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Alain Jean-Marie
INRIA and
LIRMM, Univ. Montpellier 2
161 Rue Ada
34392 Montpellier Cedex 05

Mabel Tidball
INRA
LAMETA
2 Place Viala
34360 Montpellier

France

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Abstract

This paper provides a set of necessary conditions for the existence of credible incentive equilibria. We study the cases of static games and dynamic games with open-loop strategies. We conclude that credible incentive equilibria with differentiable incentive functions do not exist without strong conditions on the payoff functions of the players. On the other hand, for piecewise-differentiable incentive functions, an infinity of solutions is usually possible.

1 Introduction

The topic of this paper is a class of constrained equilibria in games, known as "Incentive Equilibria". The principle of incentive equilibria has been developed for dynamic games by Ehtamo and Hämäläinen [1, 2], inspired from the work of Osborne [9] about the definition of a "quota rule" able to explain the stability of a Cartel. The concept has since been used for several applications in the Management of Natural Resources or in Marketing, by Ehtamo and Hämäläinen [3, 4], in discrete-time as well as in continuous-time, by Jørgensen and Zaccour in continuous-time games [5, 6, 7], and recently by Martín-Herrán and Zaccour [8].

The general reason for studying incentives is the wish to construct a game in which players are induced to cooperate. This is done by defining a desired outcome E^* (normally, a Pareto solution) upon which all players agree, and a reaction rule. Each player is assumed to retaliate to a deviation of the opponent, with respect to the agreed outcome, by applying this rule. This reaction rule is called the incentive function. If it is well chosen, the optimal behavior for player i is to play E_i^*, if she believes that her opponent implements her incentive function. This converts E^* into a (constrained) equilibrium. The issue is then to provide reasons justifying why a player would indeed chose to play according to her incentive function, when observing a deviation from the agreed outcome. A necessary condition for this, called "credibility", is used in the literature.

Credibility holds if every player, if faced with a deviation from her opponent, would prefer to follow the incentive rather than sticking to her equilibrium value. Credibility is a minimum requirement for one to expect that the incentive design will be followed by the players. It is not asked that each player's *optimal* behavior is to apply the incentive, but just that they do not lose by doing so, as compared to letting the opponent deviate without doing anything.

The literature on incentive equilibria provides a number of game situations where incentive equilibria are computed, often on a *ad hoc* basis. The issue of credibility is less often discussed in details. The purpose of this paper is to provide general first-order necessary conditions bearing on the payoff functions and the incentive functions, under which the existence of credible equilibria is possible, subject to appropriate second-order conditions.

This turns out to be possible, due to the fact that conditions for the existence of incentive equilibria, and for credibility, can be expressed as optimization problems. The first-order conditions of these problems provide conditions for the existence of credible incentive equilibria, for static games and for dynamic games with open-loop strategies.

It turns out that considering *differentiable* and credible incentive functions implies very strong conditions on profit functions. Since they are not usually met in practice, no such credible equilibria can be constructed. In order to circumvent this problem, we have considered two possibilities. The first one is to remove the differentiability condition [9, 3]. In this case, a multiplicity of solutions is indeed possible (Corollary 3.1). The second idea is to remove the requirement that the equilibrium point should be a Pareto optimum. In that case, in particular, the Nash equilibrium with constant incentive functions is a (weak) credible incentive equilibrium (Theorem 3.1).

The paper is organized as follows. We begin with the definition of incentive equilibria and credibility, in Section 2. Then we study the case of static games in Section 3. The results are then generalized to dynamic games with open-loop strategies, in Section 4. We finish the paper with an application in Section 5, and conclusions in Section 6.

2 Definitions

Consider a two-player game. The strategy of player i will be denoted by E_i. This strategy belongs to a suitable strategy space Σ_i. The payoff function of player i is a mapping:

$$J_i : \Sigma_1 \times \Sigma_2 \to \mathbb{R} .$$

Definition 2.1 (Incentive equilibrium). *Consider a Pareto optimum (E_1^*, E_2^*) of the game. An incentive equilibrium strategy at this optimum is a pair of mappings (Ψ_1, Ψ_2), with $\Psi_1 : \Sigma_2 \to \Sigma_1$, $\Psi_2 : \Sigma_1 \to \Sigma_2$, and such that:*

$$\begin{aligned} J_1(E_1, \Psi_2(E_1)) &\leq J_1(E_1^*, \Psi_2(E_1^*)) \qquad \forall E_1 \in \Sigma_1 && (1)\\ J_2(\Psi_1(E_2), E_2) &\leq J_2(\Psi_1(E_2^*), E_2^*) \qquad \forall E_2 \in \Sigma_2 && (2)\\ \Psi_1(E_2^*) = E_1^* &\qquad \Psi_2(E_1^*) = E_2^* . && (3) \end{aligned}$$

Definition 2.2 (Credible incentive equilibrium). *The pair (Ψ_1, Ψ_2) is a credible incentive equilibrium at (E_1^*, E_2^*) if it is an incentive equilibrium, and* *if there exists a subset* $\Sigma_1' \times \Sigma_2'$ *of* $\Sigma_1 \times \Sigma_2$ *such that:*

$$\begin{aligned} J_1(\Psi_1(E_2), E_2) &\geq J_1(E_1^*, E_2) , &(4) \\ J_2(E_1, \Psi_2(E_1)) &\geq J_2(E_1, E_2^*) , &(5) \end{aligned}$$

for all $E_1 \in \Sigma_1'$ *and* $E_2 \in \Sigma_2'$.

Observe that this definition depends on some subset $\Sigma' = \Sigma_1' \times \Sigma_2'$ of strategies. The ideal situation would be that the credibility conditions hold for *any* possible deviation. Asking for such a *global* credibility seems to be a very strong requirement, and most examples proposed in the literature are consistent with respect to a subset of all possible deviations. Another possibility would have been to define a *locally* credible equilibrium, by requiring that the subset Σ' be a *neighborhood* of the equilibrium, involving all possible "small" deviations from the equilibrium strategy. This is our approach for the case of static games, in Section 3. For dynamic games however, we follow the usage of the literature and we consider deviations of a particular kind, for which the term "neighborhood" is not appropriate. This way, we obtain a rather weak notion of credibility, but we avoid the complexity associated with using a topology on the space of open-loop strategies. In any cases, an interesting problem is to find, for some proposed equilibrium, the largest set of deviations for which this equilibrium is credible.

We also introduce a weaker notion of credibility, in which the principle of incentives is conserved, but where Pareto-optimality of the equilibrium is not required:

Definition 2.3 (Weak credible incentive equilibrium). *A weak credible incentive equilibrium is a couple of strategies* (E_1^*, E_2^*) *and of incentives* (Ψ_1, Ψ_2) *such that there exists a subset* $\Sigma_1' \times \Sigma_2'$ *of* $\Sigma_1 \times \Sigma_2$ *such that* (1)–(5) *are satisfied, for all* $E_1 \in \Sigma_1'$ *and* $E_2 \in \Sigma_2'$.

When there is a risk of confusion, we shall call an incentive equilibrium according to Definition 2.1 a "strong" incentive equilibrium.

3 The static case

We consider in this section the case of two-person static games, in which the strategy space of both players is an open subset of $\mathbb{R}$. Our objective is to obtain necessary conditions for the existence of credible incentive equilibria, in the sense defined in Definitions 2.2 and 2.3, but with the stronger requirement that the subset $\Sigma_1' \times \Sigma_2'$ be a neighborhood of the equilibrium.

For reasons that will become clear at the end of this section (Theorem 3.2), the proper setting for studying credibility is that of incentive functions with the following features: they are continuous, piecewise differentiable (usually, only piecewise affine functions are considered), but they are not necessarily differentiable at the incentive equilibrium point $E^* = (E_1^*, E_2^*)$.

Accordingly, we shall assume that:

$$\Psi_i(E_j) = \begin{cases} \Psi_i^+(E_j) & \text{if } E_j \geq E_j^* \\ \Psi_i^-(E_j) & \text{if } E_j \leq E_j^* , \end{cases}$$

where Ψ_i^+ and Ψ_i^- are functions that are differentiable, including at $E_j = E_j^*$. We shall denote:

$$a_i^+ \ = \ (\Psi_i^+)'(E_j^*) \ , \quad \text{and} \quad a_i^- \ = \ (\Psi_i^-)'(E_j^*) \ .$$

Throughout this section, the payoff functions J_i will be assumed to be differentiable, with continuous partial derivatives. We shall denote in addition

$$A_i \ = \ - \ \frac{\partial J_i / \partial E_i}{\partial J_i / \partial E_j}(E_1^*, E_2^*) \ , \tag{6}$$

when the denominator is not zero.

3.1 Necessary conditions for equilibria

We first begin by considering weak incentive equilibria. We state a lemma characterizing the possible values of the derivatives $a_i^\pm$ of the credible incentive functions, according to the sign of the partial derivatives $\partial J_i / \partial E_j$. Next, we specialize this result to the case of (strong) incentive equilibria, that is, when E^* is a Pareto optimum (Corollary 3.1). Finally, we discuss the case where the incentive function is required to be differentiable at the equilibrium point (Corollary 3.2).

Lemma 3.1. *Let (E_1^*, E_2^*) and (Ψ_1, Ψ_2) be a weak credible incentive equilibrium. Then it is necessary that one of the following cases hold for $a_1^\pm$, the left and right derivatives of Ψ_1, evaluated at (E_1^*, E_2^*), and for A_2 defined by (6):*

a/ Either $\partial J_1/\partial E_1 < 0$ and $\partial J_2/\partial E_1 < 0$, or $\partial J_1/\partial E_1 > 0$ and $\partial J_2/\partial E_1 > 0$. Then necessarily $A_2 = 0$ and $a_1^+ = a_1^- = 0$.

b/ $\partial J_1/\partial E_1 < 0$ and $\partial J_2/\partial E_1 > 0$. Then

$$a_1^+ \leq \min(A_2, 0) \leq \max(A_2, 0) \leq a_1^- \ .$$

c/ $\partial J_1/\partial E_1 > 0$ and $\partial J_2/\partial E_1 < 0$. Then

$$a_1^- \leq \min(A_2, 0) \leq \max(A_2, 0) \leq a_1^+ \ .$$

d/ $\partial J_1/\partial E_1 = 0$ and $\partial J_2/\partial E_1 < 0$. Then

$$a_1^- \leq A_2 \leq a_1^+ \ .$$

e/ $\partial J_1/\partial E_1 = 0$ and $\partial J_2/\partial E_1 > 0$. Then

$$a_1^+ \leq A_2 \leq a_1^- \ .$$

f/ $\partial J_1/\partial E_1 > 0$ and $\partial J_2/\partial E_1 = 0$. Then necessarily $A_2 = 0$, and

$$a_1^- \leq 0 \leq a_1^+ \ .$$

g/ $\partial J_1/\partial E_1 < 0$ and $\partial J_2/\partial E_1 = 0$. Then necessarily $A_2 = 0$, and

$$a_1^+ \leq 0 \leq a_1^- \ .$$

h/ $\partial J_1/\partial E_1 = \partial J_2/\partial E_1 = 0$. *Then necessarily* $A_2 = 0$*, but all values for* a_1^+ *and* a_1^- *are allowed.*

A symmetric classification exists for Ψ_2*.*

Proof. First, consider Equation (2) in the form:

$$J_2(\Psi_1(E_2), E_2) \; - \; J_2(\Psi_1(E_2^*), E_2^*) \; \leq \; 0 \; , \qquad \forall E_2 \; .$$

Since equality obtains for $E_2 = E_2^*$ (Ψ_1 satisfies (3)), this is equivalent to writing:

$$E_2^* \; \in \; \mathrm{argmax}_{E_2}\{J_2(\Psi_1(E_2), E_2) \; - \; J_2(\Psi_1(E_2^*), E_2^*)\} \; . \tag{7}$$

Consider the case $E_2 \geq E_2^*$. The first order condition of this constrained maximization problem is

$$\frac{\partial J_2}{\partial E_2}(E_1^*, E_2^*) \; + \; a_1^+ \; \frac{\partial J_2}{\partial E_1}(E_1^*, E_2^*) \; \leq \; 0 \; . \tag{8}$$

In other words, the function is necessarily nonincreasing locally for $E_2 \geq E_2^*$. Next, consider Inequality (4), the credibility condition for player 1, in the form:

$$J_1(\Psi_1(E_2), E_2) \; - \; J_1(E_1^*, E_2) \; \geq \; 0 \; ,$$

for all E_2 in a neighborhood of E_2^*. Since equality obtains with $E_2 = E_2^*$, E_2^* is solution of the associated minimization problem. Assume again $E_2 \geq E_2^*$. The first order condition of this constrained optimization problem is:

$$\begin{aligned} 0 \; &\leq \; a_1^+ \; \frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \; + \; \frac{\partial J_1}{\partial E_2}(E_1^*, E_2^*) \; - \; \frac{\partial J_1}{\partial E_2}(E_1^*, E_2^*) \\ &= \; a_1^+ \; \frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \; . \end{aligned}$$

Consider for instance case *c/*. Since $\partial J_1/\partial E_1 > 0$, this last condition reduces to: $a_1^+ \geq 0$. On the other hand, since $\partial J_2/\partial E_1 < 0$, Condition (8) is equivalent to $a_1^+ \geq A_2$. Combining the two constraints, we obtain:

$$a_1^+ \; \geq \; \max(A_2, 0) \; .$$

Consider now the case where $E_2 \leq E_2^*$. The first order condition for the optimization problem (7) under this constraint is:

$$\frac{\partial J_2}{\partial E_2}(E_1^*, E_2^*) \; + \; a_1^- \; \frac{\partial J_2}{\partial E_1}(E_1^*, E_2^*) \; \geq \; 0 \; . \tag{9}$$

Similarly, the credibility condition for $E_2 \leq E_2^*$ becomes:

$$0 \; \geq \; a_1^- \; \frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \; .$$

Still for case *c/*, these two constraints can be summarized as: $a_1 \leq \min(A_2, 0)$. Hence the conclusion for this case. The other cases are easily obtained by inspection. □

Observe that we have not imposed restrictions *a priori* on the sign of the derivatives of the functions J_i in order to be as exhaustive as possible. In usual applications in Economics, the function J_i will be increasing with respect to E_i, in which case only cases $a/$, $c/$ or perhaps $f/$ are relevant.

Remark 3.1. The proof of Lemma 3.1 can be adapted to provide a *sufficient* condition for a pair of functions Ψ_i to be an equilibrium: assume that in case $c/$, the inequalities constraining $a_1^{\pm}$ hold in the strict sense. Then the function Ψ_1 is a weak incentive equilibrium. Indeed, under this condition, the function to be maximized in (7) is *strictly* increasing for $E_2 \nearrow E_2^*$, and strictly decreasing for $E_2 \searrow E_2^*$. This implies that E_2^* is a local maximum. Similarly for the minimization problem which represents credibility.

The first corollary of Lemma 3.1 is a set of necessary conditions for the existence of a *strong* credible incentive equilibrium.

Corollary 3.1. *Let (E_1^*, E_2^*) and (Ψ_1, Ψ_2) be a credible incentive equilibrium. Then, necessarily, one of the following five cases must hold:*

a/ $\partial J_1/\partial E_1 > 0$, $\partial J_1/\partial E_2 < 0$, $\partial J_2/\partial E_1 < 0$ *and* $\partial J_2/\partial E_2 > 0$. *Then*

$$a_1^- \leq 0 \leq A_2 \leq a_1^+, \quad \textit{and} \quad a_2^- \leq 0 \leq A_1 \leq a_2^+ .$$

b/ $\partial J_1/\partial E_1 < 0$, $\partial J_1/\partial E_2 > 0$, $\partial J_2/\partial E_1 > 0$ *and* $\partial J_2/\partial E_2 < 0$. *Then*

$$a_1^+ \leq 0 \leq A_2 \leq a_1^-, \quad \textit{and} \quad a_2^+ \leq 0 \leq A_1 \leq a_2^- .$$

c/ $\partial J_1/\partial E_1 > 0$, $\partial J_1/\partial E_2 > 0$, $\partial J_2/\partial E_1 < 0$ *and* $\partial J_2/\partial E_2 < 0$. *Then*

$$a_1^- \leq A_2 \leq 0 \leq a_1^+, \quad \textit{and} \quad a_2^+ \leq A_1 \leq 0 \leq a_2^- .$$

d/ $\partial J_1/\partial E_1 < 0$, $\partial J_1/\partial E_2 < 0$, $\partial J_2/\partial E_1 > 0$ *and* $\partial J_2/\partial E_2 > 0$. *Then*

$$a_1^+ \leq A_2 \leq 0 \leq a_1^-, \quad \textit{and} \quad a_2^- \leq A_1 \leq 0 \leq a_2^+ .$$

e/ $\partial J_1/\partial E_1 = \partial J_1/\partial E_2 = \partial J_2/\partial E_1 = \partial J_2/\partial E_2 = 0$. *Then all values for $a_i^{\pm}$ are allowed.*

Proof. Consider the maximization problem for a Pareto optimum of the game: for some $\alpha \in (0, 1)$,

$$(E_1^*, E_2^*) \;=\; \mathrm{argmax}_{(E_1,E_2)}\{\alpha J_1(E_1, E_2) \;+\; (1-\alpha) J_2(E_1, E_2)\} .$$

The first order conditions for this optimum are:

$$\alpha \frac{\partial J_i}{\partial E_i}(E_1^*, E_2^*) \;+\; (1-\alpha)\frac{\partial J_j}{\partial E_i}(E_1^*, E_2^*) \;=\; 0, \qquad i = 1, 2,\ j \neq i . \tag{10}$$

Assume first that none of the four partial derivatives $\partial J_i/\partial E_j$ is zero. The conditions above is equivalent to:

$$\frac{\partial J_1/\partial E_1}{\partial J_2/\partial E_1} \;=\; \frac{\partial J_2/\partial E_1}{\partial J_2/\partial E_2} \;=\; \frac{\alpha - 1}{\alpha} .$$

This in turn implies that:

$$A_1 A_2 \; = \; \frac{\partial J_1/\partial E_1}{\partial J_1/\partial E_2} \, \frac{\partial J_2/\partial E_2}{\partial J_2/\partial E_1} \; = \; \frac{\partial J_1/\partial E_1}{\partial J_2/\partial E_1} \, \frac{\partial J_2/\partial E_2}{\partial J_1/\partial E_2} \; = \; \frac{\alpha - 1}{\alpha} \, \frac{\alpha}{\alpha - 1} \; = \; 1 \, . \tag{11}$$

These two equalities impose sign constraints on the partial derivatives so that only the four combinations listed in cases $a/$ to $d/$ are possible. For each of them, the constraints on $a_i^{\pm}$ and A_j are obtained from Lemma 3.1. For instance, case $b/$ is the combination of cases $b/$ of this Lemma, for both players 1 and 2.

Next, suppose that one of the four derivatives, say $\partial J_1/\partial E_1$, is zero. Then, because of (10), $\partial J_2/\partial E_1$ is null as well. According to Lemma 3.1, cases $f/$, $g/$ or $h/$, then $A_2 = 0$ necessarily, that is: $\partial J_2/\partial E_2 = 0$. Still because of (10), the fourth partial derivative $\partial J_2/\partial E_1$ is zero. It follows, that the relevant case of Lemma 3.1 is case $h/$: all values of $a_i^{\pm}$ are possible. The conclusion holds if any other $\partial J_i/\partial E_j$ is assumed to be zero initially. $\square$

Remark 3.2. In all the cases listed in Corollary 3.1, there exists an infinity of possibilities for $a_i^{\pm}$. Consider for instance case $a/$. According to Remark 3.1, if the $a_i^{\pm}$ are chosen such that inequalities are strict: $a_1^- < 0$, $A_2 < a_1^+$, $a_2^- < 0$ and $A_1 < a_2^+$, then the equilibrium indeed credible, whatever the exact form of the functions Ψ_i, provided that their derivatives at E^* are $a_i^{\pm}$. Similarly for the other cases.

On the other hand, all examples in the cited literature select the particular values $a_i^+ = A_j$ or $a_i^- = A_j$, according to the situation. This choice has the interesting consequence that

$$a_i \, a_j \; = \; 1,$$

which derives from the identity (11). However, there is no guarantee that this choice will ensure credibility, without additional assumptions on payoff functions.

The second corollary of Lemma 3.1 is a set of necessary conditions for the existence of a weak incentive equilibrium with *differentiable* incentive functions.

Corollary 3.2. *Let* (E_1^*, E_2^*) *and* (Ψ_1, Ψ_2) *be a weak credible incentive equilibrium, with incentive functions that are differentiable. Then, denoting* $a_i = (\Psi_i')(E_j^*)$*, we have:*

a/ if $a_1 = 0$ *and* $a_2 = 0$*, then necessarily*

$$\frac{\partial J_i}{\partial E_i}(E_1^*, E_2^*) \; = \; 0 \; , \qquad i = 1, 2 \; ;$$

b/ if $a_1 \neq 0$ *and* $a_2 \neq 0$*, then necessarily*

$$\frac{\partial J_i}{\partial E_j}(E_1^*, E_2^*) \; = \; 0 \; , \qquad i, j = 1, 2 \; ;$$

c/ if $a_1 = 0$ *and* $a_2 \neq 0$*, then necessarily*

$$\frac{\partial J_2}{\partial E_2}(E_1^*, E_2^*) \; = \; 0 \; , \qquad \frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \; + \; a_2 \, \frac{\partial J_1}{\partial E_2}(E_1^*, E_2^*) \; = \; 0 \; ;$$

d/ if $a_1 \neq 0$ and $a_2 = 0$, then necessarily

$$\frac{\partial J_1}{\partial E_1}(E_1^*, E_2^*) \;=\; 0 \;, \qquad \frac{\partial J_2}{\partial E_1}(E_1^*, E_2^*)\; a_1 \;+\; \frac{\partial J_2}{\partial E_2}(E_1^*, E_2^*) \;=\; 0 \;.$$

Proof. If the incentive function is differentiable, then $a_i^+ = a_i^- = a_i$. Imposing this condition to the different cases of Lemma 3.1 gives the result. □

3.2 Applications

3.2.1 Nash equilibria

One recognizes in case $a/$ of Corollary 3.2 the first-order condition for a Nash equilibrium. In this case, the derivatives of both incentive functions at the equilibrium vanish. This suggests that among candidate weak credible incentive equilibria, one finds Nash equilibria with constant incentives. Indeed, we have:

Theorem 3.1. *A Nash equilibrium (E_1^*, E_2^*) is a weak credible incentive equilibrium for the (constant) incentive functions: $\Psi_i(E_j) = E_i^*$.*

Proof. A Nash equilibrium (E_1^*, E_2^*) is defined by Equations (1) and (2) with $\Psi_i(E_j)$ replaced by E_i^*. The constant incentives $\Psi_i(E_j) = E_i^*$, evidently satisfy Equations (3). According to the previous lemma, the credibility conditions are also satisfied. □

3.2.2 Differentiable incentive equilibria

We now state the result that the existence of a credible incentive equilibrium, with *differentiable* functions, requires strong properties on the payoff functions.

Theorem 3.2. *Let (Ψ_1, Ψ_2) be a credible incentive equilibrium at a Pareto optimum, where the incentive functions Ψ_i are differentiable. Then, necessarily:*

$$\frac{\partial J_i}{\partial E_j}(E_1^*, E_2^*) \;=\; 0 \;, \qquad i, j = 1, 2 \;.$$

Proof. If (E_1^*, E_2^*) is a Pareto optimum of the game, then Conditions (10) hold. Joining these conditions to either of the four cases of Corollary 3.2 yields the result. □

Remark 3.3. The condition stated in Theorem 3.2 is equivalent to saying that E^* is the simultaneous critical points of both payoff functions. Conversely, if a simultaneous maximum exists, then it is an incentive equilibrium, for any incentive functions Ψ_i. Indeed, for any pair of functions Ψ_i which satisfy Equation (3), then Inequalities (1) and (2) hold.

For credibility however, although Corollary 3.1 does not impose any condition on the derivatives of Ψ_i, all incentive functions are not credible. Consider for instance the payoff function $J_1(E_1, E_2) = -(E_1 + E_2)^2 - E_1^2$. Obviously, $E^* = (0, 0)$ is the unique global maximum. Assume player 1 has the incentive function Ψ_1. Then the credibility condition (4) writes as:

$$0 \;\leq\; -\left(\Psi_1(E_2) + E_2\right)^2 - \Psi_1(E_2) + E_2^2 \;=\; -\,2\Psi_1(E_2)(\Psi_1(E_2) + E_2) \;.$$

Therefore, only incentive functions such that $0 \leq \Psi_1(E_2) \leq -E_2$ or $-E_2 \leq \Psi_1(E_2) \leq 0$ are credible.

The practical consequence of Theorem 3.2 is that no *credible* incentive equilibria with *differentiable* functions can exist, *unless* both payoff functions have the (rare) feature to possess a common global maximum. In that case, a second-order analysis is necessary to determine which incentive functions are credible.

3.2.3 Osborne's example

The example developed in [9] is that of a static oligopoly. In this case, the strategies E_i are the level of production of the firms, and $J_i(\cdot)$ are their profit functions. They have the property that $\partial J_i/\partial E_i > 0$ and $\partial J_i/\partial E_j < 0$. The incentive function[1] proposed by Osborne is:

$$\Psi_i(E_j) \;=\; \max\left\{ E_i^*, E_i^* + \frac{E_i^*}{E_j^*}(E_j - E_j^*) \right\} \;, \tag{12}$$

where $(E_1^*, \ldots, E_n^*)$ is a particular Pareto outcome of the oligopoly, the "Cartel point". This point is assumed to have the property that, for all i and j,

$$E_i^* \, \frac{\partial J_j}{\partial E_i}(E^*) \;=\; E_j^* \, \frac{\partial J_i}{\partial E_j}(E^*) \;.$$

Restricting the analysis to a duopoly, the necessary condition for being the maximum of the joint profit imposes (see (10)):

$$\frac{\partial J_i}{\partial E_i}(E^*) \;+\; \frac{\partial J_j}{\partial E_i}(E^*) \;=\; 0 \;,$$

for $i, j = 1, 2$. Accordingly, the quantities A_i defined in (6) are: $A_i = E_j^*/E_i^*$, and are positive, since the E_j^* are positive. Applying Corollary 3.1, we see that case $a/$ is the only relevant one given the assumptions on J^i, and we conclude that the incentive may be credible only if:

$$a_i^- \le 0 \le \frac{E_i^*}{E_j^*} \le a_i^+ .$$

The incentive function Ψ_i defined in (12) is such that: $a_i^- = 0$ and $a_i^+ = E_i^*/E_j^*$. This equilibrium is indeed credible, in the local sense of Definition 2.2, under suitable concavity assumptions on J_i. Osborne observes that this credibility is usually not global: for large deviations with respect to E^*, Condition (4) is not satisfied anymore.

4 The case of Nash Open Loop equilibria

Assume now that the game played is dynamic, and that players have open-loop strategies. Accordingly, the strategy space Σ_i of player i is the set of measurable functions from $\mathbb{R}_+$ to some open set $\mathcal{E}_i$ of $\mathbb{R}$. The state of the system is denoted by $x(t)$, and evolves according to the differential equation

$$\dot{x}(t) \;=\; f(E_1(t), E_2(t), x(t)) \;, \quad x(0) = x_0 \;, \tag{13}$$

[1] The topic of Osborne's paper is the stability of a Cartel. In this context, the "incentive" function is actually a threat function, with which members of the Cartel would retaliate to potential cheaters.

where $E_i(t)$ is the action of player i at time t according to his strategy E_i. The payoff of player i is given by:

$$J_i(E_1, E_2; x_0) = \int_0^T e^{-\rho t} F_i(E_1(t), E_2(t), x(t))\, dt \ , \qquad (14)$$

with a time horizon $T < +\infty$ and a discount factor $\rho \geq 0$.

According to Definition 2.1, an incentive function Ψ_i in this context should be a function mapping any measurable functions from $\mathbb{R}$ to $\mathcal{E}_j$ to a measurable function from $\mathbb{R}$ to $\mathcal{E}_i$. A simpler form of incentives results from the idea that a static incentive equilibrium is required at each instant in time. We shall restrict here our discussion to *affine* incentive equilibria, defined as follows:

Definition 4.1 (Affine incentive equilibrium). *An incentive equilibrium (Ψ_1, Ψ_2) at (E_1^*, E_2^*) is said to be affine if Ψ_1 and Ψ_2 are of the form:*

$$\begin{aligned} \Psi_1(E_2)(t) &= E_1^*(t) + v_1(t)(E_2(t) - E_2^*(t)) \ , \\ \Psi_2(E_1)(t) &= E_2^*(t) + v_2(t)(E_1(t) - E_1^*(t)) \ , \end{aligned}$$

for some scalar functions $v_1(t)$ and $v_2(t)$.

In this section, we shall establish sufficient conditions for an affine equilibrium to be a credible incentive equilibrium, in the weak or the strong sense. We focus the analysis on affine incentive functions which we are differentiable. It will turn out that the analysis of the open-loop case is very similar to that of the static case. The extension of the results of Section 3 to *piecewise* affine incentive equilibria is actually straightforward (see Remark 4.3).

We begin with a basic result, in which the problem of finding a credible affine incentive equilibrium is seen as a dynamic optimization problem.

Lemma 4.1. *A credible affine incentive equilibrium at a Pareto optimum is a* *solution of the following system of equations, for some $\alpha_1 > 0$ and $\alpha_2 > 0$:*

$$\left\{\begin{array}{lcl} 0 & = & \alpha_1 \dfrac{\partial F_1}{\partial E_i} + \alpha_2 \dfrac{\partial F_2}{\partial E_i} + \lambda^* \dfrac{\partial f}{\partial E_i} \qquad i = 1,2 \\ \dot{\lambda}^* & = & -\alpha_1 \dfrac{\partial F_1}{\partial x} - \alpha_2 \dfrac{\partial F_2}{\partial x} - \lambda^* \dfrac{\partial f}{\partial x} + \rho\lambda^* \; ; \qquad \lambda^*(T) = 0 \\ \dot{x}^* & = & f \; ; \qquad x(0) = x_0 \end{array}\right. \tag{15}$$

$$\left\{\begin{array}{lcl} 0 & = & \dfrac{\partial F_1}{\partial E_1} + v_2 \dfrac{\partial F_1}{\partial E_2} + \lambda^1 \left(\dfrac{\partial f}{\partial E_1} + v_2 \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^1 & = & -\dfrac{\partial F_1}{\partial x} - \lambda^1 \dfrac{\partial f}{\partial x} + \rho\lambda^1 \; ; \qquad \lambda^1(T) = 0 \\ 0 & = & v_1 \dfrac{\partial F_2}{\partial E_1} + \dfrac{\partial F_2}{\partial E_2} + \lambda^2 \left(v_1 \dfrac{\partial f}{\partial E_1} + \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^2 & = & -\dfrac{\partial F_2}{\partial x} - \lambda^2 \dfrac{\partial f}{\partial x} + \rho\lambda^2 \; ; \qquad \lambda^2(T) = 0 \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{lcl} 0 & = & -v_1 \dfrac{\partial F_1}{\partial E_1} + \lambda^1 \dfrac{\partial f}{\partial E_2} + \lambda^{1c} \left(v_1 \dfrac{\partial f}{\partial E_1} + \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^{1c} & = & \dfrac{\partial F_1}{\partial x} - \lambda^{1c} \dfrac{\partial f}{\partial x} + \rho\lambda^{1c} \; ; \qquad \lambda^{1c}(T) = 0 \\ 0 & = & -v_2 \dfrac{\partial F_2}{\partial E_2} + \lambda^2 \dfrac{\partial f}{\partial E_1} + \lambda^{2c} \left(\dfrac{\partial f}{\partial E_1} + v_2 \dfrac{\partial f}{\partial E_2} \right) \\ \dot{\lambda}^{2c} & = & \dfrac{\partial F_2}{\partial x} - \lambda^{2c} \dfrac{\partial f}{\partial x} + \rho\lambda^{2c} \; ; \qquad \lambda^{2c}(T) = 0 \; , \end{array}\right. \tag{17}$$

all functions being evaluated at (E_1^*, E_2^*, x^*). The unknowns are

- *the cooperative solution E_1^* and E_2^*;*
- *the incentive coefficients $v_1(t)$ and $v_2(t)$;*
- *the state variable $x^*(t)$;*
- *the adjoint variables $\lambda^*(t), \lambda^1(t), \lambda^2(t), \lambda^{1c}(t)$ and $\lambda^{2c}(t)$.*

Proof. First, E_i^* is required to be the solution for the cooperative problem. Equations (15) correspond to the first order necessary conditions to the optimization problem which defines the cooperative solution for the joint payoff:

$$\max_{E_1,E_2} \left[J_1(E_1, E_2, x_0) + J_2(E_1, E_2, x_0) \right] = \max_{E_1,E_2} \sum_{i=1}^{2} \alpha_i \int_0^T e^{-\rho t} F_i(E_1(t), E_2(t), x(t)) \, dt \, ,$$

such that

$$\dot{x}(t) = f(E_1(t), E_2(t), x(t)), \quad x(0) = x_0.$$

In fact, the Hamiltonian for this problem is

$$H^*(E_1, E_2, x, \lambda^*) = \alpha_1 F_1(E_1, E_2, x) + \alpha_2 F_2(E_1, E_2, x) + \lambda^* f(E_1, E_2, x)$$

and we easily recognize (15) as the first order necessary conditions for the optimization of the Hamiltonian.

Second, (Ψ_1, Ψ_2) must be an incentive equilibrium at (E_1^*, E_2^*). So $v_i(t)$, $i = 1, 2$, must be the solution of the following optimization problems:

$$\max_{E_1} J_1(E_1, \Psi_2(E_1), x_0) \; = \; \max_{E_1} \int_0^T e^{-\rho t} F_1(E_1(t), \Psi_2(E_1), x^1(t)) \; dt \; ,$$

such that

$$\dot{x}^1(t) \; = \; f(E_1(t), \Psi_2(E_1)(t), x^1(t)), \quad x^1(0) = x_0.$$

and

$$\max_{E_2} J_2(\Psi_1(E_2), E_2, x_0) \; = \; \max_{E_2} \int_0^T e^{-\rho t} F_2(\Psi_1(E_2), E_2, x^2(t)) \; dt \; ,$$

such that

$$\dot{x}^2(t) \; = \; f(\Psi_1(E_2)(t), E_2(t), x^2(t)), \quad x^2(0) = x_0.$$

For this problems the corresponding Hamiltonians are:

$$\begin{aligned} H^1(E_1, x, \lambda^1) &= F_1(E_1, \Psi_2(E_1), x) + \lambda^1 f(E_1, \Psi_2(E_1), x) \\ H^2(E_2, x, \lambda^2) &= F_2(\Psi_1(E_2), E_2, x) + \lambda^2 f(\Psi_1(E_2), E_2, x) \; . \end{aligned}$$

and (16) gives the necessary conditions for optimization. Since the solution of both optimization problems must be E_1^* and E_2^*, the optimal trajectory for the state is

$$\dot{x}^i(t) \; = \; f(E_1^*(t), E_2^*(t), x^i(t)), \quad x^i(0) = x_0 \; ,$$

which coincides with the differential equation defining x^* in (15). By uniqueness of the solution of this equation, we have $x^1 \equiv x^2 \equiv x^*$.

Finally (17) corresponds to the credibility problem. Consider the point of view of player 1. Following the argument in the proof of Lemma 3.1, the following conditions must hold:

$$D_1(E_2) \; \stackrel{\Delta}{=} \; J_1(E_1^*, E_2; x_0) \; - \; J_1(\Psi_1(E_2), E_2; x_0) \; \leq \; 0 \; ,$$

for E_2 in the neighborhood of E_2^*, where

$$\begin{aligned} J_1(\Psi_1(E_2), E_2; x_0) &= \int_0^T e^{-\rho t} F_1(\Psi_1(E_2), E_2, x^{1a}) \; dt \; , \\ \dot{x}^{1a}(t) &= f(\Psi_1(E_2), E_2, x^{1a}(t)), \quad x^{1a}(0) = x_0 \; , \end{aligned}$$

and

$$\begin{aligned} J_1(E_1^*, E_2, x_0) &= \int_0^T e^{-\rho t} F_1(E_1^*, E_2, x^{1c}(t)) \; dt \; , \\ \dot{x}^{1c}(t) &= f(E_1^*, E_2, x^{1c}(t)), \quad x^{1c}(0) = x_0. \end{aligned}$$

Since $D_1(E_2^*) = 0$, credibility is equivalent to the optimization problem:

$$\max_{E_2} D_1(E_2) \; = \; 0 \; .$$

For this problem the Hamiltonian H^{1c} is:

$$\begin{aligned} H^{1c}(E_2, x^{1a}, x^{1c}, \lambda^{1a}, \lambda^{1c}) \; = \; & F_1(E_1^*, E_2, x^{1c}) - F_1(\Psi_1(E_2), E_2, x^{1a}) \\ & + \lambda^{1c} f(E_1^*, E_2, x^{1c}) + \lambda^{1a} f(\Psi_1(E_2), E_2, x^{1a}) \; . \end{aligned}$$

Conditions $\partial H^{1c}/\partial E_2 = 0$ and $\dot\lambda^{1c} = -\partial H^{1c}/\partial x^{1c}$ give the two first equations of (17). Finally, the condition for the adjoint variable λ^{1a} reads as:

$$\dot\lambda^{1a} \;=\; -\frac{\partial H^{1c}}{\partial x^{1a}} \;=\; -\frac{\partial F_1}{\partial x} \;-\; \lambda^{1a}\,\frac{\partial f}{\partial x}\,,$$

evaluated at (E_1^*, E_2^*, x^*). This equation coincides with that of λ^1 in (16), so that by uniqueness of the solution, $\lambda^{1a} \equiv \lambda^1$. A symmetric reasoning applies for player 2, finally yielding (17). □

We conclude from this analysis that finding a credible incentive equilibrium with two players requires the solution of 12 equations with 10 unknowns. As in the static case, the problem is therefore overconstrained, and is likely to have solutions only if special conditions on the payoff functions are met.

On the other hand, removing the condition that the objective strategy be a solution of the joint optimization problems removes 4 equations and two unknowns (the state variable x^* and the adjoint variable λ^*). This results in a system of 8 equations and 8 unknowns which may be easier to solve.

Remark 4.1. Observe that if cost functions are independent of the state, that is, $F_i(E_1, E_2, x) \equiv F_i(E_1, E_2)$, then all adjoint variables are zero. The remaining system of equations has no solution unless $\partial F_i/\partial E_j = 0$ for all i, j. This result is Theorem 3.2.

We continue with a technical step towards the solution of the system of equations (15)– (17).

Lemma 4.2. *A credible affine incentive equilibrium at a Pareto optimum is a solution of the system of equations* (15)*,* (16) *and*

$$\left\{\begin{array}{lcl} 0 &=& v_1\left(\dfrac{\partial F_1}{\partial E_1} + \lambda^1\dfrac{\partial f}{\partial E_1}\right) \\ 0 &=& v_2\left(\dfrac{\partial F_2}{\partial E_2} + \lambda^2\dfrac{\partial f}{\partial E_2}\right) \end{array}\right. \tag{18}$$

Proof. Considering the differential equations for $\lambda^1(t)$ and $\lambda^{1c}(t)$ in Equations (16) and (17), one gets that

$$\frac{d}{dt}(\lambda^1 + \lambda^{1c}) \;=\; (\lambda^1 + \lambda^{1c})(t)\,\frac{\partial f}{\partial x}\,; \qquad (\lambda^1 + \lambda^{1c})(T) = 0\,.$$

The solution of this differential equation is unique and identically zero. Therefore, $\lambda^1(t) = -\lambda^{1c}(t)$ for all t. A symmetric situation holds for λ^{2c}. Replacing in Equations (17) and simplifying yields (18). □

We can now state the principal result, analogous to Corollary 3.2.

Theorem 4.1. *A weak credible affine incentive equilibrium at E^* may hold only if one of the four following conditions is met at each time instant t:*

i/ $v_1(t) = v_2(t) = 0$ *and*

$$\frac{\partial F_i}{\partial E_i} \;+\; \lambda^i\frac{\partial f}{\partial E_i} \;=\; 0\,, \qquad i = 1, 2\,.$$

ii/ $v_1(t) \neq 0$ *and* $v_2(t) \neq 0$ *and*

$$\frac{\partial F_i}{\partial E_j} \;+\; \lambda^i \frac{\partial f}{\partial E_j} \;=\; 0 \;, \qquad i,j = 1,2 \;.$$

iii/ $v_1(t) = 0$ *and* $v_2(t) \neq 0$ *and*

$$\begin{aligned} 0 &= \frac{\partial F_2}{\partial E_2} \;+\; \lambda^2 \frac{\partial f}{\partial E_2} \\ 0 &= \frac{\partial F_1}{\partial E_1} + v_2 \frac{\partial F_1}{\partial E_2} + \lambda^1 \left(\frac{\partial f}{\partial E_1} + v_2 \frac{\partial f}{\partial E_2} \right) \end{aligned}$$

iv/ $v_1(t) \neq 0$ *and* $v_2(t) = 0$ *and*

$$\begin{aligned} 0 &= \frac{\partial F_1}{\partial E_1} \;+\; \lambda^1 \frac{\partial f}{\partial E_1} \\ 0 &= \frac{\partial F_2}{\partial E_2} + v_1 \frac{\partial F_2}{\partial E_1} + \lambda^2 \left(\frac{\partial f}{\partial E_2} + v_1 \frac{\partial f}{\partial E_1} \right) \;. \end{aligned}$$

Proof. To simplify the notation, let:

$$C_{ij} \;=\; \frac{\partial F_i}{\partial E_j} \;+\; \lambda^i \frac{\partial f}{\partial E_j} \;.$$

The equations necessarily satisfied by a weak credible incentive equilibrium are:

$$\begin{aligned} C_{11} + v_2 C_{12} = 0 \;, &\qquad C_{22} + v_1 C_{21} = 0 \;, \\ v_1 C_{11} = 0 \;, &\qquad v_2 C_{22} = 0 \;. \end{aligned}$$

If $v_1 = v_2 = 0$, then the last equations hold, and the two first imply $C_{11} = C_{22} = 0$. Hence *i/*.

If $v_1 \neq 0$ and $v_2 \neq 0$, the last equations imply $C_{11} = C_{22} = 0$. The two first ones then imply $C_{12} = C_{21} = 0$. Hence *ii/*.

Finally, if $v_1 = 0$ and $v_2 \neq 0$, then $C_{22} = 0$. This implies that the second equation holds. Hence *iii/*. A symmetric argument holds in case *iv/*. □

Corollary 4.1. *A credible affine incentive equilibrium at a Pareto optimum* E^* *may hold only if, at all time instants,*

$$\frac{\partial F_i}{\partial E_j}(E_1^*, E_2^*) \;+\; \lambda^i \frac{\partial f}{\partial E_j}(E_1^*, E_2^*, x^*) \;=\; 0 \;, \qquad i,j = 1,2 \;,$$

where λ^i *is solution of the differential equation:*

$$\dot{\lambda}^i \;=\; -\frac{\partial F_i}{\partial x} - \lambda^i \frac{\partial f}{\partial x} + \rho \lambda^i \;, \tag{19}$$

with $\lambda^i(T) = 0$.

Proof. The conditions for Pareto optimality write, with the notation of the previous proof, as:

$$\alpha_1 C_{11} + \alpha_2 C_{21} = 0 \;, \qquad \alpha_2 C_{22} + \alpha_1 C_{12} = 0 \;,$$

where the α_i are nonnegative. Combining these constraints with any of the four cases in Theorem 4.1 results in $C_{ij} = 0$ for all i and j. □

We have therefore a conclusion extending that of the static case. The necessary conditions stated in Corollary 4.1 are that of a simultaneous maximum for both payoff functions J_i. Therefore, the existence of a *credible* affine incentive equilibrium is possible only if this very strong property holds.

We conclude this analysis with notes on extensions of the results stated.

Remark 4.2. The analysis can be extended to incentive functions Ψ_i of the form:

$$\Psi_i(E_j)(t) \;=\; E_i^*(t) + V_i(t, E_j(t)) \;, \quad \text{with } V_i(t, E_j^*(t)) \;\equiv\; 0 \;,$$

of which the Affine incentive functions of Definition 4.1 are a special case. In that case, Lemma 4.1 and Lemma 4.2 apply with

$$v_i(t) \;=\; \frac{\partial V_i}{\partial E_j}(t, E_j^*(t)) \;.$$

Remark 4.3. Lemma 4.1 corresponds, for the dynamic case, to Corollary 3.2 of the static case. It is straightforward to establish results corresponding to Lemma 3.1 and Corollary 3.1 for piecewise-differentiable incentive functions. Assuming the general form of Remark 4.2, we define: $V_i(t, E_j(t)) = V_i^+(t, E_j(t))$ if $E_j(t) \geq E_j^*(t)$ and $V_i(t, E_j(t)) = V_i^-(t, E_j(t))$ if $E_j(t) \leq E_j^*(t)$. The left and right-derivatives are denoted as: $v_i^\pm(t) = \partial V_i^\pm / \partial E_j(t, E_j^*(t))$.

The transposition of the results of Section 3 consists in replacing "$\partial J_i/\partial E_j$" by "$\partial F_i/\partial E_j + \lambda^i \partial f/\partial E_j$", where λ^i is solution of the differential equation (19), and $a_i^\pm$ by $v_i^\pm(t)$.

Remark 4.4. The analysis can be extended to the infinite-horizon case, by substituting the transversality condition

$$\lim_{T\to\infty} e^{-\rho T} \lambda(T) \;=\; 0$$

to "$\lambda(T) = 0$" whenever relevant.

5 Case study

We apply in this section the previous result to a model borrowed to Martín-Herrán and Zaccour [8]. We apply the results of Sections 3 and 4 to this case, and identify the possible (locally) credible incentive equilibria. Next, taking advantage of the existence of closed formulas, we investigate the extent of the credibility: what are the maximal deviations allowed in order to preserve credibility.

The model is described by the following elements:

$$J_i(E_1(\cdot), E_2(\cdot); x_0) \;=\; \int_0^\infty e^{-\rho t} \left(\log(E_i(t)) - \phi_i x(t)\right) \, dt \;, \tag{20}$$

$$\dot{x}(t) \;=\; E_1(t) + E_2(t) - \delta x(t), \quad x(0) = x_0 \;.$$

Using the calculations reported in [8], it is easy to establish that the Pareto solution corresponding to the maximization of $\sum_i \alpha_i J_i$, is:

$$E_i^* \;=\; \frac{\alpha_i(\delta + \rho)}{\alpha_1 \phi_1 + \alpha_2 \phi_2} \;. \tag{21}$$

The Pareto-optimal control therefore does not depend on time[2], a fact which prompts the study of time-invariant strategies. We shall therefore successively consider credible incentives *restricted* to such strategies, then the general case.

5.1 Static credibility

In this section, we consider only time-invariant strategies. The game is then reduced to a two-player static game, and we can apply to it the results of Section 3.

For two given real values e_1 and e_2, the total payoff of player i is given by:

$$J_i(e_1, e_2; x_0) \;=\; \frac{1}{\rho}\log(e_i) \;-\; \frac{\phi_i}{\rho(\rho+\delta)}(e_1+e_2) \;-\; \frac{\phi_i x_0}{\rho+\delta}\;.$$

Given two values for α_i, the Pareto optimum is given by Equation (21). Computing the derivatives of J_i, evaluated at this Pareto outcome, we obtain:

$$\frac{\partial J_i}{\partial e_i} \;=\; \frac{1}{\rho e_i^*} \;-\; \frac{\phi_i}{\rho(\rho+\delta)} \;=\; \frac{\alpha_j}{\alpha_i}\,\frac{\phi_j}{\rho(\rho+\delta)}\;, \qquad \frac{\partial J_i}{\partial e_j} \;=\; -\;\frac{\phi_i}{\rho(\rho+\delta)}\;,$$

so that the value of A_i defined by Equation (6) is:

$$A_i \;=\; \frac{\alpha_j\phi_j}{\alpha_i\phi_i}\;.$$

Applying Corollary 3.1, we find that we are in case $a/$, and we conclude that an incentive function Ψ_i is credible if and only if, with a_i^- and a_i^+ the left and right-derivatives of Ψ_i evaluated at $e_j = e_j^*$:

$$a_i^- \le 0 \le \frac{\alpha_i\phi_i}{\alpha_j\phi_j} \le a_i^+. \tag{22}$$

The case $\alpha_1 = \alpha_2$ has been studied in [8], where it is proved that ϕ_i/ϕ_j is indeed the slope of a credible incentive function.

The conditions provided by Corollary 3.1 being only sufficient, we now investigate whether incentive functions with left and right-slopes constrained by (22) are credible, locally or for any deviation. To that end, we select the piecewise affine function:

$$\Psi_i(e_j) \;=\; \max\left\{e_i^*, e_i^* + \frac{\alpha_i\phi_i}{\alpha_j\phi_j}(e_j - e_j^*)\right\}\;.$$

For Player 1, the credibility condition (4) becomes: for $e_2 \ge e_2^*$:

$$0 \;\le\; \log\frac{e_1^* + \alpha_1\phi_1/\alpha_2\phi_2(e_2 - e_2^*)}{e_1^*} \;-\; \phi_1\,\frac{\alpha_1\phi_1}{\alpha_2\phi_2}\,(e_2 - e_2^*)\;.$$

The right-hand side of this inequality is a function of e_2, say $h(e_2)$, which vanishes at $e_2 = e_2^*$, has a positive derivative at this point, and tends to $-\infty$ when e_2 grows. Consequently, there exists an interval $[e_2^*, \bar{e}_2]$ where the condition is satisfied. In order to compute a closed-form lower bound for $\bar{e}_2$, we use two

[2]It is shown in [8] that this is a feature common to all linear-state differential games.

methods. The first one is to find the maximal interval over which the function h_2 is increasing, therefore positive. The second one consists in using the bound $\log(1+x) \geq x - x^2/2$. After straightforward calculations, one finds:

$$\bar{e}_2 \;\geq\; e_2^* \;\max\left\{1 + \frac{\alpha_2}{\alpha_1}\left(\frac{\phi_2}{\phi_1}\right)^2, 1 + \frac{2\alpha_2(\phi_2)^2}{\phi_1(\alpha_1\phi_1 + \alpha_2\phi_2)}\right\} .$$

5.2 Dynamic credibility

We now turn to the dynamic case, and establish what are the credible affine incentive functions. According to Remark 4.3, we can find credibility conditions on the derivatives of the functions Ψ_i. These turn out to be the same as in the static case: see Condition (22).

Next, in order to check the extent of the credibility, we restrict the computations to the case $\alpha_1 = \alpha_2 = 1$. This implies that $e_1^* = e_2^* = e^* = (\delta+\rho)/(\phi_1+\phi_2)$. We select the incentive function:

$$\Psi_i(E_j)(t) \;=\; e^* \;+\; \max\left\{\frac{\phi_i}{\phi_j}(E_j(t) - e^*), 0\right\} .$$

According to Condition (4), credibility for player 1 holds if and only if, for any strategy $E_2(\cdot)$ of player 2 in some vicinity of e^*,

$$\int_0^{+\infty} \left[\log(\Psi_1(E_2(t))) - \phi_1 x^{\Psi}(t) - \log(e^*) + \phi_1 x^*(t)\right]\; e^{-\rho t}\; dt \;\geq\; 0\;, \qquad (23)$$

where the two trajectories $x^{\Psi}(\cdot)$ and $x^*(\cdot)$ are the respective solutions of

$$\begin{aligned} \dot{x} &= e^* \;+\; \frac{\phi_1}{\phi_2}\max(0, E_2(t) - e^*) \;+\; E_2(t) - \delta x(t)\;, \\ \dot{x} &= e^* \;+\; E_2(t) - \delta x(t)\;, \end{aligned}$$

with initial condition $x(0) = x_0$. Assume that there exists a constant $M \geq 1$ such that for all t,

$$E_2(t) \;\leq\; M\; e^* . \qquad (24)$$

Integrating the differential equations, we have:

$$\begin{aligned} x^{\Psi}(t) - x^*(t) &= \int_0^t \frac{\phi_1}{\phi_2}\max(E_2(t) - e^*, 0)e^{\delta(u-t)}\; dt \\ &\leq \frac{\phi_1}{\phi_2}(M-1)\frac{1 - e^{\delta t}}{\delta} . \end{aligned}$$

Denote by H the integral in Condition (23). Using the inequality above, and the bound $\log(1 + \max(x, 0)) \geq x - x^2/2$, we have:

$$\begin{aligned} H &\geq \int_0^{+\infty} \left[\frac{\phi_1}{\phi_2}(\frac{E_2(t)}{e^*} - 1) - \frac{\phi_1^2}{2\phi_2^2}(\frac{E_2(t)}{e^*} - 1)^2 - \frac{\phi_1}{\phi_2}(M-1)\frac{1 - e^{\delta t}}{\delta}\right]\; e^{-\rho t}\; dt \\ &\geq \int_0^{+\infty} \left[\frac{\phi_1}{\phi_2} + \frac{2\phi_1^2}{\phi_2^2}\right]\frac{E_2(t)}{e^*}e^{-\rho t}dt \;+\; \frac{1}{\rho}\left[-\frac{\phi_1^2}{\phi_2^2}M^2 - \frac{\phi_1}{\phi_2} + \frac{\phi_1^2}{\phi_2^2} - \frac{\phi_1^2}{\phi_2^2}\frac{M-1}{\rho+\delta}e^*\right] . \end{aligned}$$

A sufficient condition for credibility is therefore that the strategy $E_2(\cdot)$ satisfies simultaneously: (24) and

$$\int_0^{+\infty} \left[\frac{\phi_1}{\phi_2} + \frac{2\phi_1^2}{\phi_2^2}\right] \frac{E_2(t)}{e^*} e^{-\rho t} dt \;\geq\; \frac{1}{\rho}\left[\frac{\phi_1^2}{\phi_2^2}(M^2-1) + \frac{\phi_1}{\phi_2} + \frac{\phi_1^2}{\phi_2^2}\frac{M-1}{\rho+\delta}e^*\right] ,$$

for some constant M.

For instance, it can be checked that the equilibrium is credible with respect to strategies of the form

$$E_2(t) \;=\; e^N + (e^* - e^N)e^{-\alpha t} , \quad \text{or} \quad E_2(t) \;=\; e^* + (e^N - e^*)e^{-\alpha t} ,$$

where $e^N = (\rho+\delta)/2$ is the Nash equilibrium of the game (a time-invariant strategy as well).

6 Conclusion

The principal conclusions we have reached is that:

- Credibility is difficult to obtain in static and continuous-time games: at a Pareto solution as well as elsewhere (weak credibility), *if the incentive function is required to be differentiable*. Strong and credible incentive equilibria may happen only at critical points of both payoff functions simultaneously. Weak credible incentive equilibria may happen at outcomes (E_1^*, E_2^*) for which at least one E_i^* is a Nash-best-response to E_j^*.

- On the other hand, with piecewise-differentiable incentive functions, (local) credibility is rather easy to obtain, and many slopes are generally allowed for these incentive functions. The actual challenge is to find incentive functions that provide a "domain of credibility" as large as possible.

As logical continuations of this work, we mention:

- Study whether credibility of open-loop strategies may hold in a neighborhood of the equilibrium, not only in a particular subset of deviations.

- Extend the analysis to discrete-time problems such as the one studied in [4].

- Investigate incentives defined on Nash-Feedback strategies.

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