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# Interval Completion is Fixed Parameter Tractable

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16th November 2006

## Abstract

We give an algorithm with runtime  $O(k^{2k}n^3m)$  for the NP-complete problem [GT35 in 6] of deciding whether a graph on  $n$  vertices and  $m$  edges can be turned into an interval graph by adding at most  $k$  edges. We thus prove that this problem is fixed parameter tractable (FPT), settling a long-standing open problem [13, 5, 19, 11]. The problem has applications in Physical Mapping of DNA [9] and in Profile Minimization for Sparse Matrix Computations [7, 20]. For the first application, our results show tractability for the case of a small number  $k$  of false negative errors, and for the second, a small number  $k$  of zero elements in the envelope.

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# 1 Introduction and motivation

Interval graphs are the intersection graphs of intervals of the real line and have a wide range of applications [10]. Connected with interval graphs is the following problem, called *Interval Completion*: Given an arbitrary graph  $G$ , what is the minimum number of edges that must be added to  $G$  in order to obtain an interval graph? This problem is NP-hard [14, 6] and it arises in both Physical Mapping of DNA and sparse matrix computations. In Physical Mapping of DNA a set of contiguous intervals of the DNA chain, called clones, are given together with experimental information on their pairwise overlaps. The goal is to build a map describing the relative position of the clones. In the presence of false negative errors, the problem of building a map with fewest errors is equivalent to finding the smallest edge set whose addition to the input graph will form an interval graph [9]. In sparse matrix computations, one of the standard methods for reordering a matrix to get as few non-zero elements as possible during Gaussian elimination, is to permute the rows and columns of the matrix so that non-zero elements are gathered close to the main diagonal [7]. The *profile* of a matrix is the smallest number of entries that can be enveloped within off-diagonal non-zero elements of the matrix. Translated to graphs, the profile of a graph  $G$  is exactly the minimum number of edges in an interval supergraph of  $G$  [20].

In this paper, we present an algorithm with runtime  $O(k^{2k}n^3m)$  for the  $k$ -Interval Completion problem of deciding whether a graph on  $n$  vertices and  $m$  edges can be made into an interval graph by adding at most  $k$  edges. This NP-complete problem is thus fixed parameter tractable (FPT), *i.e.* there is an algorithm with runtime  $f(k)n^{O(1)}$ , which settles a long-standing open problem [13, 5, 19, 11]. An early paper (first appearance FOCS '94) in this line of research by Kaplan, Shamir, and Tarjan [13] gives FPT algorithms for Chordal Completion, Strongly Chordal Completion, and Proper Interval Graph Completion. In all these cases a fairly straightforward bounded-search tree algorithm works, that identifies a witness of non-membership in the desired class of graphs, and branches on all possible ways of adding an inclusion-minimal set of edges that gets rid of the witness. The existence of an FPT algorithm for  $k$ -Interval Completion is left as an open problem by [13], with the explanation: “An arbitrarily large obstruction  $X$  could exist in a graph that is not interval but could be made interval with the addition of any one out of  $O(|X|)$  edges.” Surprisingly, in this paper we are able to show that a bounded search tree algorithm will nevertheless give an FPT algorithm for  $k$ -Interval Completion.

Let us mention some related work. Cai presented a simple FPT algorithm for Chordal Completion, and he also proved that minimum completion into hereditary graph classes with a finite set of forbidden subgraphs is FPT [2]. Note that interval graphs do not have a finite set of forbidden subgraphs [15]. Natanson, Shamir and Sharan [16] study three types of edge modification problems for various graph classes, where completion is one type, deletion a second type, and the third type being editing where both additions and deletions are allowed, showing both hardness results and approximation algorithms. Heggernes, Telle and Villanger [12] gave an  $O(n^{2.376})$  algorithm for minimal chordal completion, breaking the long-standing  $O(n^3)$  bound of the LEX-M algorithm from [18] for this problem. Reed, Smith and Vetta [17] gave an FPT algorithm for the bipartite edge deletion problem, deciding if a graph can be made bipartite by the removal of  $k$  edges. In a recent paper, Gutin, Szeider, and Yeo [11] showed that deciding whether a graph  $G$  has profile at most  $k + |V(G)|$  is FPT, while leaving open the parameterized complexity of the more natural question if  $G$  has profile at most  $k + |E(G)|$ , which we resolve in this paper.

Our search tree algorithm for  $k$ -Interval Completion circumvents the problem of large obstructions (witnesses) by first getting rid of all small witnesses, in particular witnesses for the existence of an asteroidal triple. An asteroidal triple (AT) in a graph consists of three pairwise non-adjacent vertices with the property that between any two of them there exists a path avoiding the neighborhood of the third. A graph is an interval graph if and only if it is both chordal and AT-free [15]. Thus, to complete into an interval graph we must destroy witnesses for non-chordality and witnesses for existence of an AT. Witnesses for non-chordality (chordless cycles of length  $> 3$ ) must have size  $O(k)$  and do not present a problem with respect to an FPT algorithm. Likewise, if an AT is witnessed by an induced subgraph  $S$  of size  $O(k)$  it does not present a problem. The difficult case is when we have a chordal non-interval graph  $G$  with no AT having a small witness. For this case we use the concept of a *thick* AT-witness, consisting of an AT  $\{a, b, c\}$  and all vertices on any chordless path between any two vertices of the AT avoiding the neighborhood of the third. A minimal thick AT-witness for the AT  $\{a, b, c\}$  is one where removing any vertex of the AT gives an interval graph. We show that for an AT  $\{a, b, c\}$  having a minimal thick AT-witness, one of the vertices in the AT is *shallow*, meaning that there is a short path (of length at most 4) from it to each of the other two vertices of the AT. For the difficult case, *i.e.*  $G$  a chordal graph having no small AT-witnesses, we are able to compute a set of vertices  $C$  consisting of shallow vertices of some AT such that removing  $C$  from the graph gives an interval graph. Based on the cardinality of  $C$  we handle this case by branching in one of several different ways of getting rid of the minimal thick AT-witness of a vertex in  $C$ . In particular, we show that when no bounded branching is possible the instance has enough structure that the best solution is a completion computed in a greedy manner.

The paper is organized as follows. In the next section we give some standard definitions. Section 3 is devoted to the first two branching rules, which will establish chordality and destroy small simple AT-witnesses. In Section 4 we introduce shallow vertices and minimal thick AT-witnesses, leading to the third branching rule. The fourth and final branching rule is given in Section 5 which also contains the proof that the greedy completion is minimum.

## 2 Preliminaries

We work with simple and undirected graphs  $G = (V, E)$ , with vertex set  $V(G) = V$  and edge set  $E(G) = E$ , and we let  $n = |V|$ ,  $m = |E|$ . For a given vertex set  $X \subset V$ ,  $G[X]$  denotes the subgraph of  $G$  induced by the vertices in  $X$ . We will use  $G - x$  to denote  $G[V \setminus \{x\}]$  for  $x \in V$ , and  $G - S$  to denote  $G[V \setminus S]$  for  $S \subseteq V$ .

The set of *neighbors* of a vertex  $x$  is denoted by  $N_G(x) = \{y \mid xy \in E\}$ , and the *closed neighborhood* is  $N_G[x] = N_G(x) \cup \{x\}$ . For a vertex set  $X$ , similarly,  $N_G[X] = \bigcup_{x \in X} N_G[x]$ , and  $N_G(X) = N_G[X] \setminus X$ . We will omit the subscript when the graph is clear from the context. A vertex set  $X$  is a *clique* if  $G[X]$  is a complete graph, and a *maximal clique* if no superset of  $X$  is a clique. A vertex  $x$  is called *simplicial* if  $N(x)$  is a clique.

A *path*  $P$  is a sequence  $v_1, v_2, \dots, v_p$  of distinct vertices of  $G$ , where  $v_i v_{i+1} \in E$  for  $1 \leq i < p$ , in which case we say that  $P$  is a path *between*  $v_1$  and  $v_p$ , and we call it a  $v_1, v_p$ -*path*. A path  $v_1, v_2, \dots, v_p$  is called a *cycle* if  $v_1 v_p \in E$ . The *length* of a path (cycle)  $P$  is the number of vertices in it, and it is denoted by  $|P|$ . We will use  $P - v_p$  and  $P + v_{p+1}$  to denote the paths  $v_1, v_2, \dots, v_{p-1}$  and  $v_1, v_2, \dots, v_p, v_{p+1}$ , respectively. We say that a path  $P$  *avoids* a vertex set  $S$  if  $P$  contains no vertex of

$S$ . A *chord* of a cycle (path) is an edge connecting two non-consecutive vertices of the cycle (path). A *chordless* cycle (path) is an induced subgraph that is isomorphic to a cycle (path). A graph is *chordal* if it contains no chordless cycle of length at least 4.

A graph is an *interval graph* if intervals can be associated to its vertices such that two vertices are adjacent if and only if their corresponding intervals overlap. Three non-adjacent vertices form an *asteroidal triple (AT)* if there is a path between every two of them that does not contain a neighbor of the third. A graph is *AT-free* if it contains no AT. A graph is an interval graph if and only if it is chordal and AT-free [15]. A vertex set  $S \subseteq V$  is called *dominating* if every vertex not contained in  $S$  is adjacent to some vertex in  $S$ . A pair of vertices  $\{u, v\}$  is called a *dominating pair* if every  $u, v$ -path is dominating. Every interval graph has a dominating pair [3], and thus also a dominating chordless path.

A *clique tree* of a graph  $G$  is a tree  $T$  whose nodes (also called *bags*) are maximal cliques of  $G$  such that for every vertex  $v$  in  $G$ , the subtree  $T_v$  of  $T$  that is induced by the bags that contain  $v$  is connected. A graph is chordal if and only if it has a clique tree [1].

A *clique path*  $Q$  of a graph  $G$  is a clique tree that is a path. Hence for every vertex  $v$  in  $G$ , the subpath  $Q_v$  of  $Q$  induced by the bags that contain  $v$  is connected. A graph  $G$  is an interval graph if and only if it has a clique-path [8]. An interval graph has at most  $n$  maximal cliques.

Clique trees and paths are useful also in connection with vertex separators. A vertex set  $S \subset V$  is a *separator* if  $G - S$  is disconnected. Given two vertices  $u$  and  $v$ ,  $S$  is a  *$u, v$ -separator* if  $u$  and  $v$  belong to different connected components of  $G - S$ . A  *$u, v$ -separator*  $S$  is *minimal* if no proper subset of  $S$  is a  *$u, v$ -separator*. In general,  $S$  is a *minimal separator* of  $G$  if there exist two vertices  $u$  and  $v$  in  $G$  such that  $S$  is a minimal  *$u, v$ -separator*. For a chordal graph  $G$ , a set of vertices  $S$  is a minimal separator of  $G$  if and only if  $S$  is the intersection of two neighboring bags in any clique tree of  $G$  [1]. Consequently, for interval graphs, a set of vertices  $S$  is a minimal separator if and only if  $S$  is the intersection of two neighboring bags in any clique path.

An interval supergraph  $H = (V, E \cup F)$  of a given graph  $G = (V, E)$ , with  $E \cap F = \emptyset$ , is called an *interval completion* of  $G$ . The set  $F$  is called the set of *fill edges* of  $H$ . If there is no set  $F'$  with  $|F'| < |F|$  such that  $(V, E \cup F')$  is an interval graph, then  $H$  is called a *minimum interval completion* of  $G$ .

The following gives a formal definition of the problem that we solve:

*k*-INTERVAL COMPLETION

**Input:** A graph  $G$ , a parameter  $k$ .

**Problem:** Is there an interval completion of  $G$  with at most  $k$  fill edges?

If such an interval completion exists, we will call it a *k-interval completion* of  $G$ .

### 3 Non-chordality and small simple AT-witnesses: Rules 1,2

**Branching Rule 1:**

If  $G$  is not chordal, find a chordless cycle  $C$  of length at least 4. If  $|C| > k + 3$  answer no. Otherwise:

- Branch on the at most  $4^{|C|}$  different ways to add an inclusion minimal set of edges between the vertices of  $C$  to make it chordal.

The correctness of the first rule is well understood [13, 2].

**Observation 3.1** *Given a graph  $G$ , let  $\{a, b, c\}$  be an AT in  $G$ . Let  $P'_{ab}$  be the set of vertices on a path between  $a$  and  $b$  in  $G - N[c]$ , let  $P'_{ac}$  be the set of vertices on a path between  $a$  and  $c$  in  $G - N[b]$ , and let  $P'_{bc}$  be the set of vertices on a path between  $b$  and  $c$  in  $G - N[a]$ . Then any interval completion of  $G$  contains at least one fill edge from the set  $\{cx \mid x \in P'_{ab}\} \cup \{ax \mid x \in P'_{bc}\} \cup \{bx \mid x \in P'_{ac}\}$ .*

**Proof.** Otherwise  $\{a, b, c\}$  would still an independent set of vertices with a path between any two avoiding the neighborhood of the third, in other words it would be an AT. ■

We introduce simple AT-witnesses and give a branching rule for small such witnesses.

**Definition 3.2** *Let  $a, b, c$  be three vertices of a graph  $G$ . We define  $P_{ab}$  to be the set of vertices on a shortest path between  $a$  and  $b$  in  $G - N[c]$ ,  $P_{ac}$  the set of vertices on a shortest path between  $a$  and  $c$  in  $G - N[b]$ , and  $P_{bc}$  the set of vertices on a shortest path between  $b$  and  $c$  in  $G - N[a]$ . Note that the three paths exist if and only if  $\{a, b, c\}$  is an AT. We define  $G_{abc}$  to be the subgraph of  $G$  induced by the vertices of  $P_{ab} \cup P_{bc} \cup P_{ac}$ , and call it a simple AT-witness for this AT.*

### Branching Rule 2:

If  $G$  is chordal: For each triple  $\{a, b, c\}$  check if  $\{a, b, c\}$  is an AT. For each AT  $\{a, b, c\}$ , find a simple AT-witness  $G_{abc}$  for it. If there exists an AT  $\{a, b, c\}$ , such that  $|\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}| \leq k + 15$  for the simple AT-witness  $G_{abc}$ , then:

- Branch on each of the fill edges in the set  $\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}$ .

By Observation 3.1, any interval completion contains at least one edge from the set branched on by Rule 2.

**Lemma 3.3** *Let  $G$  be a graph to which Rule 1 cannot be applied (i.e.  $G$  is chordal). There exists a polynomial time algorithm that finds a simple AT-witness  $G_{abc}$ , where  $|\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}| \leq k + 15$ , if such an AT-witness exists.*

**Proof.** A simple AT-witness can be found in polynomial time: for a triple of vertices, check if there exists a shortest path between any two of them that avoids the neighborhood of the third vertex. Since shortest paths are used to define simple AT-witnesses, then  $|\{cx \mid x \in P_{ab}\} \cup \{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}|$  will be the same for all simple AT-witnesses for an AT  $\{a, b, c\}$ . ■



## 4 Thick AT-witnesses and shallow vertices: Rule 3

In this section we introduce minimal simple AT-witnesses and show that they each have a shallow vertex. We then introduce thick AT-witnesses showing that also minimal thick AT-witnesses have a shallow vertex, and give a branching rule based on this.

Note that our recursive search tree subroutine will apply only a single branching rule, and the rules will be applied in the order they are introduced. Thus, if Rule 1 applies we apply it and branch, else if Rule 2 applies we apply it and branch, else if Rule 3 applies we apply it and branch, else apply Rule 4. In Subsection 4.1 we consider graphs to which Rule 1 cannot be applied (chordal graphs) and in Subsection 4.2 graphs to which neither Rule 1 nor Rule 2 can be applied.

### 4.1 $G$ is a chordal graph

**Observation 4.1** *Let  $G_{abc}$  be a simple AT-witness in a chordal graph  $G$ . Then  $a, b, c$  are simplicial vertices in  $G_{abc}$ .*

**Proof.** By the definition of  $G_{abc}$ ,  $|N(a)| \leq 2$ . As  $P_{bc}$  avoids  $N(a)$  and as any vertex of  $N(a)$  has a neighbor in the connected component of  $G_{abc} - N(a)$ ,  $N(a)$  is a minimal  $a, c$ -separator.  $G_{abc}$  is a chordal graph (since it is an induced subgraph of a chordal graph), and by [4] every minimal separator of a chordal graph is a clique, thus  $N(a)$  is a clique. ■

**Definition 4.2** *A simple AT-witness  $G_{abc}$  is minimal if  $G_{abc} - x$  is AT-free for any  $x \in \{a, b, c\}$ .*

**Observation 4.3** *Let  $G_{abc}$  be a minimal simple AT-witness in a chordal graph. Then for any  $x \in \{a, b, c\}$ ,  $G_{abc} - x$  is an interval graph, where  $\{a, b, c\} \setminus \{x\}$  is a dominating pair.*

**Proof.** We prove the observation for  $x = c$ , and the other two possibilities are symmetric. Clearly,  $G' = G_{abc} - c$  is an interval graph, since  $G$  is chordal and  $G_{abc}$  is a minimal simple AT-witness. For a contradiction assume that  $\{a, b\}$  is not a dominating pair in  $G'$ ; thus there exists a path  $P'_{ab}$  from  $a$  to  $b$  in  $G' - N[y]$  for some vertex  $y \in V(G') \setminus \{a, b\}$ . Let  $Q$  be a clique path of  $G'$ . Vertex  $y$  does not appear in any bag of  $Q$  that contains  $a$  or  $b$ , and it does not appear in any bag between subpaths  $Q_a$  and  $Q_b$  of  $Q$ . Let us without loss of generality assume that  $Q_a$  appears between  $Q_y$  and  $Q_b$  in  $Q$ . Because of this  $y$  is not contained in the component  $C_b$  of  $G' - N[a]$  that contains  $b$ . Furthermore,  $a$  is a simplicial vertex by Observation 4.1, and  $P'_{ab}$  contains vertices from  $N(a)$ , thus  $y \notin N(a)$  since  $P'_{ab}$  would not avoid the neighborhood of  $y$  otherwise. The path  $P_{bc} - c$  is contained in  $C_b$ , and thus  $y$  is not adjacent to any vertex in  $P_{bc} - c$ . We know that  $cy \notin E$ , since by Observation 4.1  $N(c)$  is a clique, and thus  $y$  would be adjacent to the neighbor of  $c$  in  $P_{ab}$  if  $cy$  were an edge. None of the paths  $P_{ab}, P_{ac}, P_{bc}$  contains  $y$ , since  $P_{bc}$  is strictly contained in  $C_{bc}$ , and since any shortest path from  $a$  to either  $b$  or  $c$  only contains one neighbor of  $a$ . Thus,  $G_{abc}$  is not a simple AT-witness for  $\{a, b, c\}$ , since it contains  $y$ . ■

**Definition 4.4** Given a graph  $G$ , and let  $\{a, b, c\}$  be an AT in  $G$ . Vertex  $c$  is called shallow if  $|P_{ac}| \leq 4$  and  $|P_{bc}| \leq 4$ .

**Lemma 4.5** Let  $G_{abc}$  be a minimal simple AT-witness in a chordal graph, where  $|P_{ab}| \geq |P_{bc}| \geq |P_{ac}|$ . If  $|P_{ab}| \geq 6$  then  $c$  is shallow.

**Proof.** Let us on the contrary assume that  $|P_{bc}| \geq 5$ . Let  $a = v_1, v_2, \dots, v_r = b$  be the path  $P_{ab}$ , and let  $c'$  be the neighbor of  $c$  in the path  $P_{bc}$ . Notice that, by Observation 4.3  $G_{abc} - a$  is an interval graph, where  $\{b, c\}$  is a dominating pair, and thus  $v_2$  is adjacent to at least one vertex in any  $b, c$ -path in  $G_{abc} - a$ , and  $c'$  is adjacent to at least one vertex in any  $a, b$ -path in  $G_{abc} - c$ . Let  $i$  be the smallest integer such that  $v_i$  is a neighbor of  $c'$ . We can now conclude that  $i \leq 3$ , since  $v_2$  is adjacent to some vertex on the path  $c, c', v_i, v_{i+1}, \dots, v_r, b$ . Vertex set  $\{a, v_{r-1}, c\}$  is an independent set;  $av_{r-1} \notin E$  since  $r \geq 6$ ;  $cv_{r-1} \notin E$  since  $r \geq 6$  and  $i \leq 3$ ; and  $ac \notin E$  since  $a, b, c$  is an independent set. The path  $a, v_2, \dots, v_{r-1}$  avoids  $N(c)$  by definition of  $P_{ab}$ ; either path  $a, v_2, c', c$  or path  $a, v_2, v_3, c', c$  avoids  $N(v_{r-1})$ , since  $r \geq 6$  and since the shortest  $b, c$ -path in  $G_{abc} - a$  contains at least five vertices ( $|P_{bc}| \geq 5$ ); and  $P_{bc} - b + v_{r-1}$  is a path that avoids  $N(a)$ , since  $r \geq 6$  and  $b$  is by Observation 4.1 a simplicial vertex in  $G_{abc}$ . The simple witness for the AT  $\{a, v_{r-1}, c\}$  induced by the paths  $(a, v_2, \dots, v_{r-1})$ ,  $(a, v_2, c', c)$  or  $(a, v_2, v_3, c', c)$ , and  $P_{bc} - b + v_{r-1}$  is now a contradiction to  $G_{abc}$  being a minimal simple AT-witness. ■

**Observation 4.6** A vertex  $v$  is simplicial only if  $v$  is an end vertex of every chordless path that contains  $v$ .

**Proof.** Any vertex that appears as a non end vertex in a chordless path, has two neighbors that are not adjacent. ■

**Definition 4.7** Let  $\{a, b, c\}$  be an AT in a chordal graph  $G$ , and let  $W = \{w \mid w \text{ is a vertex of a chordless } a, b\text{-path, } a, c\text{-path, or } b, c\text{-path in } G\}$ . The graph  $G_{Tabc} = G[W]$  is a thick AT-witness for the AT  $\{a, b, c\}$ .

We denote the neighborhoods of  $a, b$ , and  $c$  in  $G_{Tabc}$  by respectively  $S_a$ ,  $S_b$ , and  $S_c$ , since these are minimal separators in  $G_{Tabc}$  and also in  $G$  by the following two observations.

**Observation 4.8** Let  $G_{Tabc}$  be a thick AT-witness in a chordal graph  $G$ . For any  $x \in \{a, b, c\}$ ,  $x$  is a simplicial vertex and  $S_x = N_{G_{Tabc}}(x)$  is a minimal separator in  $G_{Tabc}$ .

**Proof.** We prove the observation for  $x = a$ ; the other possibilities are symmetric. Every neighbor of  $a$  in  $G_{Tabc}$  appears in a chordless path from  $a$  to either  $b$  or  $c$  or both. Because of the existence of the path  $P_{bc}$  avoiding  $S_a$ , it follows that  $S_a$  is a minimal separator. In a chordal graph, every minimal separator is a clique [4]. Hence  $a$  is simplicial in  $G_{Tabc}$ . ■

**Observation 4.9** Let  $G_{Tabc}$  be a thick AT-witness in a chordal graph  $G$ . Then the set of minimal separators of  $G_{Tabc}$  are exactly the set of minimal  $a, b$ -separators,  $a, c$ -separators, and  $b, c$ -separators of  $G$ .

**Proof.** Every minimal separator of  $G_{T_{abc}}$  separates two simplicial vertices appearing in two different leaf bags of any clique tree of  $G_{T_{abc}}$ . Since  $a, b, c$  are the only simplicial vertices in  $G_{T_{abc}}$ , every minimal separator of  $G_{T_{abc}}$  is a minimal  $a, b$ -separator,  $b, c$ -separator, or  $a, c$ -separator.

Let  $S$  be a minimal  $a, b$ -separator in  $G$ . Then there exist two connected components  $C_a$  and  $C_b$  of  $G - S$ , containing respectively  $a$  and  $b$ , such that  $N_G(C_a) = N_G(C_b) = S$ . For any vertex  $z \in S$  we can now find a chordless shortest path in  $G$  from  $z$  to each of  $a$  and  $b$ , where every intermediate vertex is contained in respectively  $C_a$  and  $C_b$ . By joining these two paths, we get a chordless path from  $a$  to  $b$  that contains  $z$ . Since this holds for any vertex in  $S$ , it follows by the way we defined  $G_{T_{abc}}$  that any minimal  $a, b$ -separator of  $G$  is a minimal  $a, b$ -separator of  $G_{T_{abc}}$ . The argument can be repeated with  $a, c$  and  $b, c$  to show that every minimal  $a, c$ -separator or  $b, c$ -separator of  $G$  is also a minimal separator of  $G_{T_{abc}}$ .

Let  $S$  be a minimal  $a, b$ -separator in  $G_{T_{abc}}$ . Vertex set  $S$  is a subset of a minimal  $a, b$ -separator of  $G$ , since the same chordless paths exist in  $G$ . But  $S$  cannot be a proper subset of a minimal  $a, b$ -separator of  $G$ , since every minimal  $a, b$ -separator of  $G$  is a minimal  $a, b$ -separator in  $G_{T_{abc}}$ , and thus  $S$  would not be a minimal separator in  $G_{T_{abc}}$  otherwise. The argument can be repeated with  $a, c$  and  $b, c$ . ■

**Definition 4.10** *A thick AT-witness  $G_{T_{abc}}$  is minimal if  $G_{T_{abc}} - x$  is AT-free for every  $x \in \{a, b, c\}$ .*

**Observation 4.11** *Let  $G_{T_{abc}}$  be a minimal thick AT-witness in the chordal graph  $G$ . Then  $G_{T_{abc}} - c$  is an interval graph, where  $\{a, b\}$  is a dominating pair.*

**Proof.** The graph  $G' = G_{T_{abc}} - c$  is by definition an interval graph, since  $G_{T_{abc}}$  is a minimal thick AT-witness. For a contradiction assume that  $\{a, b\}$  is not a dominating pair, and thus there exists a path  $P'_{ab}$  from  $a$  to  $b$  in  $G' - N[y]$  for some vertex  $y \in V(G') \setminus \{a, b\}$ . Let  $Q$  be a clique path of  $G'$ . Vertex  $y$  does not appear in any bag of  $Q$  that contains  $a$  or  $b$ , and it does not appear in any bags between the subpaths  $Q_a$  and  $Q_b$  of  $Q$ . Let us without loss of generality assume that  $Q_a$  appears between  $Q_y$  and  $Q_b$  in  $Q$ . We show that  $y$  is then not in any chordless path between any pair of  $a, b, c$ , giving the contradiction. Due to the above assumptions,  $y$  is not contained in the component  $C_b$  of  $G' - N[a]$  that contains  $b$ . Furthermore,  $a$  is a simplicial vertex by Observation 4.8, and  $P'_{ab}$  contains vertices from  $N_{G'}(a)$ , thus  $y \notin N_{G'}(a)$  since  $P'_{ab}$  would not avoid the neighborhood of  $y$  otherwise. The path  $P_{bc} - c$  is contained in  $C_b$  since it contains no vertex of  $N[a]$ , and thus  $y$  is not adjacent to any vertex in  $P_{bc} - c$ . We know that  $cy \notin E(G_{T_{abc}})$ , since by Observation 4.8,  $N_{G_{T_{abc}}}(c)$  is a clique, and thus  $y$  would be adjacent to the neighbor of  $c$  in  $P_{ab}$  if  $cy$  were an edge in  $E(G_{T_{abc}})$ . Now we have a contradiction since  $y$  is not in any chordless path between any pair of  $a, b, c$ . ■

## 4.2 $G$ is chordal and Rule 2 does not apply

**Lemma 4.12** *Let  $G_{T_{abc}}$  be a minimal thick AT-witness in a graph  $G$  to which neither Rule 1 (i.e.  $G$  chordal) nor Rule 2 can be applied. Then at least one of the vertices in the AT  $\{a, b, c\}$  is shallow, and there exists a minimal simple AT-witness  $G_{abc}$ , where  $V(G_{abc}) \subseteq V(G_{T_{abc}})$ .*

**Proof.** Let  $P_{ab}, P_{ac}, P_{bc}$  be shortest chordless paths contained in  $G_{T_{abc}}$ , and let  $G_{abc}$  be defined by  $P_{ab}, P_{ac}, P_{bc}$ . It is clear that  $G_{T_{abc}}$  is minimal only if  $G_{abc}$  is

minimal. By Lemma 4.5, Rule 2, and the fact that  $G_{abc}$  is a minimal AT-witness, we know that at least one of the vertices in  $\{a, b, c\}$  are shallow. ■

**Lemma 4.13** *Let  $G$  be a graph to which neither Rule 1 nor Rule 2 can be applied, and let  $G_{Tabc}$  be a minimal thick AT-witness in  $G$  where  $c$  is shallow. Then every vertex in  $S_c$  is adjacent to every vertex in  $S_a \cup S_b$ .*

**Proof.** Let  $E' = E(G_{Tabc})$ , and let us on the contrary and without loss of generality assume that  $c'a' \notin E'$  for  $c' \in S_c$  and  $a' \in S_a$ . Let  $P_{ab} = (a = v_1, v_2, \dots, v_r = b)$ ,  $P_{bc}$ , and  $P_{ac}$  be the shortest paths used to define a simple AT-witness for  $\{a, b, c\}$ . We will show that either  $\{a', b, c\}$  or  $\{a, v_{r-1}, c\}$  is an AT in a subgraph of  $G_{Tabc}$ , contradicting its minimality.

Vertex set  $\{a', b, c\}$  is an independent set since  $cb \notin E'$ ,  $a'b \notin E'$  due to  $|P_{ab}| > 15-8$  (Rule 2), and  $a'c \notin E'$  because  $c$  is simplicial in  $G_{Tabc}$ , and thus  $c'a' \in E'$  if  $a'c \in E'$ . Either  $v_2 = a'$ , or  $a'v_2 \in E'$  since  $a$  is simplicial in  $G_{Tabc}$ .  $P_{ab} - a + a'$  is a path from  $a'$  to  $b$  that avoids the neighborhood of  $c$ . In the same way  $P_{ac} - a + a'$  is a path from  $a'$  to  $c$ , and since  $|P_{ab}| > 7$  this path avoids the neighborhood of  $b$ . By Observation 4.11,  $c'$  is adjacent to some vertex on the path  $P_{ab} = (a = v_1, v_2, \dots, v_r = b)$ . If  $c'$  is adjacent to some vertex  $v_i$  where  $i > 3$ , then there is a path  $c, c', v_i, \dots, v_r = b$  that avoids the neighborhood of  $a'$ , and we have a contradiction since  $a', b, c$  would be an AT in  $G_{Tabc} - a$ . We can therefore assume that  $v_j c' \in E'$ , where  $j \in \{2, 3\}$ , and that there exists no  $v_i c' \in E'$  for any  $i > 3$ . The set  $\{a, v_{r-1}, c\}$  is an independent set, since  $cv_{r-1}, av_{r-1} \notin E'$ . The path  $a, v_2, \dots, v_{r-1}$  avoids the neighborhood of  $c$ , the path  $c, c', v_j, \dots, a$  avoids the neighborhood of  $v_{r-1}$ , and  $P_{bc} - b + v_{r-1}$  is a path from  $c$  to  $v_{r-1}$  that avoids the neighborhood of  $a$ , since  $b$  is simplicial in  $G_{Tabc}$ . This is a contradiction since  $G_{Tabc} - b$  contains the AT  $\{a, v_{r-1}, c\}$ . ■

**Lemma 4.14** *Let  $G = (V, E)$  be graph to which neither Rule 1 nor Rule 2 can be applied. Let  $G_{Tabc}$  be a minimal thick AT-witness in  $G$  where  $c$  is shallow. Let  $C_c$  be the connected component of  $G - S_c$  that contains  $c$ . Then every vertex of  $C_c$  has in  $G$  the same set of neighbors  $S_c$  outside  $C_c$ , in other words  $\forall u \in C_c : N_G(u) \setminus C_c = S_c$ .*

**Proof.** By definition  $N_G(u) \setminus C_c \subseteq S_c$ . Let us assume for a contradiction that  $ux \notin E$  for some  $x \in S_c$  and  $u \in C_c$ . Since  $C_c$  is a connected component there exists a path from  $u$  to  $c$  inside  $C_c$ . Let  $u', c'$  be two consecutive vertices on this path, such that  $S_c \subseteq N_G(c')$  and  $u'x' \notin E$  for some  $x' \in S_c$ . This is a contradiction, since by Lemma 4.13  $x'$  creates a short path from  $a$  to  $b$  that avoids the neighborhood of  $u'$ , and by using  $P_{ac} - c$  and  $P_{bc} - c$  and the vertices  $c'$  and  $u'$  we can create short paths from  $a$  to  $u'$  and from  $b$  to  $u'$  that avoid the neighborhoods of  $b$  and  $a$ . This is now a contradiction, since  $\{a, b, u'\}$  is an AT with a simple AT-witness where the number of branching fill edges are 5 for the path  $a, a', x', b', b$ , 5 for  $P_{ac} - c$  and  $c', u'$ , and 5 for  $P_{bc} - c$  and  $c', u'$ , giving a total of 15 branching edges. ■

**Lemma 4.15** *Let  $G$  be graph to which neither Rule 1 nor Rule 2 can be applied, and let  $G_{Tabc}$  be a thick AT-witness in  $G$ . Then there exists a minimal thick AT-witness  $G_{Txyz}$  in  $G$ , where  $V(G_{Txyz}) \subseteq V(G_{Tabc})$  and  $z$  is shallow, such that  $z \in \{a, b, c\}$ .*

**Proof.**  $G_{Txyz}$  will be obtained from  $G_{Tabc}$  by deleting one of the simplicial vertices in the AT that defines  $G_{Tabc}$ , and repeat this until a minimal thick AT-witness  $G_{Txyz}$  is obtained. Note that only neighbors of the deleted vertex can become

simplicial after each deletion, by Observation 4.6. As a result, the deleted vertices induce at most three connected components, where each of the components is adjacent to one of the vertices  $x, y, z$ . By Lemma 4.12 and Lemma 4.5, one of the vertices  $x, y, z$  is shallow. Let us without loss of generality assume that  $z$  is the shallow vertex in  $G_{T_{xyz}}$ . By Lemma 4.9, minimal separators of  $G_{T_{xyz}}$  are also minimal separators of  $G_{T_{abc}}$ , so let us assume without loss of generality that  $z$  and  $c$  is contained in the same connected component of  $G_{T_{abc}} - N_{G_{T_{xyz}}}(z)$ . Notice that  $z$  and  $c$  might be the same vertex. By Lemma 4.14,  $c$  is shallow in the minimal thick AT-witness  $G_{T_{xyc}}$ . ■

**Definition 4.16** *Given a graph  $G$  to which Rules 1 and 2 do not apply we compute a set  $C(G) = C_1 \cup C_2 \cup \dots \cup C_r$  of vertices that are shallow in some minimal thick AT-witness, with  $G \setminus C(G) = R_r$  an interval graph, as follows:*

```

 $R_0 := G; i := 0; C(G) := \emptyset;$ 
while  $R_i$  is not an interval graph do
   $i := i + 1;$ 
  Find  $G_{T_{a_i b_i c_i}}$  a minimal thick AT-witness in  $R_{i-1}$  with  $c_i$  shallow;
  Let  $C_i$  be the connected component of  $R_{i-1} - N_{G_{T_{a_i b_i c_i}}}(c_i)$  that contains  $c_i$ ;
  for each  $c \in C_i$  do  $G_{T_{a_i b_i c}} := G_{T_{a_i b_i c_i}} - c_i + c$ ;
   $R_i := R_{i-1} - C_i;$ 
   $C(G) := C(G) \cup C_i;$ 
end-while
 $r := i$ 

```

One operation remains unexplained, namely how a minimal thick AT-witness is computed. For each triple  $\{a, b, c\}$  check if the triple is an AT in  $R_i$ . Let  $R'_i = R_i$ . While there exists a simplicial vertex in  $R'_i$  different from  $a, b, c$  remove the vertex from  $R'_i$ . Due to Observation 4.6 we have now obtained a thick AT-witness  $G_{T_{abc}}$  for the AT  $\{a, b, c\}$ . Check if  $G_{T_{abc}} - a$ ,  $G_{T_{abc}} - b$ , and  $G_{T_{abc}} - c$  are interval graphs, and thus if  $G_{T_{abc}}$  is a minimal thick AT-witness.

Note that we also computed graphs  $G = R_0 \supset R_1 \supset \dots \supset R_r$ , with  $R_r$  interval, and a minimal thick AT-witness for each  $c \in C(G)$  (from the thick minimal AT-witness  $G_{T_{a_i b_i c_i}}$  with  $c_i \in C_i$ , we defined, for any  $c \in C_i$ , the graph  $G_{T_{a_i b_i c}} := G_{T_{a_i b_i c_i}} - c_i + c$ , which will be a thick minimal AT-witness for  $\{a_i, b_i, c\}$  with  $c$  shallow by Lemma 4.14) that will be used in the next section. First we give Branching Rule 3.

### Branching Rule 3:

This rule applies if Rules 1 and 2 do not apply and  $|C(G)| > k$ , in which case we let  $B$  be a subset of  $C(G)$ , where  $|B| = k + 1$ . For each  $c \in B$  find a simple AT-witness  $G_{abc}$  where  $c$  is shallow, with shortest paths  $P_{bc}$  and  $P_{ac}$  avoiding  $N(a)$  and  $N(b)$ , respectively.

- For each  $c \in B$ , branch on the at most 8 fill edges  $\{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}$ .
- Branch on the at most  $|B|(|B| - 1)/2$  possible fill edges  $\{uv \mid u, v \in B \text{ and } uv \notin E\}$ .

Observe that Rule 3, only requires that  $|C(G)| > k$ , and thus an algorithm can stop the computation of  $C(G)$  when this size is reached.

**Lemma 4.17** *Any  $k$ -interval completion of a graph  $G$ , where neither Rule 1 or 2 can be applied contains a fill edge which is branched on by Rule 3.*

**Proof.** In a  $k$ -interval completion we cannot add more than  $k$  fill edges. Thus, since  $|B| = k + 1$  any  $k$ -interval completion  $H$  of  $G$  either contains a fill edge between two vertices in  $B$  (and all these are branched on by Rule 3), or there exists a vertex  $c \in B$  with no fill edge incident to it (since the opposite would require  $k + 1$  fill edges). If  $c \in B$  does not have a fill edge incident to it, then by Observation 3.1 one of the edges in  $\{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\}$  must be a fill edge (and all these are branched on for each  $c \in B$  by Rule 3). ■

## 5 More branching and a greedy completion: Rule 4

In this section we present the fourth and final rule and prove correctness of the resulting FPT algorithm. We now consider graphs  $G$  to which none of the Rules 1, 2, or 3 can be applied. This means that  $G$  is chordal (Rule 1), that  $|C(G)| \leq k$  (Rule 3), implying that (the components of)  $G[C(G)]$  is an interval graph (Rule 2). As in the second and third rules the fourth rule will branch on single fill edges, but it will also consider minimal separators, based on the following two basic observations.

**Observation 5.1** *If  $G$  has a minimal thick AT-witness  $G_{Tabc}$  in which  $P_{ac}, P_{bc}$  are shortest paths avoiding  $N(b)$  and  $N(a)$  respectively, then any interval completion of  $G$  either contains a fill edge from the set  $\{bx \mid x \in P_{ac}\} \cup \{ax \mid x \in P_{bc}\}$  or contains one of the edge sets  $\{\{cx \mid x \in S\} \mid S \text{ is a minimal } a, b\text{-separator in } G_{Tabc}\}$ .*

**Proof.** By Observation 3.1, we know that at least one of the edges in  $\{ax \mid x \in P_{bc}\} \cup \{bx \mid x \in P_{ac}\} \cup \{cx \mid x \in P_{ab}\}$  for the paths  $P_{ab}, P_{ac}, P_{bc}$  defined in the proof of Lemma 4.12, is a fill edge of any interval completion of  $G$ . If an interval completion  $H$  does not contain any fill edge from the set  $\{bx \mid x \in P_{ac}\} \cup \{ax \mid x \in P_{bc}\}$ , then  $H$  contains at least one fill edge from the set  $\{cx \mid x \in P'_{ab}\}$ , where  $P'_{ab}$  is any chordless  $a, b$ -path in  $G$  that avoids the neighborhood of  $c$ . Thus,  $N_H(c)$  contains a minimal  $a, b$ -separator in  $G$  (which by Observation 4.9 is also a minimal  $a, b$ -separator in  $G_{Tabc}$ ) since every chordless and thus every  $a, b$ -path in  $G - N[c]$  contains a vertex of  $N_H(c)$ . ■

**Observation 5.2** *Let  $G$  be a graph to which neither Rule 1 nor 2 can be applied, and let  $G_{Tabc}$  be a minimal thick AT-witness in  $G$  where  $c$  is shallow. Then  $S_c \subset S$  for every minimal  $a, b$ -separator  $S$  different from  $S_a$  and  $S_b$ .*

**Proof.** Let  $S$  be a minimal  $a, b$ -separator different from  $S_a$  and  $S_b$ .  $S$  is then also a minimal  $a', b'$ -separator for some  $a' \in S_a$  and some  $b' \in S_b$ , since no minimal  $a, b$ -separator contains another minimal  $a, b$ -separator as a subset. It then follows from Lemma 4.13 that  $S_c \subset N(a') \cap N(b')$ , and thus  $S_c \subset S$ . ■

Recall that  $C(G) = C_1 \cup C_2 \cup \dots \cup C_r$  was computed in Definition 4.16 by removing from  $G$  the vertex sets  $C_i$  in order from  $i = 1$  to  $r$ . A priori we have no guarantee that there are no edges between a vertex in  $C_i$  and a vertex in  $C_j$ , for some  $i \neq j$ , but when  $|C(G)| \leq k$  this indeed holds, as shown in the following lemma.

**Lemma 5.3** *Let  $G = (V, E)$  be a graph to which none of Rules 1, 2, 3 can be applied, and let  $C(G) = C_1 \cup C_2 \cup \dots \cup C_r$  from Definition 4.16. Then  $C_i$  induces an interval graph that is a connected component of  $G[C(G)]$ , for each  $1 \leq i \leq r$ .*

**Proof.** Firstly, since  $|C_i| \leq k$  and Rules 1, 2 do not apply, it must induce an interval graph. To argue that it is a connected component, note first that by definition  $G[C_i]$  is connected and  $C_i \cap C_j = \emptyset$  for any  $i \neq j$ . For a contradiction we assume that  $cz \in E$  for some  $c \in C_i$  and  $z \in C_j$  with  $i < j$ . Let  $G_{Tabc}$  be the minimal thick AT-witness in  $R_{i-1}$  with  $c$  the shallow vertex and  $S_c = N_{G_{Tabc}}(c)$ , and let likewise  $G_{Txyz}$  be the minimal thick AT-witness in  $R_{j-1}$  with  $z$  shallow and  $S_z = N_{G_{Txyz}}(z)$ . Let  $P_{ab}$  be a path from  $a$  to  $b$  in  $G_{Tabc} \setminus N(c)$ . There are now two cases:

Case I: There is a vertex  $w \in P_{ab} \cap S_z$ . By Observation 4.9 both  $S_c$  and  $S_z$  are minimal separators in the chordal graph  $G$ , and thus  $S_c, S_z$  are cliques [4]. Thus, since  $cw \notin E$  we must have  $c \notin S_z$ . But then we have  $c$  and  $z$  in the same component  $C_z$  of  $G \setminus S_z$ . By Lemma 4.14  $c$  and  $z$  must therefore have the same neighbors outside  $C_z$ . But this contradicts the fact that  $zw \in E$  while  $cw \notin E$ .

Case II:  $P_{ab} \cap S_z = \emptyset$ . Let  $C_z$  be the connected component of  $G \setminus S_z$  that contains  $z$ . By Lemma 4.13 we have  $zw \in E$  for some  $w \in P_{ab}$  and therefore  $V(P_{ab}) \subseteq C_z$ . By Lemma 4.5 and the fact that Rule 2 cannot be applied we have at least  $k + 16 - 8$  vertices in  $P_{ab}$  and thus  $|C_z| \geq |P_{ab}| > k$ . Assuming we can show the subset-property  $C_z \subseteq C_1 \cup C_2 \cup \dots \cup C_j$  we are done with the proof since this will lead to the contradiction  $k < |C_z| \leq |C_1 \cup C_2 \cup \dots \cup C_j| \leq |C(G)| \leq k$ . Let us prove the subset-property.  $G$  has a perfect elimination ordering starting with the vertices of  $C_1$ , as these vertices are a component resulting from removing a minimal separator from  $G$ . By induction, we have that  $G$  has a perfect elimination ordering  $\alpha$  starting with the vertices in  $C_1 \cup C_2 \cup \dots \cup C_{j-1}$ . For a contradiction assume there exists a vertex  $w \in C_z \setminus (C_1 \cup C_2 \cup \dots \cup C_j)$ . As  $w \in C_z$  there is a shortest  $w, z$ -path  $P_{wz}$  in  $C_z$ . Since  $zw \notin E$ ,  $P_{wz}$  contains at least 3 vertices and one of these vertices belongs to some  $C_i$ , if not  $w$  would belong to  $C_j$ . Let  $s$  be the first vertex in the ordering  $\alpha$  that belongs to the path  $P_{wz}$ . This is now a contradiction since a none end vertex of a chordless path cannot be simplicial. ■

Rule 4 will branch on a bounded number of single fill edges and it will also compute a greedy completion by choosing for each shallow vertex a minimal separator minimizing fill and making the shallow vertex adjacent to all vertices of that separator. We will prove that if none of the single fill edges branched on in Rule 4 are present in any  $k$ -interval completion, then the greedy completion gives a minimum interval completion. The greedy choices of separators are made as follows:

**Definition 5.4** *Let  $G$  be a graph where Rules 1,2,3 do not apply. Let Definition 4.16 give  $C(G) = C_1 \cup C_2 \cup \dots \cup C_r$ , representative vertices  $c_1, c_2, \dots, c_r$  and minimal thick AT-witnesses  $G_{T_{a_i b_i c_i}}$  and graphs  $G = R_0 \supset R_1 \supset \dots \supset R_r$ , with  $R_r$  interval. We compute fill-minimizing minimal separators  $M_1$  to  $M_r$  as follows:*

```

for  $i := 1$  to  $r$  do
   $M_i := \text{null}$ ;
  for each minimal  $a_i, b_i$ -separator  $S$  in  $G_{T_{a_i b_i c_i}}$  do
    if  $S \cap C(G) = \emptyset$  and  $S \neq S_{a_i}$  and  $S \neq S_{b_i}$  and  $S \neq N(C_j)$  for all  $1 \leq j \leq r$  then
      if  $M_i = \text{null}$  or  $|S \setminus N(C_i)| < |M_i \setminus N(C_i)|$  then  $M_i := S$ ;
end-for

```

**Lemma 5.5** *If  $M_i \neq \text{null}$  then  $M_i$  is a minimal separator in  $R_r$  for any  $1 \leq i \leq r$ .*

**Proof.** The vertex set  $M_i$  is a minimal separator in  $G_{T_{a_i b_i c_i}}$  by construction and since  $G_{T_{a_i b_i c_i}}$  is a subgraph of the chordal graph  $R_i$  it is by Observation 4.9 also a minimal separator of  $R_i$ . We prove that  $M_i$  is also a minimal separator in  $R_j$  for any  $i+1 \leq j \leq r$  by induction on  $j$ . Recall that  $R_j$  is obtained by removing  $C_j$  from  $R_{j-1}$ , where  $C_j$  is a component of  $R_{j-1} \setminus S_{c_i}$  for a minimal separator  $S_{c_i}$  of  $R_{j-1}$ , and  $S_{c_i} = N(C_j)$  by Lemma 4.14. Consider a clique tree of  $R_{j-1}$  and observe that any minimal separator of  $R_{j-1}$  that is not a minimal separator of  $R_j$  is either equal to  $N(C_j)$  or it contains a vertex of  $C_j$ . Finally, note that the minimal separator  $M_i$  has been chosen so that it is not of this type. ■

**Branching Rule 4:**

Rule 4 applies if none of Rules 1, 2, 3 apply, in which case we compute, as in Definitions 4.16 and 5.4,  $C_1, C_2, \dots, C_r$  (which are components of  $G[C(G)]$  by Lemma 5.3), the minimal thick AT-witnesses  $G_{T_{a_i b_i c}}$  with  $c$  shallow for each  $c \in C_i$ , and  $M_1, \dots, M_r$  (which are separators of  $R_r$  by Lemma 5.5). For each  $1 \leq i \leq r$  and each  $c \in C_i$  choose  $a'_i \in S_{a_i} \setminus S_c$  and  $b'_i \in S_{b_i} \setminus S_c$  and find  $P_{a_i c}$  and  $P_{b_i c}$  (shortest paths in  $G_{T_{a_i b_i c}}$  avoiding  $N(b_i)$  and  $N(a_i)$ , respectively, of length at most 4 by Lemma 4.12). For each pair  $1 \leq i \neq j \leq r$ , choose a vertex  $v_{i,j} \in N(C_j) \setminus N(C_i)$  (if it exists).

- For  $1 \leq i \leq r$  and  $c \in C_i$ , branch on the at most 8 fill edges  $\{a_i x \mid x \in P_{b_i c}\} \cup \{b_i x \mid x \in P_{a_i c}\}$  and also on the 2 fill edges  $\{ca'_i, cb'_i\}$ .
- Branch on the at most  $|C(G)|(|C(G)|-1)/2$  fill edges  $\{uv \mid u, v \in C(G) \text{ and } uv \notin E\}$ .
- Branch on the at most  $|C(G)|r$  fill edges  $\bigcup_{1 \leq i \neq j \leq r} \{cv_{i,j} \mid c \in C_i\}$ .
- Finally, compute  $H = (V, E \bigcup_{1 \leq i \leq r} \{cx \mid c \in C_i \text{ and } x \in M_i\})$  and check if it is a  $k$ -interval completion of  $G$  (note that we do not branch on  $H$ ).

**Lemma 5.6** *If  $G$  has a  $k$ -interval completion, and Rules 1, 2, or 3 do not apply to  $G$ , and no  $k$ -interval completion of  $G$  contains any single fill edge branched on by Rule 4, then the graph  $H$ , which Rule 4 obtains by adding fill edges from every vertex in  $C_i$  to every vertex in  $M_i$  for every  $1 \leq i \leq r$ , is a  $k$ -interval completion of  $G$ .*

**Proof.** By Observation 5.1, for each  $c \in C_i$  either one of the edges in  $\{a_i x \mid x \in P_{b_i c}\} \cup \{b_i x \mid x \in P_{a_i c}\}$  is a fill edge (and all these are branched on by Rule 4) or else the  $k$ -interval completion contains the edge set  $\{cx \mid x \in S\}$  for some minimal  $a_i, b_i$ -separator in  $G_{T_{a_i b_i c}}$ . Such an edge set in a  $k$ -interval completion is one of four types depending on the separator  $S$  used to define it. For each type and any  $c \in C_i$  we argue that Rule 4 considers it. Observe that  $N(C_i) \setminus C_i = N(c) \setminus C_i$  by Lemma 4.14, and thus the fill edges from  $c$  will go to vertices in  $S \setminus N(C_i)$ , which is nonempty since there is an  $a_i, b_i$ -path avoiding  $N(c)$ . We now give the four types of minimal separators  $S$ , and show that the first three are branched on by a single fill edge:

1.  $S \cap C(G) \neq \emptyset$ . Since  $N(C_i) \cap C(G) = \emptyset$  by Lemma 5.3, we have in this case a fill edge between two vertices in  $C(G)$  (between  $c \in C_i$  and a vertex in  $C(G) \cap S \setminus N(C_i)$ ) and all these are branched on by Rule 4.
2.  $S = S_{a_i}$  or  $S = S_{b_i}$ , where  $S_{a_i}, S_{b_i}, S_c$  defined by  $G_{T_{a_i b_i c}}$ . We found in Rule 4 a pair of vertices  $a'_i \in S_{a_i} \setminus S_c$  and  $b'_i \in S_{b_i} \setminus S_c$  and branched on the fill edges  $ca'_i$  and  $cb'_i$ .



3.  $S = N(C_j)$  for some  $1 \leq j \leq r$ . If  $S = N(C_j)$  then  $N(C_j) \setminus N(C_i) \neq \emptyset$  and we found in Rule 4 a vertex  $v_{i,j} \in N(C_j) \setminus N(C_i)$  and branched on the fill edge  $cv_{i,j}$ .
4.  $S$  is neither of the three types above. Note that  $M_i$  was chosen in Definition 5.4 by looping over all minimal  $a_i, b_i$ -separators  $S$  in  $G_{Ta_i b_i c_i}$  (which by Lemma 4.14 are exactly the minimal  $a_i, b_i$ -separators of  $G_{Ta_i b_i c}$ ) satisfying  $S \cap C(G) = \emptyset$ ,  $S \neq S_a$ ,  $S \neq S_b$ , and  $S \neq N(C_j)$  for any  $j$ . Thus, of all separators of this fourth type,  $M_i$  is the one minimizing the fill.

The assumption is that  $G$  has a  $k$ -interval completion but no single edge branched on by Rule 4 is present in any  $k$ -interval completion. This means that only separators of the fourth type are used in any  $k$ -interval completion. Since  $H$  added the minimum possible fill while using only separators of the fourth type any interval completion of  $G$  must add at least  $|E(H) \setminus E(G)|$  edges. It remains to show that  $H$  is an interval graph.  $H$  is constructed from an interval graph  $R_r$  and the components  $G[C_1], \dots, G[C_r]$  of  $G[C(G)]$ , which are interval graphs by Lemma 5.3, and  $M_1, \dots, M_r$  which are minimal separators of  $R_r$  by Lemma 5.5. Since  $M_i \neq S_{a_i}$  and  $M_i \neq S_{b_i}$  we have by Observation 5.2 that  $S_c = N(C_i) \subset M_i$  so that adding all edges between  $C_i$  and  $M_i$  for  $1 \leq i \leq r$  gives the graph  $H$ . We show that  $H$  is an interval graph by induction on  $0 \leq i \leq r$ . Let  $H_0 = R_r$  and let  $H_i$  for  $i \geq 1$  be the graph we get from  $H_{i-1}$  and  $C_i$  by making all vertices of  $C_i$  adjacent to all vertices of the minimal separator  $M_i$  of  $R_r$ .  $H_0$  is an interval graph by induction, and its minimal separators include all minimal separators of  $R_r$ . If  $(K_1, K_2, \dots, K_q)$  is a clique path of  $H_{i-1}$  with  $M_i = K_j \cap K_{j+1}$ , and  $(K'_1, K'_2, \dots, K'_p)$  is a clique path of  $G[C_i]$  then  $(K_1, K_2, \dots, K_j, K'_1 \cup M_i, K'_2 \cup M_i, \dots, K'_p \cup M_i, K_{j+1}, \dots, K_q)$  is a clique path of  $H_i$ , and hence  $H_i$  is an interval graph. Finally, observe that the minimal separators of  $H_{i-1}$  and hence of  $R_r$  are also minimal separators of  $H_i$ . ■

**Theorem 5.7** *The search tree algorithm applying Rules 1,2,3,4 in that order will decide in  $O(k^{2k}n^3m)$  time whether an input graph  $G$  on  $n$  vertices and  $m$  edges can be completed into an interval graph by adding at most  $k$  edges. Thus  $k$ -Interval Completion is FPT.*

**Proof.** At least one of the rules will apply to any graph which is not interval. Each rule takes time at most  $O(n^3m)$  (e.g. checking for each triple if it is an AT, and if so computing a thick AT-witness) and has branching factor at most  $k^2$  (e.g. branching on fill edges between pairs of shallow vertices). A branching performed by Rule 1 adds a set of fill edges and drops the value of  $k$  by the cardinality of this set. Its correctness is well understood [13, 2]. A branching performed by the other rules adds a single fill edge and drops  $k$  by one. The height of the tree is thus no more than  $k$ , before  $k$  reaches 0 and we can answer “no”. If an interval graph is found we answer “yes”. The correctness of Rules 2 and 3 follow by Observations 3.1 and 5.1. The Correctness of Rule 4 follows by Lemma 5.6. ■

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