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To cite this version:

HAL Id: lirmm-00121842
https://hal-lirmm.ccsd.cnrs.fr/lirmm-00121842
Submitted on 22 Dec 2006

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A 4-approximation for the line-column paths colouring problem in bi-directed meshes networks

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december 2006
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Abstract: We study the row-column chain coloring problem in directed meshes (each directed chain is of one out of eight possible types). The decision problem is known to be NP-complete, and an 8-approximation algorithm has been provided for the associated optimization problem [KT03]. We improve on this result by providing a 4-approximation algorithm, thus catching up with the best non directed result known to us [BCP06].

1. Introduction

Motivated by all-optical networks applications [KT03, BCP06], the input to the directed paths colouring decision problem is a digraph $G$, a collection $P$ of directed paths of $G$ (called a routing of $G$) and a positive integer $k$. The answer is yes when $k$ colours suffice to colour each directed path so that no two paths using a common directed edge bear the same colour.

This problem is NP-complete [GJMP80] and, unless $P = NP$, there are no polynomial algorithms to solve the associated optimization problem, namely the minimum directed paths colouring problem, within a constant-ratio approximation (a proof is given when the problem is restricted to paths in meshes [Pal02]). Thus, attention has been paid to subproblems such as, for instance, restricting to paths in trees [EJ01] or row-column paths in meshes [Pal02, KT03].

Given some digraph $G$, $V(G)$ and $E(G)$ denote its set of vertices and its set of directed edges respectively. The conflicts graph $H(G, P)$ of a collection $P$ of directed paths of a given digraph $G$ is the graph whose vertices are the directed paths of $P$, two directed paths being adjacent in $H(G, P)$ iff they use a common directed edge of $G$. Thus, an instance of the directed paths coloring decision problem has a yes answer iff $k \geq X(H(G, P))$, where $X(H(G, P))$ denotes the chromatic number of $H(G, P)$.

Given some collection $P$ of directed paths in a digraph $G$, the load of a directed edge $e$ of $G$ is the number of paths of $P$ which $e$ belongs to. The load of the directed paths $P$ is then the load of one of its orientd edge of maximal load. If $L$ is the load of $P$ and $X$ is the chromatic number of the conflicts graph, then clearly $X \geq L$. Different $k$-approximating algorithms have taken advantage from the fact that they use $k \times L$ colours at most.
In this paper, we focus on so-called row-column directed paths in symmetrically directed meshes (see below). To our knowledge, the best approximation algorithms are 8-approximation algorithms [KT03]. We here provide an algorithm which uses 4L colours at most, therefore a 4-approximation algorithm.

Given some integer \( i \), \( P_i \) is the digraph where \( V(P_i) = \{0, 1, ..., i - 1, i\} \) and \( E(P_i) = \{(0, 1), (1, 2), ..., (i - 1, i)\} \). Any digraph isomorphic to \( P_i \) is called an directed path. The first vertex (resp. last vertex) of a directed path is its vertex of null in-going degree (resp. out-going degree). The directed edge incident to its first vertex (resp. last vertex) is its first directed edge (resp. last directed edge). A symmetrical path is a graph isomorphic to \( P_i \) where \( V(P_i) \) = \( V(P_i) \) and where \( E(P_i) = E(P_i) \cup \{(i, i - 1), ..., (2, 1), (1, 0)\} \).

The cartesian product \( G \times G' \) of two digraphs \( G \) and \( G' \) is the digraph whose vertices are directed pairs \((x, x')\) where \( x \) is a vertex of \( G \) and \( x' \) a vertex of \( G' \) and such that \(((x, x'), (y, y'))\) is a directed edge iff \( x = y \) and \((x', y')\) is a directed edge of \( G' \), or \( x' = y' \) and \((x, y)\) is a directed edge of \( G \).

Given two positive integers \( i \) and \( j \), \( M_{i,j} \) is the cartesian product of the directed path \( P_i \) with the directed path \( P_j \). Every digraph isomorphic to \( M_{i,j} \) is a symmetrically directed mesh, or simply a mesh in order to make it short in the sequel. A directed edge of \( M_{i,j} \) is a \( R \)-edge if it is of the form \(((i, j), (i, j + 1))\), a \( L \)-edge if it is of the form \(((i, j), (i, j - 1))\), a \( U \)-edge if it is of the form \(((i,j),(i-1,j))\) and a \( D \)-edge if it is of the form \(((i,j),(i+1,j))\).

A \( R \)-path (resp. \( L \)-path, \( U \)-path, \( D \)-path) is a directed path whose all edges are \( R \)-edges (resp. \( L \)-edges, \( U \)-edges, \( D \)-edges).

A \( RU \)-path is a directed path which is the union of a \( R \)-path and a \( U \)-path. The corner of a \( RU \)-path is its vertex common to its \( R \)-path and its \( U \)-path. The \( R \)-edge incident to the corner of a \( RU \)-path is its corner-in-going edge and the \( U \)-edge incident to its corner is its corner-out-going) edge. \( RD \)-paths, \( LU \)-paths, \( LD \)-paths, \( UR \)-paths, \( UL \)-paths, \( DR \)-paths and \( DL \)-paths are defined accordingly, with their corners, corner-in-going edges and corner-out-going edges. Paths of these eight different types are the row-column directed paths of a mesh.

2. The algorithm

In this section we provide a 4-approximate algorithm to solve the directed paths colouring problem when restricted to row-column paths in directed meshes. The row-column paths are partitioned in two classes, the \( UR \), \( RD \), \( UL \) and \( LD \)-paths on the one hand, and the \( DL \), \( RU \), \( DR \) and \( LU \)-paths on the other hand. The algorithm deals with each class at a time, colouring the paths of a class with at most \( 2L \) colours, where \( L \) is the routing load. As colours are different from class to class, the algorithm uses at most \( 4L \) colours on the whole, and from now on, we focus on one class only, namely the class of \( UR \)-paths, \( RD \)-paths, \( UL \)-paths and \( LD \)-paths.
For each of these four directed paths types, a particular vertex is elected to stand for its anchor:

- the anchor of an UR-path is its last vertex;
- the anchor of a RD-path is its corner;
- the anchor of an UL-path is its corner;
- the anchor of a LD-path is its first vertex.

Given some instance \((M, P)\) of the directed paths colouring problem restricted to row-column paths in directed meshes, the A algorithm given below calls a procedure \(\text{Colour}(P, i, j)\) which greedily colours every directed paths, either a UR, RD, UL or a LD-path, whose anchor is vertex \((i, j)\).

For \(i=0\) to \(i=n\)
  For \(j=n\) to \(j=0\)
    \(\text{Colour}(P, i, j)\)
  EndFor
EndFor

**Proposition:**

Algorithm A is polynomial and uses \(2L-1\) colours at most to colour every UR, RD, UL and LD-path of any given routing, where \(L\) is the routing load.

**Proof:**

The algorithm is clearly polynomial. We prove by induction on \((i, j)\), increasing as specified in algorithm A, that no more than \(2L\) colours are needed to colour each of the directed paths (of the specified types) whose anchor is \((i, j)\). When \((i, j)\) is \((0, n)\), one can check that \(L\) colours suffice. Now let us assume that \(2L-1\) colours at most have been used by algorithm A to colour every directed path whose anchor precedes vertex \((i, j)\) and consider colouring all directed paths whose anchor is \((i, j)\) : let us call this the current step, and let \(P\) be one of the directed paths to be coloured at current step.

**Facts:**

- if \(P\) is an UR-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with \(P\) must be an UR-path, a RD-path or an UL-path which uses \(P\) last or corner-in-going directed edge;
- if \(P\) is an RD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with \(P\) must be an LD-path, a RD-path or an UR-path which uses \(P\) corner-in-going or corner-out-going directed edge;
- if \(P\) is an UL-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with \(P\) must be an LD-path, a UR-path or an UL-path which uses \(P\) corner-in-going or corner-out-going directed edge;
- if \(P\) is an LD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with \(P\) must be an LD-path, a RD-path or an UL-path which uses \(P\) first or corner-out-going directed edge.

As in each case no more than \(L-1\) paths use each of the two addressed directed edges, current step is performed using no more than \(2L-1\) colours. Which leads to conclusion.
Applying algorithm A successively to the UR, RD, UL and LD-paths on the one hand, and, using new colours, to the UD, RU, UR and LU-paths on the other hand straightforwardly leads to the following:

**Theorem:**

The directed paths colouring problem restricted to row-column paths in directed meshes is 4-approximable by an algorithm which solves the problem using $4L-2$ colours at most.

### 3. Conclusion

In the directed case, the colouring problem for row-column directed paths in a symmetrically directed mesh is known to be NP-complete and 8-approximation algorithms have already been proposed. We have presented here a 4-approximate algorithm for this problem. To our knowledge, this result improves by a factor 2 previously known results, and catch on with unoriented case results where meshes and paths are unoriented ones and for which a 4-approximate result is known [BCP06].

Conveying our proof to the unoriented case leads to another proof of the 4-approximation result in the unoriented mesh. But conversely, the proof given in [BCP06] to a 4-approximation in the unoriented case, which generalizes to meshes of dimension $d$, has not yet seem to apply to the directed case. This could be investigated further more.

### 4. References


