A 4-approximation for the line-column paths colouring problem in bi-directed meshes networks

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Abstract : *We study the row-column chain coloring problem in directed meshes (each directed chain is of one out of eight possible types). The decision problem is known to be NP-complete, and an 8-approximation algorithm has been provided for the associated optimization problem [KT03]. We improve on this result by providing a 4-approximation algorithm, thus catching up with the best non directed result known to us [BCP06].*

1. Introduction

Motivated by all-optical networks applications [KT03, BCP06], the input to the *directed paths colouring* decision problem is a digraph *G*, a collection $\mathcal{P}$ of directed paths of *G* (called a routing of *G*) and a positive integer *k*. The answer is *yes* when *k* colours suffice to colour each directed path so that no two paths using a common directed edge bear the same colour.

This problem is NP-complete [GJMP80] and, unless P = NP, there are no polynomial algorithms to solve the associated optimization problem, namely the *minimum directed paths colouring* problem, within a constant-ratio approximation (a proof is given when the problem is restrected to paths in meshes [Pal02]). Thus, attention has been paid to subproblems such as, for instance, restricting to paths in trees [EJ01] or row-column paths in meshes [Pal02, KT03].

Given some digraph *G*, *V(G)* and *E(G)* denote its set of vertices and its set of directed edges respectively. The *conflicts graph* $H(G,\mathcal{P})$ of a collection $\mathcal{P}$ of directed paths of a given digrah *G* is the graph whose vertices are the directed paths of $\mathcal{P}$, two directed paths being adjacent in $H(G,\mathcal{P})$ iff they use a common directed edge of *G*. Thus, an instance of the *directed paths coloring* decision problem has a *yes* answer iff $k \geq X(H(G, P))$, where $X(H(G, P))$ denotes the chromatic number of $H(G,\mathcal{P})$.

Given some collection $\mathcal{P}$ of directed paths in a digraph *G*, the *load of a directed edge e* of *G* is the number of paths of $\mathcal{P}$ which *e* belongs to. The *load* of the directed paths $\mathcal{P}$ is then the load of one of its orientd edge of maximal load. If *L* is the load of $\mathcal{P}$ and $\mathcal{X}$ is the chromatic numbre of the conflicts graph, then clearly $X \geq L$. Different *k*-approximating algorithms have taken advantage from the fact that they use $k \times L$ colours at most.

In this paper, we focus on so-called row-column directed paths in symmetrically directed meshes (see below). To our knowledge, the best approximation algorithms are 8-approximation algorithms [KT03]. We here provide an algorithm which uses $4L$ colours at most, therefore a 4-approximation algorithm.

Given some integer i, $\overrightarrow{P_{[i]}}$ is the digraph where $V(P_{[i]}) = \{0, 1, ..., i-1, i\}$ and $E(P_{[i]}) = \{(0, 1), (1, 2), ..., (i-1, i)\}$. Any digraph isomorphic to $P_{[i]}$ is called an *directed path*. The *first vertex* (resp. *last vertx*) of a directed path is its vertex of null in-going degree (resp. out-going degree). The directed edge incident to its first vertex (resp. last vertex) is its *first directed edge* (resp. *last directed edge*). A *symmetrical path* is a graph isomorphic to $\overleftrightarrow{P_{[i]}}$ where $V(\overleftrightarrow{P_{[i]}}) = V(\overrightarrow{P_{[i]}})$ and where $E(\overleftrightarrow{P_{[i]}}) = E(\overrightarrow{P_{[i]}}) \cup \{(i, i-1), ..., (2, 1), (1, 0)\}$.

The *cartesian product* $G \times G'$ of two digraphs G and G' is the digraph whose vertices are directed pairs (x,x') where x is a vertex of G and x' a vertex of G' and such that $((x, x'), (y, y'))$ is a directed edge iff $x = y$ and (x', y') is a directed edge of G', or $x' = y'$ and (x, y) is a directed edge of G.

Given two positive integers i and j, $\overleftrightarrow{M_{[i,j]}}$ is the cartesian product of the directed path $\overrightarrow{P_{[i]}}$ with the directed path $\overrightarrow{P_{[j]}}$. Every digraph isomorphic to $\overleftrightarrow{M_{[i,j]}}$ is a *symmetrically directed mesh*, or simply a *mesh* in order to make it short in the sequel. A directed edge of $\overleftrightarrow{M_{[i,j]}}$ is a *R-edge* if it is of the form $((i, j), (i, j+1))$, a *L-edge* if it is of the form $((i, j), (i, j-1))$, a *U-edge* if it is of the form ((i,j),(i-1,j)) and a *D-edge* if it is of the form ((i,j),(i+1,j)).

A *R-path* (resp. *L-path*, *U-path*, *D-path*) is a directed path whose all edges are R-edges (resp. L-edges, U-edges, D-edges).

A *RU-path* is a directed path which is the union of a R-path and a U-path. The *corner* of a RU-path is its vertex common to its R-path and its U-path. The R-edge incident to the corner of a RU-path is its *corner-in-going* edge and the U-edge incident to its corner is its *corner-out-going*) edge. *RD-paths*, *LU-paths*, *LD-paths*, *UR-paths*, *UL-paths*, *DR-paths* and *DL-paths* are defined accordingly, with their corners, corner-in-going edges and corner-out-going edges. Paths of these eight different types are the *row-column* directed paths of a mesh.

2. The algorithm

In this section we provide a 4-approximate algorithm to solve the directed paths colouring problem when restricted to row-column paths in directed meshes. The row-column paths are partitionned in two classes, the UR, RD, UL and LD-paths on the one hand, and the DL, RU, DR and LU-paths on the other hand. The algorithm deals with each class at a time, colouring the paths of a class with at most $2L$ coulours, where L is the routing load. As colours are different from class to class, the algorithm uses at most $4L$ coulours on the whole, and from now on, we focus on one class only, namely the class of UR-paths, RD-paths, UL-paths and LD-paths.

For each of these four directed paths types, a particular vertex is elected to stand for its *anchor*:

- the anchor of an UR-path is its last vertex;
- the anchor of a RD-path is its corner;
- the anchor of an UL-path is its corner;
- the anchor of a LD-path is its first vertex.

Given some instance (M, $\mathcal{P}$) of the directed paths colouring problem restricted to row-column paths in directed meshes, the *A* algorithm given below calls a procedure `Colour(P,i,j)` which greedily colours every directed paths, either a UR, RD, UL or a LD-path, whose anchor is vertex (i,j).

```
For i←0 to i=n
        For j←n to j=0
                Colour(P,i,j)
        EndFor
EndFor
```

Proposition :

Algorithm A is polynomial and uses $2L$-1 coulours at most to colour every UR, RD, UL and LD-path of any given routing, where L is the routing load.

Proof :

The algorithm is clearly polynomial. We prove by induction on (i,j), increasing as specified in algorithm *A*, that no more than $2L$ coulours are needed to colour each of the directed paths (of the specified types) whose anchor is (i,j). When (i, j) is (0, n), one can check that L coulours suffice. Now let us assume that $2L$-1 coulours at most have been used by algorithm *A* to colour every directed path whose anchor precedes vertex (i, j) and consider colouring all directed paths whose anchor is (i, j) : let us call this the current step, and let P be one of the directed paths to be coloured at current step.

Facts :

- if P is an UR-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an UR-path, a RD-path or an UL-path which uses P last or corner-in-going directed edge;
- if P is an RD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an LD-path, a RD-path or an UR-path which uses P corner-in-going or corner-out-going directed edge;
- if P is an UL-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an LD-path, a UR-path or an UL-path which uses P corner-in-going or corner-out-going directed edge;
- if P is an LD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an LD-path, a RD-path or an UL-path which uses P first or corner-out-going directed edge.

As in each case no more than L-1 paths use each of the two addressed directed edges, current step is performed using no more than $2L$-1 colours. Which leads to conclusion.

Applying algorithm *A* successively to the UR, RD, UL and LD-paths on the one hand, and, using new colours, to the UD, RU, UR and LU-paths on the other hand straightforwardly leads to the following :

Theorem :

The directed paths colouring problem restricted to row-column paths in directed meshes is 4-approximable by an algorithm which solves the problem using 4*L*-2 coulours at most.

3. Conclusion

In the directed case, the colouring problem for row-column directed paths in a symmetrically directed mesh is known to be NP-complete and 8-approximation algorithms have already been proposed. We have presented here a 4-approximate algorithm for this problem. To our knowledge, this result improves by a factor 2 previously known results, and catch on with unoriented case results where meshes and paths are unoriented ones and for which a 4-approximate result is known [BCP06].

Conveying our proof to the unoriented case leads to another proof of the 4-approximation result in the unoriented mesh. But conversely, the proof given in [BCP06] to a 4-approximation in the unoriented case, which generalizes to meshes of dimension *d*, has not yet seem to apply to the directed case. This could be investigated further more.

4. References

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