Discrete Geometry and Symbolic Dynamics

Valerie Berthe

DISCRETE GEOMETRY AND SYMBOLIC DYNAMICS

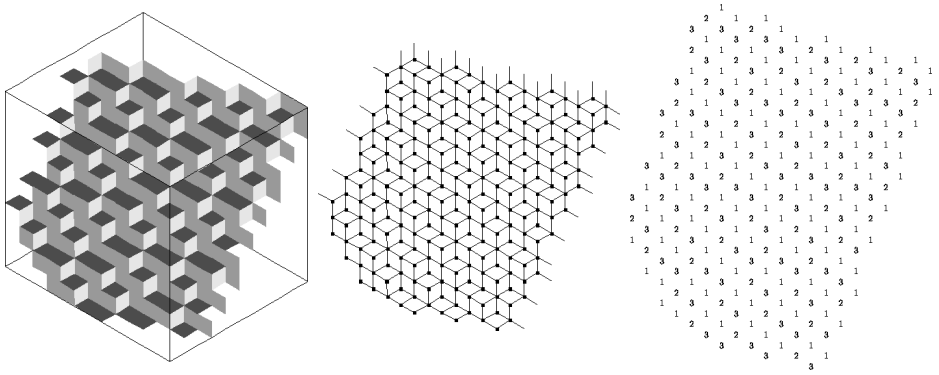
VALÉRIE BERTHÉ

Avec mes pensées les plus amicales pour Christer Kiselman.

1. Introduction

The aim of this survey is to illustrate various connections that exist between word combinatorics and arithmetic discrete geometry through the discussion of some discretizations of elementary Euclidean objects (lines, planes, surfaces). We will focus on the rôle played by dynamical systems (toral rotations mainly) that can be associated in a natural way with these discrete structures. We will see how classical techniques in symbolic dynamics applied to some codings of such discretizations allow one to obtain results concerning the enumeration of configurations and their statistical properties. Note that we have no claim to exhaustivity: the examples that we detail here have been chosen for their simplicity.

Let us illustrate this interaction with the following figure where a piece of an arithmetic discrete plane in $\mathbb{R}^3$ is depicted, as well as its orthogonal projection onto the antidiagonal plane $\Delta\colon x_1+x_2+x_3=0$ in $\mathbb{R}^3$, which can be considered as a piece of a tiling of the plane by three kinds of lozenges, and lastly, its coding as a two-dimensional word over a three-letter alphabet.



This paper is organized as follows. We first start with the most simple situation, namely discrete lines and Sturmian words (see Section 2). Section 3 is devoted to the higher-dimensional case, i.e., to the study of arithmetic discrete planes. We generalize this study performed mainly in the so-called naive case, first, to a broader class of arithmetic discrete planes in Section 4, and second, to functional stepped surfaces in Section 5. Section 6 is

concerned with the generation of arithmetic discrete planes by generalized substitutions. Special focus is given to the Rauzy fractal associated with the cubic Pisot number of minimal polynomial $X^3 - X^2 - X - 1 = 0$.

2. Sturmian words and discrete lines

This section is devoted to the connections between arithmetic discrete lines and Sturmian words. A wide literature has been devoted to the study of discrete lines, as illustrated for instance in the surveys [KR04, BCK04]. Let us start by recalling the definition of an arithmetic discrete line, introduced by Reveillès in [Rev91].

Definition 1. Let $\mathbf{v} \in \mathbb{R}^2$, and $\mu, \omega \in \mathbb{R}$. The (lower) arithmetic *discrete line* $\mathfrak{D}(\mathbf{v}, \mu, \omega)$ is defined as

$$\mathfrak{D}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^2;\ 0 \leq \langle \mathbf{v}, \mathbf{v} \rangle + \mu < \omega\}.$$

Parameter μ is called the *translation parameter* of $\mathfrak{D}(\mathbf{v}, \mu, \omega)$, and ω is called the *width* of $\mathfrak{D}(\mathbf{v}, \mu, \omega)$.

Two natural cases are more particularly studied, namely if $\omega = ||\mathbf{v}||_\infty$ then $\mathfrak{D}(\mathbf{v}, \mu, \omega)$ is said *naive*, and if $\omega = ||\mathbf{v}||_1$, then $\mathfrak{D}(\mathbf{v}, \mu, \omega)$ is said *standard*. One checks that a naive (resp. standard) line is made of of horizontal and

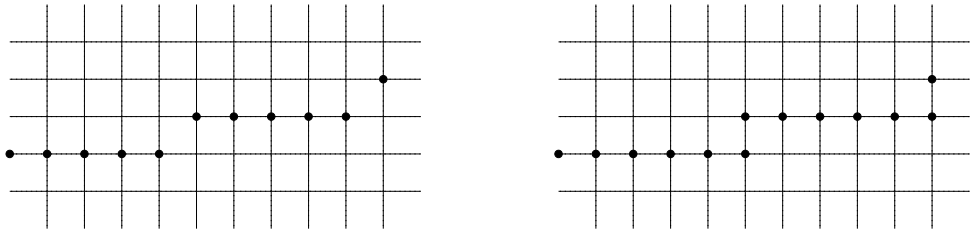


Figure 2.1. Left: a naive line. Right: a standard line.

vertical (resp. horizontal and diagonal) steps. One can code such a standard line by using the *Freeman code* [Fre70] over the two-letter alphabet $\{0, 1\}$ as follows: one codes horizontal steps by a 0, and vertical ones by a 1. One gets a *so-called* Stumian word $(u_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$.

More precisely, Sturmian words are defined as follows:

Definition 2 (Morse-Hedlund [MH40])**.** Let $R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z},\ x \mapsto x + \alpha \bmod 1$ be the rotation of angle α of the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $u = (u_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$. The infinite word u is a Sturmian word if there exist $\alpha \in (0, 1)$, $\alpha \notin \mathbb{Q}$, $x \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\ u_n = i \Longleftrightarrow R_\alpha^n(x) = n\alpha + x \in I_i \pmod 1,$$

with $I_0 = [0, 1 - \alpha[,\ I_1 = [1 - \alpha, 1[$ or $I_0 =]0, 1 - \alpha],\ I_1 =]1 - \alpha, 1]$.

One thus checks that Sturmian words are codings of naive arithmetic discrete lines. For more on Sturmian words, see the surveys [AS02, Lot02, PF02] and the references therein.

The following lemma is classical for the study of Sturmian words. Its interest for further generalizations is stressed in the survey [BFZ05].

Lemma 1. *The word $w = w_1 \cdots w_n$ over the alphabet $\{0,1\}$ is a factor the Sturmian word u if and only if*

$$I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n} \neq \emptyset.$$

Proof. By definition, one has

$$\forall i \in \mathbb{N},\ u_n = i \iff n\alpha + x \in I_i \pmod 1.$$

One first notes that $u_k u_{k+1} \cdots u_{n+k-1} = w_1 \cdots w_n$ if and only if

$$\begin{cases} k\alpha + x \in I_{w_1} \\ (k+1)\alpha + x \in I_{w_2} \\ \dots \\ (k+n-1)\alpha + x \in I_{w_n} \end{cases}$$

One then applies the density of $(n\alpha)_{n\in\mathbb{N}}$ in $\mathbb{R}/\mathbb{Z}$ (recall that α is assumed to be an irrational number). □

One easily checks that the sets $I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n}$ are intervals of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Furthermore, the factors of u are in one-to-one correspondence with the $n+1$ intervals of $\mathbb{T}$ whose end-points are given by $-k\alpha \bmod 1$, for $0 \leq k \leq n$. This implies that Sturmian words have exactly $n+1$ factors of length n, for every $n \in \mathbb{N}$. This is even a characterization of Stumian words:

Theorem 1 (Coven-Hedlund [CH73])**.** *A word $u \in \{0,1\}^{\mathbb{N}}$ is Sturmian if and only if it has eactly $n+1$ factors of length n.*

The function that associates with a word the number of its factors of a given length is called the *complexity function*. For more on this function, see for instance [AS02]. One more generally deduces from Lemma 1 various combinatorial properties of Sturmian words, such as the expression of densities of factors [Ber96], that can be deduced from the equidistribution of the sequence $(n\alpha)_{n\in\mathbb{N}}$.

Let us note that Definition 2 can be restated in terms of dynamical systems as follows. A *dynamical system* (X,T) is defined as the action of a continuous and onto map T on a compact space X. An example of a geometric dynamical system is given by $(\mathbb{T}, R_\alpha)$. In other words, a Sturmian word is a coding of a dynamical system of the form $(\mathbb{T}, R_\alpha)$ with respect either to the two-interval partition $\{I_0 = [0, 1-\alpha[,\ I_1 = [1-\alpha, 1[\}$ or to $\{I_0 =]0, 1-\alpha],\ I_1 =]1-\alpha, 1]\}$.

As another example, let us consider symbolic dynamical systems. Let $\mathcal{A}$ be a finite set. Let $u \in \mathcal{A}^{\mathbb{N}}$. Let $\mathcal{L}(u)$ be the set of its factors. The *shift* S is defined as $S : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}},\ (u_n)_{n\in\mathbb{N}} \mapsto (u_{n+1})_{n\in\mathbb{N}}$. The *symbolic dynamical*

system generated by u is (X_u, S) with $X_u := \overline{\{S^n(u);\ n \in \mathbb{N}\}} = \{v \in \mathcal{A}^{\mathbb{N}};\ \mathcal{L}(v) \subset \mathcal{L}(u)\} \subset \mathcal{A}^{\mathbb{N}}$. One deduces from Lemma 1 that two Sturmian words coding the same rotation have the same set of factors, and thus, that the symbolic dynamical system generated by a Sturmian word coding the rotation R_α consists of *all* the Sturmian words that code the same rotation.

Note that several combinatorial properties of Sturmian words or of naive arithmetic discrete lines respectively, have been studied and stated independently: for instance, the notion of balance, and the chord property respectively, have been considered in [MH38, MH40, Lot02, PF02] for Sturmian words, and in [Fre74, Ros74, Hun85, Mel05] in discrete geometry. For more details on the connections between Sturmian words and discrete lines, see Chap. 1 of [Jam05b], and more generally, for references on discrete lines, see the surveys [KR04, BCK04].

3. Discrete planes

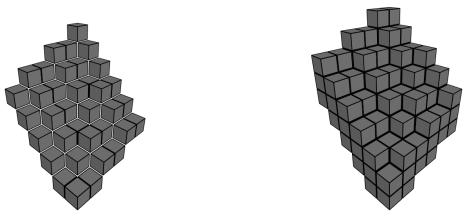
Let us consider now the higher-dimensional case.

Definition 3. Let $\mathbf{v} \in \mathbb{R}^3$, and $\mu, \omega \in \mathbb{R}$. The *arithmetic discrete hyperplane* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is defined as

$$\mathfrak{P}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^3;\ 0 \leq \langle \mathbf{x}, \mathbf{v} \rangle \mu < \omega\}.$$

If $\omega = ||v||_\infty$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said *naive*. If $\omega = ||v||_1$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said *standard*.

A piece of a naive plane (left) as well as a piece of a standard plane (right) are depicted in the figure below



Let us see now how to associate with a standard arithmetic discrete plane a coding as two-dimensional word on a three-letter alphabet that plays the rôle of the Freeman code for arithmetic discrete lines.

Let $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ stand for the canonical basis of $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$. Let E_1, E_2 and E_3 be the three following faces:

$$\begin{aligned} E_1 &= \left\{\lambda \mathbf{e_2} + \mu \mathbf{e_3} \mid (\lambda, \mu) \in [0, 1[^2\right\}, \\ E_2 &= \left\{-\lambda \mathbf{e_1} + \mu \mathbf{e_3} \mid (\lambda, \mu) \in [0, 1[^2\right\}, \\ E_3 &= \left\{-\lambda \mathbf{e_1} - \mu \mathbf{e_2} \mid (\lambda, \mu) \in [0, 1[^2\right\}. \end{aligned}$$

We call *pointed face* the set $\mathbf{x}+E_i$. The point $\mathbf{x}$ is called the *distinguished vertex* of the face $\mathbf{x}+E_i$. Note that each pointed face includes exactly one integer point, namely, its *distinguished vertex*.

Let $\mathfrak{P}(\mathbf{v},||v||_1) = \mathfrak{P}(\mathbf{v},0,||v||_1)$ be a standard arithmetic discrete plane. One associates with $\mathfrak{P}$ a so-called *stepped plane* $\mathcal{P}$ defined as the union of faces of integral cubes that connect the points points of $\mathfrak{P}_\sigma$, as depicted in Fig. 3.1. By *integral cube*, we mean a translate by a vector with integral entries of the fundamental unit cube $\mathcal{C} = \{\sum_{1\leq i\leq n} \lambda_i \mathbf{e}_i;\ \lambda_i \in [0,1],\ \text{ for all } i\}$ with integral vertices. The stepped plane $\mathcal{P}$ is thus defined as the boundary of the set of integral cubes that intersect the lower open half-space $\{\mathbf{x}\in\mathbb{Z}^3;\ \langle \mathbf{x},\mathbf{v}_\beta\rangle \leq 0\}$. The vertices of $\mathcal{P}$ (that is, the points with integer coordinates of $\mathcal{P}$) are exactly the points of the arithmetic discrete plane $\mathfrak{P}$, according for instance to [BV00].

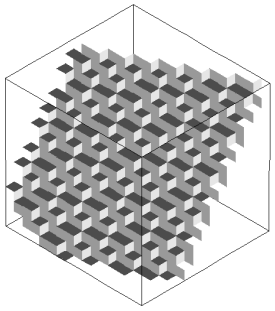


FIGURE 3.1. Stepped surface.

Let Δ be the diagonal plane of equation $x_1+x_2+x_3=0$ and let π_0 be the orthogonal projection onto Δ. Note that $\pi_0(\mathbb{Z}^3)$ is a lattice in Δ with basis $(\pi_0(e_1),\ \pi_0(e_2))$, and that $\pi_0(e_3) = -\pi_0(e_1) - \pi_0(e_2)$. If we use this basis for $\pi_0(\mathbb{Z}^3)$, then the restriction of π_0 to $\mathbb{Z}^3$ becomes the following map, also denoted by π_0 by abuse of notation:

$$\pi_0 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2,\ \mathbf{x}\mapsto (x_1-x_3, x_2-x_3).$$

According to [BV00, ABI02], the restriction of the projection map π_0 to $\mathfrak{P}(\mathbf{v},\mu)$ is one-to-one and onto Δ:

$$\forall (m_1,m_2)\in\mathbb{Z}^2,\ \exists!\ \mathbf{x}+E_i\subset\mathcal{P}(\mathbf{v},\mu),\ \pi(\mathbf{x})=(m_1,m_2).$$

Furthermore, the projections of the faces of the stepped plane $\mathcal{P}(\mathbf{v},\mu)$ tile the diagonal plane Δ with three kinds of lozenges (see Figure 1).

We thus provide each stepped plane with a two-dimensional coding as follows. The *two-dimensional coding* of the stepped plane $\mathcal{P}(\mathbf{v},\mu)$ is the two-dimensional word $U\in\{1,2,3\}^{\mathbb{Z}^2}$ defined, for all $(m_1,m_2)\in\mathbb{Z}^2$ and all $i\in\{1,2,3\}$, by

$$U_{m_1,m_2} = i \iff \exists \mathbf{x}+E_i\subset\mathcal{P}(\mathbf{v},\mu) \text{ such that } (m_1,m_2)=\pi(\mathbf{x}).$$

One checks (e.g., see [BV00, ABI02, ABS04]) that for $(m_1, m_2) \in \mathbb{Z}^2$ and $i \in \{1, 2, 3\}$, then $U_{m_1,m_2} = i$ if and only if:

$$m_1 v_1 + m_2 v_2 + \mu \bmod v_1 + v_2 + v_3 \in [v_1 + \cdots + v_{i-1}, v_1 + \cdots + v_i[. \tag{3.1}$$

Let us now introduce an analogue of the dynamical system $(\mathbb{T}, R_\alpha)$ coded by the two-dimensional word U. Given two continuous and onto maps T_1 and T_2 acting on X and satisfying $T_1 \circ T_2 = T_2 \circ T_1$, the $\mathbb{Z}^2$-*action* by T_1 and T_2 on X, that we denote by (X, T_1, T_2), is defined as

$$\forall (m, n) \in \mathbb{Z}^2,\ \forall x \in X,\quad (m, n) \cdot x = T_1^m \circ T_2^n(x).$$

As an example, consider a $\mathbb{Z}^2$-action by two rotations on the $\mathbb{R}/\mathbb{Z}$, that is, the $\mathbb{Z}^2$-action defined by

$$(m, n) \cdot x = R_\alpha^m R_\beta^n(x) = x + m\alpha + n\beta \mod 1.$$

Given any partition $\{P_1, \cdots, P_d\}$ of the torus and a point x we can define a (two-dimensional) word $U = (U_{m,n})_{(m,n)\in\mathbb{Z}^2} \in \{1, 2, \cdots, d\}^{\mathbb{Z}^2}$ coding the orbit of x under this $\mathbb{Z}^2$-action by $U_{(m,n)} = i$ whenever $R_\alpha^m R_\beta^n x \in P_i$, for $(m, n) \in \mathbb{Z}^2$. Two-dimensional Sturmian words, as introduced in Definition 4 below, are examples of such codings. Definition 4 corresponds to (3.1) after a suitable renormalization by $||v||_1$ of the parameters involved.

Definition 4 ([BV00])**.** Let $U = (U_{m,n})_{(m,n)\in\mathbb{Z}^2} \in \{1, 2, 3\}^{\mathbb{Z}^2}$. The two-dimensional word U is said to be a two-dimensional Sturmian word if there exist $x \in \mathbb{R}$, and α, $\beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly independent and $\alpha + \beta < 1$ such that

$$\forall (m, n) \in \mathbb{Z}^2,\ Um, n = i \Longleftrightarrow R_\alpha^m R_\beta^n(x) = x + n\alpha + m\beta \in I_i \pmod 1,$$

with

$$I_1 = [0, \alpha[,\ I_2 = [\alpha, \alpha + \beta[,\ I_3 = [\alpha + \beta, 1[$$

or

$$I_1 =]0, \alpha],\ I_2 =]\alpha, \alpha + \beta],\ I_3 =]\alpha + \beta, 1].$$

We consider here finite rectangular arrays of consecutive letters, that is, rectangular words $w = \begin{matrix} w_{0,n-1} & \cdots & w_{m-1,n-1} \\ \vdots & & \vdots \\ w_{0,0} & \cdots & w_{m-1,0} \end{matrix}$; we say here that w has size (m, n). The *rectangular complexity* of the two-dimensional word U is the function $p_U(m, n)$ which associates with each $(m, n) \in \mathbb{N}^2$, m and n being nonzero, the cardinality of the set $L_{(m,n)}(u)$ of rectangular factors of size (m, n) occurring in u.

The analogue of Lemma 1 also holds here: the word $w = \begin{matrix} w_{0,n} & \cdots & w_{m,n} \\ \vdots & & \vdots \\ w_{0,0} & \cdots & w_{m,0} \end{matrix}$ is a factor of the two-dimensional Sturmian word coding U if and only if

$$\cap_{0\leq i\leq m, 0\leq j\leq n} R_\alpha^{-i} R_\beta^{-j} I_{w_{i,j}} \neq \emptyset. \tag{3.2}$$

We first deduce, since α, β are assumed to satisfy $\dim_{\mathbb{Q}}(1, \alpha, \beta) = 3$, that for a given (α, β), then the language of rectangular factors of U is here again the same for every x. We also deduce results concerning the counting of rectangular factors of a given size: there are exaclty $mn + m + n$ factors of size (m, n) in the two-dimensional Sturmian word U. We can not only deduce topological results from (3.2) but also metrical results such as the following: the frequencies of rectangular factors of size (m, n) of a two-dimensional Sturmian word take at most $min(m, n) + 5$ values [BV00].

Let us note that we have chosen in Definition 4 to restrict ourselves to rationally independent parameters. Usually in arithmetic discrete geometry, parameters are chosen to be integers. The results discussed above can also be obtained for standard arithmetic discrete planes $\mathfrak{P}(\mathbf{v}, \mu, \omega)$, whatever is the value taken by $\dim_{\mathbb{Q}}(1, v_1, v_2, v_3)$. Properties deduced from the density and even the equidistribution of the sequence $(n\alpha)_{n\in\mathbb{Q}}$, for α being assumed to be irrational, will be obtained by direct application of Bezout's lemma. For more details, see the complete study performed in [Jam05b].

4. Functionality

Naive planes have been widely studied (e.g., see [Rev91, DRR95, AAS97, VC99, VC00, Jac01, Jac02, BB02, BB04, BCK04, Kis04]) and are well known to be *functional*, i.e., in a one-to-one correspondence with the integer points of one of the coordinate planes by an orthogonal projection map. In other words, given a naive arithmetic discrete plane $\mathfrak{P}$ and the suitable coordinate plane, then for any integer point P of this coordinate plane, there exists a unique point of $\mathfrak{P}$ obtained from P by adding a third coordinate.

The aim of this section is to show how to extend the notion of functionality for naive arithmetic discrete planes to a larger family of arithmetic discrete planes. The results we will discuss are from [BFJ05, BFJP].

Instead of projecting on a coordinate plane, let us introduce a suitable orthogonal projection map on a plane along a direction $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, in some sense dual to the normal vector of the discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$, that is, $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \omega$, so that the projection of $\mathbb{Z}^3$ and the points of the discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ are in one-to-one correspondence.

One interest of the notion of functionality is to reduce a three-dimensional problem to a two-dimensional one, allowing a better understanding of the combinatorial and geometric properties of arithmetic discrete planes: this allows us, first, to recode arithmetic discrete planes by a two-dimensional word over the two-letter alphabet $\{0, 1\}$, and second, to exhibit from this

coding many geometric properties of arithmetic discrete planes (set of local configurations, enumeration of (m,n)-cubes, statistical properties...).

To be more precise, let $\mathfrak{P}(\mathbf{v},\mu,\omega)$ be an arithmetic discrete plane, and let $\boldsymbol{\alpha}\in\mathbb{Z}^3$ be such that $\gcd\{\alpha_1,\alpha_2,\alpha_3\}=1$. Let $\pi_{\boldsymbol{\alpha}}:\mathbb{R}^3\longrightarrow\{\mathbf{x}\in\mathbb{R}^3,\ \langle\boldsymbol{\alpha},\mathbf{x}\rangle=0\}$ be the affine orthogonal projection map onto the plane $\{\mathbf{x}\in\mathbb{R}^3,\ \langle\boldsymbol{\alpha},\mathbf{x}\rangle=0\}$ along the vector $\boldsymbol{\alpha}$. Then the map $\pi_{\boldsymbol{\alpha}}:\mathfrak{P}(\mathbf{v},\mu,\omega)\longrightarrow\pi_{\boldsymbol{\alpha}}(\mathbb{Z}^3)$ is a bijection if, and only if, $|\langle\boldsymbol{\alpha},\mathbf{v}\rangle|=\omega$. Note that for any rational arithmetic discrete plane $\mathfrak{P}(\mathbf{v},\mu,\omega)$, with $\mathbf{v}\in\mathbb{Z}^3$, $\mu\in\mathbb{Z}$, $w\in\mathbb{N}$ and $\gcd\{v_1,v_2,v_3\}=1$, then there exists a vector $\boldsymbol{\alpha}\in\mathbb{Z}^3$ such that $|\langle\boldsymbol{\alpha},\mathbf{v}\rangle|=\omega$, i.e., such that the map $\pi_{\boldsymbol{\alpha}}:\mathfrak{P}(\mathbf{v},\mu,\omega)\longrightarrow\pi_{\boldsymbol{\alpha}}(\mathbb{Z}^3)$ is a bijection. We say that any rational arithmetic discrete plane has *functional* vectors.

Let $\Gamma_{\boldsymbol{\alpha}}$ be the lattice obtained by projecting the arithmetic discrete plane $\mathfrak{P}(\mathbf{v},\mu,\omega)$ on one of the coordinate planes along the functional vector $\boldsymbol{\alpha}$. We will make in all that follows the following assumption: *there exists* $i\in\{1,2,3\}$ *such that* $\alpha_i=1$*, say* $\alpha_3=1$. Under this assumption, then $\Gamma_{\boldsymbol{\alpha}}=\mathbb{Z}\mathbf{e_1}+\mathbb{Z}\mathbf{e_2}$. This hypothesis gives an explicit and simple expression of the preimage of a point in $\Gamma_{\boldsymbol{\alpha}}$: let $\mathfrak{P}=\mathfrak{P}(\mathbf{v},\mu,\omega)$; the map $\pi_{\boldsymbol{\alpha}}^{-1}:\Gamma_{\boldsymbol{\alpha}}\to\mathfrak{P}$, satisfies for all $\mathbf{y}\in\Gamma_{\boldsymbol{\alpha}}$ with $\mathbf{y}=y_1\mathbf{e_1}+y_2\mathbf{e_2}$:

$$\pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{y})=\mathbf{y}-\Big\lfloor\frac{v_1y_1+v_2y_2+\mu}{\omega}\Big\rfloor\boldsymbol{\alpha}.$$

We define the *height* $H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})$ at $\mathbf{y}$ as the third coordinate x_3 of $\mathbf{x}=\pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{y})\in\mathfrak{P}$. One has

$$H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})=-\Big\lfloor\frac{v_1y_1+v_2y_2+\mu}{\omega}\Big\rfloor.$$

Note that, since $\omega=\alpha_1v_1+\alpha_2v_2+\alpha_3\alpha_3$, then the hypothesis $\alpha_3=1$ is equivalent to $\omega\in v_1\mathbb{Z}+v_2\mathbb{Z}+v_3$, i.e., $\omega-v_3\in\gcd(v_1,v_2)\mathbb{Z}$ in the rational case. Also note that there does not always exist a functional vector $\boldsymbol{\alpha}$ with $\alpha_3=1$. Consider for instance the case $\mathbf{v}=(6,10,15)$ with $\omega=20$: it is impossible to express ω as $\alpha_1v_1+\alpha_2v_2+\alpha_3v_3$ with one of the α_i's equal to 1.

4.1. **Local configurations and (m,n)-cubes.** Let us apply now the functionality to (m,n)-cubes and local configurations, generalizing the study performed for rational naive planes in [VC97, Sch97, Gér99, VC99, Jac02]. For the sake of consistency in the notation, we call them here $\mathbf{m}$-cubes with $\mathbf{m}=(m_1,m_2)$ rather than (m,n)-cubes. Our stategy is the folllowing: we recode arithmetic discrete planes according to a two-dimensional word $U\in\{0,1\}^{\mathbb{Z}^2}$ over the two-letter alphabet $\{0,1\}$, namely a so-called *generalized Rote word* [Rot94], following the approach of [Vui99, BV01]. Such a two-dimensional word codes a $\mathbb{Z}^2$-action by two rotations with respect to a partition of the one-dimensional torus into two intervals of length $1/2$. We then express $\mathbf{m}$-cubes as equivalence classes of rectangular factors of the two-dimensional word U, and show, for every $\mathbf{m}\in\mathbb{N}^2$, that the number of $\mathbf{m}$-cubes in $\mathfrak{P}(\mathbf{v},\mu,\omega)$ is computed by enumerating points on the one-dimensional torus.

Let $\mathfrak{P} = \mathfrak{P}(\mathbf{v}, \mu, \omega)$ be an arithmetic discrete plane and let $\boldsymbol{\alpha} \in \mathbb{Z}^3$ such that $\gcd(\boldsymbol{\alpha}) = 1$ and $\langle \boldsymbol{\alpha}, \mathbf{v} \rangle = \omega$. We assume that $\alpha_3 = 1$ in all that follows.

Let $\mathbf{m} \in (\mathbb{N}^\star)^2$ be given. By $\mathbf{m}$-cube we mean a local configuration in the discrete plane that can be observed thanks to $\pi_{\boldsymbol{\alpha}}$ through an $\mathbf{m}$-window in the functional lattice $\Gamma_{\boldsymbol{\alpha}} = \mathbb{Z}\mathbf{e_1} + \mathbb{Z}\mathbf{e_2}$ (see Figure 4.1). More precisely, the $\mathbf{m}$-*cube* $\mathcal{C}(\mathbf{y}, \mathbf{m})$ of $\mathfrak{P}$ is defined as the following subset of $\mathfrak{P}$:

$$\mathcal{C}(\mathbf{y}, \mathbf{m}) = \left\{ \pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{y} + \mathbf{z}), \mathbf{z} \in [\![0, m_1 - 1]\!]\mathbf{e_1} + [\![0, m_2 - 1]\!]\mathbf{e_2} \right\}.$$

Two $\mathbf{m}$-cubes $\mathcal{C}$ and $\mathcal{C}'$ are called *translation equivalent* if there exists a vector $\mathbf{z} \in \mathbb{Z}^3$ such that $\mathcal{C}' = \mathcal{C} + \mathbf{z}$.

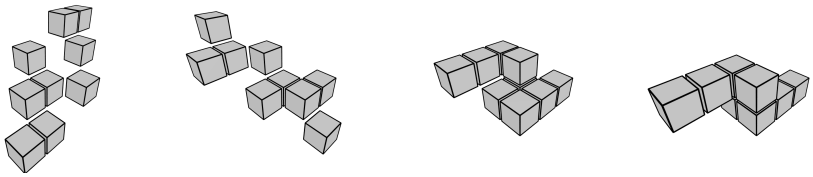


FIGURE 4.1. From left to right: the $(3,3)$-cube of $\mathfrak{P}(\mathbf{v}, 0, 9)$ (resp. $\mathfrak{P}(\mathbf{v}, 0, 11)$, $\mathfrak{P}(\mathbf{v}, 0, 21)$, $\mathfrak{P}(\mathbf{v}, 0, 37)$) centered on $(0,0,0)$, where $\mathbf{v} = 6\mathbf{e_1} + 10\mathbf{e_2} + 15\mathbf{e_3}$, and projected along the vector $-\mathbf{e_1}+\mathbf{e_3}$ (resp. $\mathbf{e_1}-\mathbf{e_2}+\mathbf{e_3}$, $\mathbf{e_1}+\mathbf{e_3}$, $2\mathbf{e_1}+\mathbf{e_2}+\mathbf{e_3}$).

In order to enumerate the different types of $\mathbf{m}$-cubes that occur in $\mathfrak{P}$, that is, the different equivalence classes for the translation equivalence, we represent them as local configurations as follows. An $m_1 \times m_2$-rectangular word $L = [L_{i_1,i_2}]_{(i_1,i_2) \in [\![0,m_1-1]\!] \times [\![0,m_2-1]\!]}$ over the infinite alphabet $\mathbb{Z}$ is called an $\mathbf{m}$-*local configuration* of $\mathfrak{P}$ if there exists $\mathbf{y} \in \mathbb{Z}^2$ such that:

$$L = \left[H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{z}) - H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y}) \right]_{\mathbf{z} \in [\![0,m_1-1]\!]\mathbf{e_1} + [\![0,m_2-1]\!]\mathbf{e_2}}.$$

Such a local configuration is denoted by $LC(\mathbf{y}, \mathbf{m})$.

Let us note that a local configuration is a plane partition. Indeed a *plane partition* of $N \in \mathbb{N}$ is a rectangular word $w = [w_{i_1,i_2}]_{(i_1,i_2) \in [\![0,m_1-1]\!] \times [\![0,m_2-1]\!]}$ over the infinite alphabet $\mathbb{N}$ satisfying $N = \sum_{i,j} w_{i,j}$ and, for all $i_1 \in [\![0, m_1 - 1]\!]$ and $i_2 \in [\![0, m_2 - 1]\!]$, $\max\{w_{i_1+1,i_2}, w_{i_1,i_2+1}\} \leq w_{i_1,i_2}$.

Notation 1. Let $L = [L_{i_1,i_2}]_{(i_1,i_2) \in [\![0,m_1-1]\!] \times [\![0,m_2-1]\!]}$ be a local configuration of size $m_1 \times m_2$. In all that follows, the notation L mod 2 stands for the $m_1 \times m_2$ rectangular word $[L_{i_1,i_2} \bmod 2]_{(i_1,i_2) \in [\![0,m_1-1]\!] \times [\![0,m_2-1]\!]}$.

4.2. **A coding as a two-dimensional word.** According to [Vui99], we introduce a two-dimensional word coding in a natural way the parity of the heights $H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})$, for $\mathbf{y}$ in the lattice $\Gamma_{\boldsymbol{\alpha}} = \mathbb{Z}\mathbf{e_1} + \mathbb{Z}\mathbf{e_2}$. Indeed, for a naive discrete plane $\mathfrak{P}$, it is well known that, given two points $\mathbf{x}$ and $\mathbf{x'}$ of $\mathfrak{P}$ such that their projections by $\pi_{\boldsymbol{\alpha}}$ are 4-connected in the functional plane, then $|x_3 - x'_3| \leq 1$. In other words, the difference between the heights of $\mathbf{x}$ and $\mathbf{x'}$ is at most 1. A quite unexpected fact is that this property holds

for any arithmetic discrete plane with $\alpha_3 = 1$. More precisely, it is easy to see that, for all $\mathbf{y} \in \Gamma_{\boldsymbol{\alpha}}$ and $i = 1, 2$, $H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y} + \mathbf{e_i}) - H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})$ takes only two values, namely $-\lfloor v_i/\omega \rfloor$ and $-\lfloor v_i/\omega \rfloor - 1$. In each case, one of these values is odd, whereas the other one is even; we define E_1 and O_1 to be respectively the even and the odd value taken by $-\lfloor v_1/\omega \rfloor$ and $-\lfloor v_1/\omega \rfloor - 1$; we similarly define E_2 and O_2. It is now natural to introduce the following two-dimensional word of parity of heights by identifying $\Gamma_{\boldsymbol{\alpha}}$ to $\mathbb{Z}^2$:

$$U = (U_{i_1,i_2})_{(i_1,i_2)\in\mathbb{Z}^2} = (H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y}) \mod 2)_{\mathbf{y}\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}. \tag{4.1}$$

The two-dimensional word U satisfies, for each $(i_1, i_2) \in \mathbb{Z}^2$

$$U_{i_1,i_2} = 0 \text{ if, and only if, } v_1 i_1 + v_2 i_2 + \mu \bmod 2\omega \in [0, \omega[.$$

Indeed, one checks that $U_{i_1,i_2} = 0$ if, and only if, $\left\lfloor \frac{v_1 i_1 + v_2 i_2 + \mu}{\omega} \right\rfloor$ is even, that is, $v_1 i_1 + v_2 i_2 + \mu \bmod 2\omega \in [0, \omega[$.

The word U is a *two-dimensional Rote word*; one-dimensional Rote words have been introduced in [Rot94]; they are defined as the infinite words over the alphabet $\{0, 1\}$ that have exactly $2n$ factors of length n for every positive integer n, and whose set of factors is closed under complementation, i.e., every word obtained by interchanging zeros and ones in a factor of the infinite word u is still a factor of u; two-dimensional Rote words have been studied for instance in [Vui99, BV01].

Let $W = [w_{i_1,i_2}]_{(i_1,i_2)\in[\![0,m_1-1]\!]\times[\![0,m_2-1]\!]}$ be a rectangular word of size $m_1 \times m_2$ over $\{0, 1\}$. We define the *complement* $\overline{W}$ of W as follows:

$$\overline{W} = [\overline{w_{i_1,i_2}}]_{(i_1,i_2)\in[\![0,m_1-1]\!]\times[\![0,m_2-1]\!]}, \quad \text{where } \overline{1} = 0 \text{ and } \overline{0} = 1.$$

We introduce the following equivalence relation defined on the set of rectangular factors of U of a given size:

$$V \sim W \text{ if, and only if, } V \in \{W, \overline{W}\}.$$

The following result holds, inspired by [Vui99] where it is stated under the assumption $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$: let $\mathfrak{P} = \mathfrak{P}(\mathbf{v}, \mu, \omega)$ be a discrete plane that admits a functional vector $\boldsymbol{\alpha}$ satisfying $\alpha_3 = 1$; there is a natural bijection between the equivalence classes of the relation $\sim$ on the rectangular factors of the two-dimensional word U of size $\mathbf{m} = (m_1, m_2)$ and the $\mathbf{m}$-local configurations of $\mathfrak{P}$; furthermore, the $\mathbf{m}$-local configurations of $\mathfrak{P}$ are in one-to-one correspondence with the translation equivalence classes of $\mathbf{m}$-cubes of $\mathfrak{P}$.

Lemma 2 plays here the rôle of our key lemma (Lemma 1).

Lemma 2. *Let $W = [w_{i_1,i_2}]_{(i_1,i_2)\in[\![0,m_1-1]\!]\times[\![0,m_2-1]\!]}$ be a rectangular word of size $m_1 \times m_2$ over $\{0, 1\}$. Let $I_0 = [0, \omega[$ and $I_1 = [\omega, 2\omega[$. Let*

$$I_W = \bigcap_{i_1=0}^{m_1-1} \bigcap_{i_2=0}^{m_2-1} \left(I_{w_{i_1,i_2}} - (v_1 i_1 + v_2 i_2) \mod 2\omega \right).$$

The set I_W is a left-closed right-open interval of $[0, 2\omega[$.
Let $\mathfrak{P} = \mathfrak{P}(\mathbf{v}, \mu, \omega)$ be a discrete plane with $\omega - v_3 \in v_1\mathbb{Z} + v_2\mathbb{Z}$. If $\dim_{\mathbb{Q}}(v_1, v_2, v_3) > 1$ or $\mathfrak{P}$ is rational and $\gcd(v_1, v_2, 2\omega) = 1$, then a rectangular word W over $\{0, 1\}$ is a factor of U if, and only if, $I_W \neq \emptyset$. Otherwise, if $\mathfrak{P}$ is rational and $\gcd(v_1, v_2, 2\omega) = 2$, then a rectangular word W over $\{0, 1\}$ is a factor of U if, and only if, I_W contains an integer with the same parity as μ.

4.3. **Enumeration of local configurations.** Let us now investigate the enumeration of $\mathbf{m}$-cubes ($\mathbf{m} = (m_1, m_2)$) occuring in a given arithmetic plane. The number of $(3, 3)$-cubes included in a given rational naive arithmetic discrete plane has been proved to be at most 9 in [VC97]. More generally, in [Rev95, Gér99], the authors proved that, given a rational naive arithmetic discrete plane $\mathfrak{P}$, $\mathfrak{P}$ contains at most m_1m_2 $\mathbf{m}$-cubes (to be more precise, translation equivalence classes of $\mathbf{m}$-cubes). In [Gér99] local configurations which are non-necessarily rectangular are also considered. In the following theorem, we show that this property also holds in our framework. For the sake of simplicity, we omit to mention that we consider translation equivalence classes of $\mathbf{m}$-cubes:

Theorem 2. *Let $\mathfrak{P} = \mathfrak{P}(\mathbf{v}, \mu, \omega)$ be a discrete plane with $\omega - v_3 \in v_1\mathbb{Z} + v_2\mathbb{Z}$. Let $\mathbf{m} = (m_1, m_2) \in (\mathbb{N}^\star)^2$. Then, $\mathfrak{P}$ contains at most m_1m_2 $\mathbf{m}$-cubes. More precisely, one has:*

1. *If $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 1$, $\mathbf{v} \in \mathbb{Z}^3$, $\mu \in \mathbb{Z}$, $\omega \in \mathbb{Z}$ and $\gcd(\mathbf{v}) = 1$, then $\mathfrak{P}$ contains at most w $\mathbf{m}$-cubes for every $\mathbf{m} = (m_1, m_2) \in (\mathbb{N}^\star)^2$. Moreover, for m_1 and m_2 large enough, $\mathfrak{P}$ contains exactly ω $\mathbf{m}$-cubes.*
2. *Let us assume $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 2$. Let $(p_1, p_2) \in \mathbb{Z}^2$ be a generator of the lattice of periods of the two-dimensional word U. Then $\mathfrak{P}$ contains at most $m_1|p_2| + m_2|p_1| - \min\{m_1, |p_1|\}\min\{m_2, |p_2|\}$ $\mathbf{m}$-cubes for $(m_1, m_2) \in \mathbb{N}^2$.*
3. *If $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$, then $\mathfrak{P}$ contains exactly m_1m_2 $\mathbf{m}$-cubes for every $\mathbf{m} = (m_1, m_2) \in (\mathbb{N}^\star)^2$.*

Let us note that the bounds upon which the previous results hold for m_1 and m_2 can be explicitly computed in terms of $\mathbf{v}$ and w. The proof is a direct application of Lemma 2. For more details, see [BFJP].

We thus can establish that, whatever the type of $\mathfrak{P}(\mathbf{v}, \mu, \omega)$, namely rational or irrational, then the computation of the frequency of occurrence of an $\mathbf{m}$-cube of $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ can be reduced to the calculation of the length of an interval on the torus $\mathbb{R}/\omega\mathbb{Z}$. We also investigate in [BFJP] the closure of the set of $\mathbf{m}$-cubes of $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ under the action of a particular geometric transformation: the *centrosymmetry*.

5. Stepped surface

Let us generalize this approach to more general discrete objects, namely the *functional stepped surfaces*, such as introduced in [Jam04]. See also [Jam05a, JP05, Jam05b, ABFJ].

A *functional discrete surface* is defined as a union of pointed faces (defined in Section 3) such that the orthogonal projection π_0 onto the diagonal plane $\Delta\colon x_1+x_2+x_3=0$ induces an homeomorphism from the discrete surface onto the diagonal plane.

As done for functional arithmetic discrete planes, one then provides a discrete surface with a coding as a two-dimensional word over a three-letter alphabet [Jam04, JP05]. Indeed, let $\mathcal{S}$ be a functional stepped surface. One has $\mathcal{S}\cap\mathbb{Z}^3=\pi_0(\{\mathbf{x}+E_i;\ (\mathbf{x}+E_i\subset\mathcal{S}\})$. Furthermore, given $(m_1,m_2)\in\mathbb{Z}^2$, there exists a unique face $\mathbf{x}+E_i\subset\mathcal{S}$ such that $(m_1,m_2)=\pi_0(\mathbf{x}+E_i)$. The following coding is thus well-defined: a two-dimensional word $U\in\{1,2,3\}^{\mathbb{Z}^2}$ is said to be the *coding* of the functional stepped surface $\mathcal{S}$ if for all $(m_1,m_2)\in\mathbb{Z}^2$ and for every $i\in\{1,2,3\}$: $U_{m_1,m_2}=i \iff \exists\mathbf{x}+E_i\in\mathcal{S}$ such that $(m_1,m_2)=\pi_0(\mathbf{x},i^*)$.

We illustrate this with the following figure where a piece of a discrete surface in $\mathbb{R}^3$ is depicted, as well as its orthogonal projection π_0 onto the plane $\Delta\colon x_1+x_2+x_3=0$, and its coding as a two-dimensional word over a three-letter alphabet.

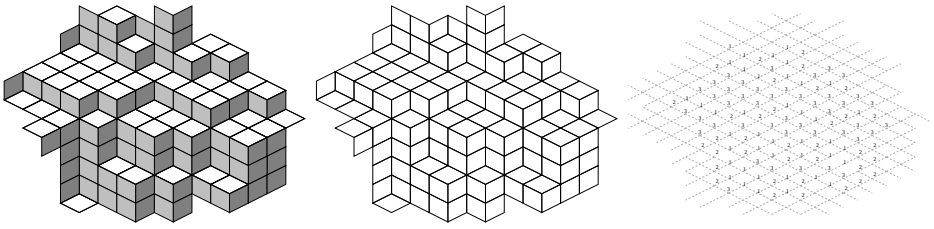


Figure 5.1. From discrete surfaces to multidimensional words via tilings

Let us quote the following nice characterization of codings of discrete surfaces. Let $U\in\{1,2,3\}^{\mathbb{Z}^2}$. Then U is a coding of a discrete surface if and only if the factors of U of the shape given in Fig. 5.2 are included in the following set of factors:

6. From discrete to continuous structures

The aim of this section, based on the surveys [Lot05, BS05, BBLT06], is to show how to generate discrete planes by means of a *generalized substitution*. We work out here in details the example of the so-called *Tribonacci substitution*.

Let $\mathcal{A}$ be a finite set. As usual in word combinatorics, we denote by $\mathcal{A}^*$ the set of words over $\mathcal{A}$ and by ε the empty word. The set $\mathcal{A}^*$ endowed with

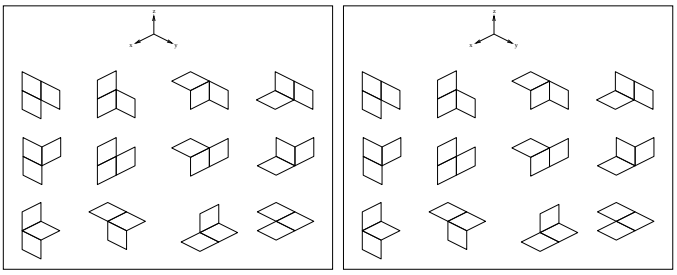


FIGURE 5.2. Permitted factors and their 3-dimensional representation.

the concatenation map is a free monoid. A *substitution* is an endomorphism of the free monoid $\mathcal{A}^*$. A substitution naturally extends to the set of one-sided words $\mathcal{A}^{\mathbb{N}}$. A *fixed point* of σ is a word $u = (u_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ that satisfies $\sigma(u) = u$.

We consider the *Tribonacci substitution* $\sigma : \{1,2,3\}^* \to \{1,2,3\}^*$ defined on the letters of the alphabet $\{1,2,3\}$ as follows: $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$. The *Tribonacci word* is the (unique) fixed point of the substitution σ. More precisely, by noticing that $\sigma^j(1)$ is a nontrivial prefix of the word $\sigma^{j+1}(1)$, the sequence of words $1,\ \sigma(1),\ \sigma^2(1),\ \ldots, \sigma^n(1),\ \ldots$ is easily seen to converge to an infinite word denoted by $\sigma^\omega(1)$. The first terms of this word are

$$1\ 2\ 1\ 3\ 1\ 2\ 1\ 1\ 2\ 1\ 3\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Note that the length, denoted by $|\sigma^j(1)|$, of $\sigma^j(1)$ satisfies the Tribonacci recurrence: $|\sigma^{j+3}(1)| = |\sigma^{j+2}(1)| + |\sigma^{j+1}(1)| + |\sigma^j(1)|$, for every $j \in \mathbb{N}$, hence the terminology.

The Tribonacci substitution has been introduced and studied in [Rau82]. For more results and references on the Tribonacci substitution, see [AR91, AY81, IK91, Lot05, Mes98, Mes00, PF02].

The *incidence matrix* $\mathbf{M}_\sigma = (m_{i,j})_{1\le i,j\le n}$ of a substitution σ has entries $m_{i,j} = |\sigma(j)|_i$, where the notation $|w|_i$ stands for the number of occurrences of the letter i in the word w. A substitution σ is called *primitive* if there exists an integer n such that $\sigma^n(a)$ contains at least one occurrence of the letter b for every pair $(a,b) \in \mathcal{A}^2$. This is equivalent to the fact that its incidence matrix is primitive, i.e., there exists a nonnegative integer n such that $\mathbf{M}_\sigma^n$ has only positive entries.

The *incidence matrix* of the Tribonacci substitution σ is $M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
This matrix is easily seen to be primitive. The characteristic polynomial of M_σ is $X^3 - X^2 - X - 1$; this polynomial admits one positive root $\beta > 1$ (the dominant eigenvalue) and two complex conjugates α and $\overline{\alpha}$, with $|\alpha| < 1$. The number β is a *Pisot number* (that is, an algebraic integer with all Galois conjugates having modulus less than 1).

If a substitution σ is primitive, then the *Perron-Frobenius theorem* ensures that the incidence matrix $\mathbf{M}_\sigma$ has a simple real positive dominant eigenvalue β. A substitution σ is called *unimodular* if $\det \mathbf{M}_\sigma = \pm 1$. A substitution σ is said to be *Pisot* if its incidence matrix $\mathbf{M}_\sigma$ has a real dominant eigenvalue $\beta > 1$ such that, for every other eigenvalue λ, one has $0 < |\lambda| < 1$. The characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$, and the dominant eigenvalue β is a Pisot number. Furthermore, it can be proved that Pisot substitutions are primitive [PF02].

The Tribonacci substitution is Pisot. The incidence matrix M_σ of the Tribonacci substitution admits as eigenspaces in $\mathbb{R}^3$ one *expanding eigenline* (generated by the eigenvector with positive coordinates $\mathbf{v}_\beta = (1/\beta, 1/\beta^2, 1/\beta^3)$ associated with the eigenvalue β) and a *contracting eigenplane* $\mathbb{H}_c$; we denote by $\mathbf{v}_\alpha$ and $\mathbf{v}_{\overline{\alpha}}$ the eigenvectors in $\mathbb{C}^3$ associated with α and $\overline{\alpha}$, normalized in such a way that the sum of their coordinates equals 1.

One associates with the Tribonacci word $u = (u_n)_{n\geq 0}$ a broken line starting from 0 in $\mathbb{Z}^3$ and approximating the expanding line $\mathbf{v}_\beta$ as follows. We introduce the *abelianization map* f of the free monoid $\{1,2,3\}^*$ defined by

$$f : \{1,2,3\}^* \to \mathbb{Z}^3,\ \ f(w) = |w|_1 \mathbf{e}_1 + |w|_2 \mathbf{e}_2 + |w|_3 \mathbf{e}_3,$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ stands for the canonical basis of $\mathbb{R}^3$. Note that for every finite word w, we have $f(\sigma(w)) = \mathbf{M}_\sigma f(w)$.

The *Tribonacci broken line* is defined as the broken line which joins with segments of length 1 the points $f(u_0 u_1 \cdots u_{N-1}),\ N \in \mathbb{N}$ (see Figure 6.1). In other words we describe this broken line by starting from the origin, and then by reading successively the letters of the Tribonacci word u, going one step in direction $\mathbf{e}_i$ if one reads the letter i.

One easily checks that the vectors $f(u_0 u_1 \dots u_N),\ N \in \mathbb{N}$, stay within bounded distance of the expanding line of $\mathbf{M}_\sigma$, which is exactly the direction given by the vector of probabilities of occurrence of the letters $1, 2, 3$ in u. It is then natural to try to represent these points by projecting them along the expanding direction onto a transverse plane, that we chose here to be the contracting plane $\mathbb{H}_c$ of $\mathbf{M}_\sigma$.

Let π stand for the projection in $\mathbb{R}^3$ onto the contracting plane along the expanding line generated by the vector $\mathbf{v}_\beta$. We thud define the set $\mathcal{R}$ as the closure of the projections of the vertices of the Tribonacci broken line:

$$\mathcal{R}_\sigma := \overline{\{\pi(f(u_0 \dots u_{N-1}));\ N \in \mathbb{N}\}}.$$

The set $\mathcal{R}_\sigma$ is called the *Rauzy fractal* associated with the Tribonacci substitution σ (see Figure 6.1). It can be divides into three pieces, called *basic pieces*, defined for $i = 1, 2, 3$ as

$$\mathcal{R}_\sigma(i) = \overline{\{\pi(f(u_0 \dots u_{N-1}));\ u_N = i,\ \ N \in \mathbb{N}\}}.$$

One checks that the the Rauzy fractal is a compact set, that is the closure of its interior; it has a non-zero measure, a fractal boundary and it is the attractor of some graph-directed iterated function system [Rau82].

One interesting feature of Rauzy fractal is that it can tile the plane in two different ways [Rau82, IR06]. These two tilings are depicted in Fig. 6.2. The first one corresponds to a periodic tiling (a lattice tiling), and the second one to a self-replicating tiling. By tiling, we mean here tilings by translation having finitely many tiles up to translation (a tile is assumed to be the closure of its interior): there exists a finite set of tiles $\mathcal{T}_i$ and a finite number of translation sets Γ_i such that $\mathbb{R}^d = \cup_i \cup_{\gamma_i \in \Gamma_i} \mathcal{T}_i + \gamma_i$, and distinct translates of tiles have non-intersecting interiors; we assume furthermore that each compact set in $\mathbb{R}^d$ intersects a finite number of tiles.

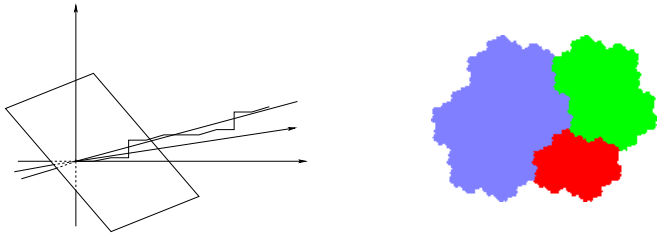


FIGURE 6.1. The Rauzy broken line and the Rauzy fractal.

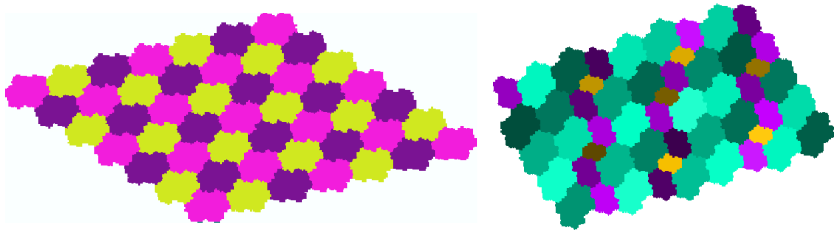


FIGURE 6.2. Lattice and self-replicating Tribonacci tilings.

6.1. **Discrete planes and tilings.** The self-replicating multiple tiling associated with the Rauzy fractal has close connections with arithmetic discrete planes. We consider indeed the standard lower arithmetic discrete plane with parameter $\mu = 0$ associated with $\mathbf{v}_\beta$ that we denote for short by $\mathfrak{P}_\sigma$:

$$\mathfrak{P}_\sigma = \{\mathbf{x} \in \mathbb{Z}^3; \quad 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \sum_{i=1,2,3} \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

We also consider the stepped plane $\mathcal{P}_\sigma$ associated with it, such as defined in Section 3. This discretisation of the contracting hyperplane $\mathbb{H}_c = \{\mathbf{x} \in \mathbb{Z}^3; \ \langle \mathbf{x}, \mathbf{v}_\beta \rangle = 0\}$ consists in approximating the plane $\mathbb{H}_c$ by selecting points with integral coordinates above and within a bounded distance of the plane. It thus can be considered as the dual of the broken line. One checks that the stepped plane $\mathcal{P}_\sigma$ is spanned by:

$$\mathcal{P}_\sigma = \bigcup_{(\mathbf{x},i) \in \mathbb{Z}^3 \times \{1,2,3\}, \ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} (\mathbf{x}, i), \tag{6.1}$$

where for $\mathbf{x} \in \mathbb{Z}^3$ and for $1 \leq i \leq 3$:

$$(\mathbf{x}, i) := \{\mathbf{x} + \sum_{j \neq i} \lambda_j \mathbf{e}_j;\ 0 \leq \lambda_j \leq 1, \quad \text{for } 1 \leq j \leq 3,\ j \neq i\}.$$

This union is a disjoint union up to the boundaries of the faces. Let us note that we have changed our definition of faces with respect to Section 3 ($(\mathbf{x}, i^*)$ versus $\mathbf{x} + E_i$). This change of notation will be more convenient for the following.

Let us project now the stepped plane $\mathcal{P}_\sigma$ on the contracting space $\mathbb{H}_c$ and replace each face $(\mathbf{x}, i)$ by the corresponding basic piece of the Rauzy fractal $\mathcal{R}_\sigma(i)$. The tiling (6.1) becomes

$$\mathbb{H}_c = \bigcup_{(\mathbf{x},i) \in \mathbb{Z}^3 \times \{1,2,3\},\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i). \tag{6.2}$$

According to [Rau82] and [IR06], (6.2) provides a tiling of the contracting plane $\mathbb{H}_c$, namely the self-substitutive tiling depicted in Figure 6.2. The terminology self-substitutive comes from the fact that it can be generated thanks to a graph-directed iterated function system given by the substitution σ.

Let us give now a construction of $\mathfrak{P}_\sigma$ based on the notion of a geometric generalized substitution due to [AI01], see also [IR06].

We define $\mathcal{F}^*$ as the $\mathbb{R}$-vector space generated by $\{(\mathbf{x}, i);\ \mathbf{x} \in \mathbb{Z}^3,\ i \in \{1, 2, 3\}\}$. We define the following generation process which can be considered as a geometric realization of the substitution σ on the geometric set $\mathcal{F}^*$ consisting of finite sums of faces:

$$\forall (\mathbf{x}, a) \in \mathbb{Z}^n \times \{1, 2, 3\},\ E_1^*(\sigma)(\mathbf{x}, a) = \sum_{\sigma(b) = pas} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{l}(p)), b).$$

Theorem 3. [AI01] *The stepped plane $\mathcal{P}_\sigma$ is stable under the action of $E_1(\sigma)^*$ and contains the unit cube*

$$\mathcal{U} := (\mathbf{0}, 1) + (\mathbf{0}, \mathbf{2} + (\mathbf{0}, 3).$$

The iterates $(E_1(\sigma)^*)^n(\mathcal{U})$ all belong to $\mathcal{P}_\sigma$, and they generate larger and larger pieces of the stepped plane $\mathcal{P}_\sigma$. By taking the limit and by projecting by π, one gets

$$\mathcal{P}_\sigma = \lim_{n \to +\infty} \pi(E_1(\sigma)^*)^n(\mathcal{U}).$$

Let us recall that $\mathfrak{P}_\sigma$ is a discrete approximation of the contracting plane of the incidence matrix $\mathbf{M}_\sigma$. After projection and renormalization, the pieces $\mathbf{M}_\sigma^n \pi(E_1(\sigma)^*)^n(\mathcal{U})$ converge and their limit is equal to the Rauzy fractal:

$$\mathcal{R}_\sigma = \lim_{n \to +\infty} \mathbf{M}_\sigma^n \pi(E_1(\sigma)^*)^n(\mathcal{U}).$$

For more on generalized substitutions and generation of discrete planes, see [ABI02, ABS04, ABFJ, Fer].

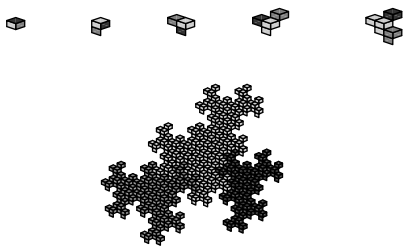


FIGURE 6.3. Generation of $\mathcal{P}_\sigma$ by iterates of $E_1^*(\sigma)$.

6.2. **Rauzy tilings.** We have seen that two tilings can be associated with the rauzy Fractal, namely, a self-substitutive tiling, and a lattice tiling, as illustrated in Figure 6.2. More precisely, one has in the lattice case

$$\mathbb{R}^2 = \cup_{\gamma\in\Gamma}(\mathcal{R}_\sigma + \gamma),$$

the union being disjoint in measure, and

$$\Gamma = \mathbb{Z}(\delta_\sigma(1) - \delta_\sigma(3)) + \mathbb{Z}(\delta_\sigma(2) - \delta_\sigma(3)).$$

This latter tiling plays an important rôle in the spectral study of the substitutive dynamical system (X_σ, S) generated by the Tribonacci word (such as defined in Section 2). Indeed, one of the main incentives behind the introduction of Rauzy fractals is the following result:

Theorem 4 ([Rau82])**.** *Let σ be the Tribonacci substitution $\sigma\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$. The Rauzy fractal $\mathcal{R}_\sigma$ (considered as a subset of $\mathbb{R}^2$) is a fundamental domain of $\mathbb{T}^2$. Let $R_\beta\colon \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto x + (1/\beta, 1/\beta^2)$. The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the toral translation $(\mathbb{T}^2, R_\beta)$.*

The Tribonacci lattice tiling has been widely studied and presents many interesting features. In particular, the Tribonacci central tile has a "nice" topological behavior (0 is an inner point and it is shown to be connected with simply connected interior [Rau82]), which leads to interesting applications in Diophantine approximation [CHM01] where points of the broken line corresponding to $\sigma^n(1)$, $n \in \mathbb{N}$, are proved to produce best approximations for the vector $(\frac{1}{\beta}, \frac{1}{\beta^2})$ for a given norm associated with the matrix $\mathbf{M}_\sigma$. See also [HM06] for the case of cubic Pisot numbers with complex conjugates satisfying the finiteness property (F).

Rauzy fractals can more generally be associated with Pisot substitutions (see [BK06, CS01a, CS01b, IR06, Mes00, Mes02, Sie03, Sie04] and the surveys [BS05, PF02]), as well as with Pisot β-shifts under the name of *central tiles* (see [Aki98, Aki99, Aki00, Aki02]), but they also can be associated with abstract numeration systems [BR05], as well as with some automorphisms of the free group [ABHS06].

Conjecture 1. Let σ be a Pisot unimodular substitution. The following equivalent conditions are conjectured to hold:

(1) the symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to a translation on the torus;
(2) (X_σ, S) has a pure discrete spectrum;
(3) the associated Rauzy fractal $\mathcal{R}_\sigma$ generates a lattice tiling, i.e.,

$$\mathbb{K}_\beta = \cup_{\gamma\in\Gamma}(\mathcal{R}_\sigma + \gamma),$$

the union being disjoint in measure, and $\Gamma = \sum_{b\in\mathcal{A},b\neq a} \mathbb{Z}(\delta_\sigma(b) - \delta_\sigma(a))$, for $a \in \mathcal{A}$.

The conjecture holds true for two-letter alphabets [BD02, HS03, Hos92]. Substantial literature is devoted to Conjecture 1 which is reviewed in [PF02], Chap.7. See also [BK06, BK05, BBK06, BS05, IR06] for recent results.

7. Conclusion

Let us conclude by giving a brief list of geometric discretizations that can be descirbed by symbolic codings of dynamical systems.

• Standard arithmetic discrete lines and Sturmian words are particular codings of rotations over the one-dimensional torus $\mathbb{T}$ with respect to a two-interval partition, one interval having as length the parameter of the rotation.

• Similarly, standard arithmetic discrete planes and two-dimensional Sturmian words are codings of a $\mathbb{Z}^2$-action by rotations over the one-dimensional torus $\mathbb{T}$ with respect to a three-interval partition, two intervals having as respective length the parameters of the $\mathbb{Z}^2$-action.

• More generally, functional arithmetic disrete planes can be coded thanks to generalized Rote words defined as codings of a $\mathbb{Z}^2$-action by rotations over the one-dimensional torus $\mathbb{T}$ with respect to a two-interval partition, with two intervals of the same length. For more examples of codings associated with naive or standard arithmetic discrete planes expressed in terms of dynamical systems, see [Jam05b] where codings by remainders, by umbrellas and by parity of heights are considered.

• In a dual way, we have seen how to associate with the Tribonacci substitution a broken line that can be considered as a discrete line in $\mathbb{R}^3$. A lattice tiling by the Rauzy fractal can then be produced that has close connection with a rotation on the two-dimensional torus $\mathbb{T}^2$.

• Lastly, let us quote [BN] as an example of a symbolic coding of discrete rotations defined as the composition of Euclidean rotations with a rounding operation, as studied in [NR03, NR04, NR05]. Indeed, it is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α called configurations. These configurations C_α and C'_α can be coded by discrete dynamical systems defined by a $\mathbb{Z}^2$-action on the

two-dimensional torus $\mathbb{T}^2$. As a consequence, results concerning densities of occurrence of symbols can be deduced.

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LIRMM - CNRS UMR 5506-161 rue Ada -34392 Montpellier Cedex 5- France
E-mail address: berthe@lirmm.fr