequence of decomposition of a strict divisor of n. This case never occurs.

□

Example 3 *For* $n := 12$*, the factorial sequences are:* $(1,2,4,12)$*,* $(1,2,6,12)$*, and* $(1,3,6,12)$*. The associated sequences of decomposition are* $(\{2\},\{4\},\{3,6,12\})$*,* $(\{2\},\{3,6\},\{4,12\})$*,* $(\{3\},\{2,6\},\{4,12\})$*. We obtain that the irreducible factors of* $1 + \cdots + X^{n-1}$ *in* $\{0,1\}[X]$ *are* Φ_2*,* Φ_3*,* Φ_4*,* $\Phi_3\Phi_6$*,* $\Phi_2\Phi_6$*,* $\Phi_3\Phi_6\Phi_{12}$*,* $\Phi_4\Phi_{12}$*.*

Theorem 7. *The number* v_n *of irreducible factors of* $1 + \cdots + X^{n-1}$ *in* $\{0,1\}[X]$ *equals*

$$\sum_{d|n} \#\{\textit{prime factors of } d\} .$$

Proof. It is enough to count the $D_{d,dp}$ that belong to a sequence of decomposition of n. *I.e.*, to enumerate the pairs (d,p) such that d is a divisor of n and p a prime divisor of d. This gives exactly the desired sum. □

The pattern associated with the polynomial $\Phi_{D_{d,dp}}$ is the arithmetic sequence starting in 0, of common difference n, and having p terms. This gives the precise structure of all tiles of a segment.

A *reciprocal polynomial* is a polynomial such that $P(X) := X^nP(1/X)$, where n is the degree of P.

Corollary 8. *Let* f *be a pattern which tiles an interval. The associated polynomial,* P_f*, is reciprocal.*

Proof. An arithmetic sequence is associated to a reciprocal polynomial and the product of reciprocal polynomial is reciprocal. □

Theorem 9. *Let* f *be a pattern which tiles a interval. Then, the smallest nonempty interval it tiles is smaller than* $2d(f)$*.*

Proof. Let $[\![n]\!]$ be the smallest segment that f tiles. One has $f \oplus \hat{f}_n = [\![n]\!]$. Assume that $d(f) \leq n/2$; then, by Proposition 7 one gets $d\left(\hat{f}_n\right) > d(f)$. Consequently, Lemma 1 implies that f tiles $[\![\pi_n(f')]\!]$, which contradicts the minimality of $[\![n]\!]$. □

Theorem 10. *Let* f *be a pattern. Algorithm 1 decides in* $O(d(f)^2)$ *time whether there is* $n \in \mathbb{N}$ *such that* f *tiles* $[\![n]\!]$*.*

Algorithm 1: recognizes if a given pattern tiles at least an interval.

Data: a pattern f

Result: the minimal $n \in \mathbb{N}$ such that f tiles $[\![n]\!]$ if it exists, and -1 otherwise

$g := \{0\}$; $i := 0$;

while $(d(f) + d(g) + 1 > \#(f)\#(g))$ *and* $(i \leq d(f))$ **do**

  $i := \min(\mathbb{N} \setminus f \oplus g)$;

  $g := g \bigcup \{i\}$;

  if f *and* g *are not in direct sum* **then return** -1;

if $(d(f) + d(g) + 1 = \#(f)\#(g))$ **then return** $d(f) + d(g) + 1$;

else return -1

Proof. Since successive values taken by i increase, Algorithm 1 stops. By application of Proposition 6 to $f \oplus g$, we are sure that if $d(f) + d(g) + 1 = \#(f)\#(g)$, then f tiles $[\![d(f) + d(g) + 1]\!]$. Otherwise, if the condition $i \leq d(f)$ is violated, Theorem 9 implies that f does not tile any segment. When the sets f, g, and $f \oplus g$ are implemented as sorted lists, computing $f \uplus g$ and checking whether it is a direct sum requires $O(d(f))$ time; this yields an overall time complexity of $O(d(f)^2)$. □

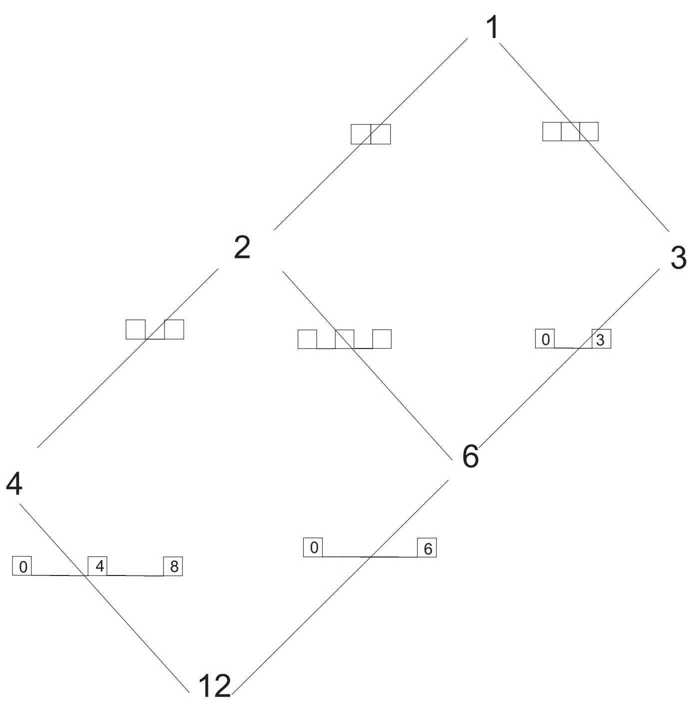

Fig. 1. The lattice of the sequences of decomposition for $n := 12$, and the associated irreducible tiles of $[\![n]\!]$. A path from 1 to 12 represents a sequence of decomposition of n. The direct sum of the subsets on the edges of the path tile $[\![12]\!]$.

5 Conclusion

This work characterizes the tilings of an interval as direct sums of arithmetic sequences. Counting results obtained also show that, surprisingly, the number of patterns that tile a segment of length n depends, not on the prime factors of n, but only on the list of their powers. *E.g.*, segments of respective lengths $n_1 := 5 \times 7^2 \times 2^4$ and $n_2 := 13 \times 3^2 \times 11^4$ (n_1 and n_2 have both $(1, 2, 4)$ as list of powers), have the same number of tiles. The regular structure of the tiles of a segment contrasts sharply with the singular structure of those tiling the torus $\mathbb{Z}/n\mathbb{Z}$. Indeed for this problem, it exists irregular sets A and B such

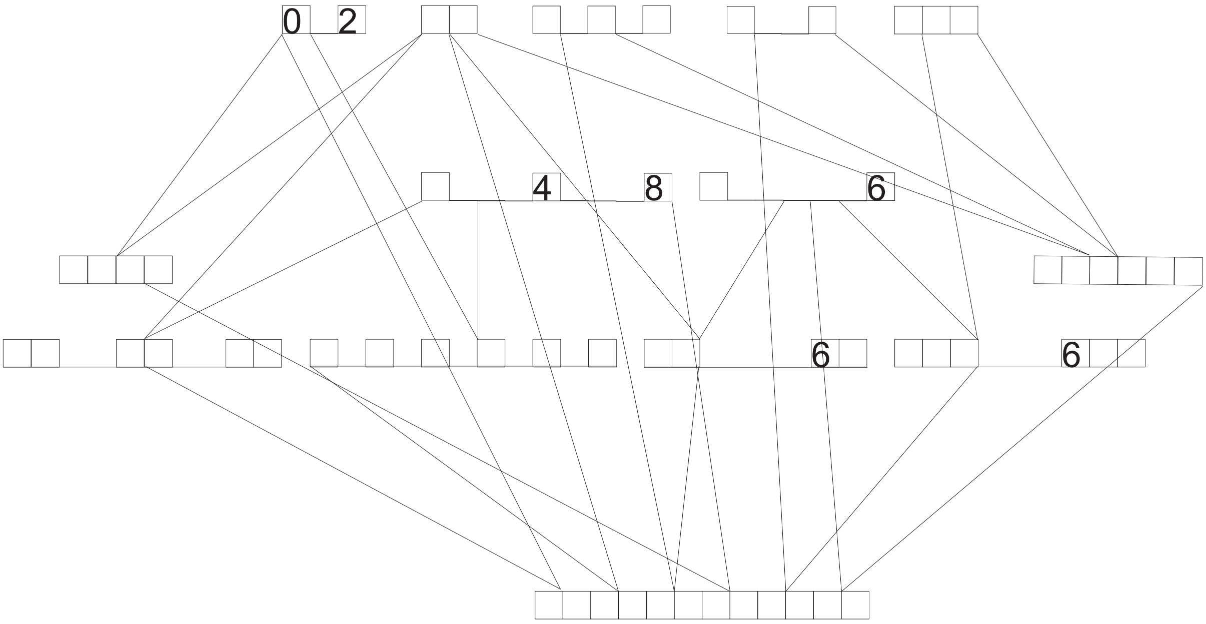

Fig. 2. The lattice of the tiles of $[\![n]\!]$ for $n := 12$. The pairs of edges having a common extremity represent the direct sum to apply to two patterns to obtain the third one.

that $A \oplus B = \mathbb{Z}/n\mathbb{Z}$ [6]. However, our results exhibit a relation between tilings, words and polynomials that opens promising directions for the tiling by a single pattern of the discrete plane or of special cases of the torus. The proofs also shed light on the complementarity of combinatorial and algebraic approaches for this types of problems.

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