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# A 4-approximation for the line-column paths colouring problem in bi-directed meshes networks

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**Abstract :** We study the row-column chain coloring problem in directed meshes (each directed chain is of one out of eight possible types). The decision problem is known to be NP-complete, and an 8-approximation algorithm has been provided for the associated optimization problem [KT03]. We improve on this result by providing a 4-approximation algorithm, thus catching up with the best non directed result known to us [BCP06].

## 1. Introduction

Motivated by all-optical networks applications [KT03, BCP06], the input to the *directed paths colouring* decision problem is a digraph  $G$ , a collection  $\mathcal{P}$  of directed paths of  $G$  (called a routing of  $G$ ) and a positive integer  $k$ . The answer is *yes* when  $k$  colours suffice to colour each directed path so that no two paths using a common directed edge bear the same colour.

This problem is NP-complete [GJMP80] and, unless  $P = NP$ , there are no polynomial algorithms to solve the associated optimization problem, namely the *minimum directed paths colouring* problem, within a constant-ratio approximation (a proof is given when the problem is restricted to paths in meshes [Pal02]). Thus, attention has been paid to subproblems such as, for instance, restricting to paths in trees [EJ01] or row-column paths in meshes [Pal02, KT03].

Given some digraph  $G$ ,  $V(G)$  and  $E(G)$  denote its set of vertices and its set of directed edges respectively. The *conflicts graph*  $H(G, \mathcal{P})$  of a collection  $\mathcal{P}$  of directed paths of a given digraph  $G$  is the graph whose vertices are the directed paths of  $\mathcal{P}$ , two directed paths being adjacent in  $H(G, \mathcal{P})$  iff they use a common directed edge of  $G$ . Thus, an instance of the *directed paths coloring* decision problem has a *yes* answer iff  $k \geq X(H(G, \mathcal{P}))$ , where  $X(H(G, \mathcal{P}))$  denotes the chromatic number of  $H(G, \mathcal{P})$ .

Given some collection  $\mathcal{P}$  of directed paths in a digraph  $G$ , the *load* of a directed edge  $e$  of  $G$  is the number of paths of  $\mathcal{P}$  which  $e$  belongs to. The *load* of the directed paths  $\mathcal{P}$  is then the load of one of its oriented edge of maximal load. If  $L$  is the load of  $\mathcal{P}$  and  $X$  is the chromatic number of the conflicts graph, then clearly  $X \geq L$ . Different  $k$ -approximating algorithms have taken advantage from the fact that they use  $k \times L$  colours at most.

In this paper, we focus on so-called row-column directed paths in symmetrically directed meshes (see below). To our knowledge, the best approximation algorithms are 8-approximation algorithms [KT03]. We here provide an algorithm which uses  $4L$  colours at most, therefore a 4-approximation algorithm.

Given some integer  $i$ ,  $\overrightarrow{P}_{[i]}$  is the digraph where  $V(\overrightarrow{P}_{[i]}) = \{0, 1, \dots, i-1, i\}$  and  $E(\overrightarrow{P}_{[i]}) = \{(0, 1), (1, 2), \dots, (i-1, i)\}$ . Any digraph isomorphic to  $\overrightarrow{P}_{[i]}$  is called an *directed path*. The *first vertex* (resp. *last vertex*) of a directed path is its vertex of null in-going degree (resp. out-going degree). The directed edge incident to its first vertex (resp. last vertex) is its *first directed edge* (resp. *last directed edge*). A *symmetrical path* is a graph isomorphic to  $\overleftrightarrow{P}_{[i]}$  where  $V(\overleftrightarrow{P}_{[i]}) = V(\overrightarrow{P}_{[i]})$  and where  $E(\overleftrightarrow{P}_{[i]}) = E(\overrightarrow{P}_{[i]}) \cup \{(i, i-1), \dots, (2, 1), (1, 0)\}$ .

The *cartesian product*  $G \times G'$  of two digraphs  $G$  and  $G'$  is the digraph whose vertices are directed pairs  $(x, x')$  where  $x$  is a vertex of  $G$  and  $x'$  a vertex of  $G'$  and such that  $((x, x'), (y, y'))$  is a directed edge iff  $x = y$  and  $(x', y')$  is a directed edge of  $G'$ , or  $x' = y'$  and  $(x, y)$  is a directed edge of  $G$ .

Given two positive integers  $i$  and  $j$ ,  $\overleftrightarrow{M}_{[i, j]}$  is the cartesian product of the directed path  $\overrightarrow{P}_{[i]}$  with the directed path  $\overrightarrow{P}_{[j]}$ . Every digraph isomorphic to  $\overleftrightarrow{M}_{[i, j]}$  is a *symmetrically directed mesh*, or simply a *mesh* in order to make it short in the sequel. A directed edge of  $\overleftrightarrow{M}_{[i, j]}$  is a *R-edge* if it is of the form  $((i, j), (i, j+1))$ , a *L-edge* if it is of the form  $((i, j), (i, j-1))$ , a *U-edge* if it is of the form  $((i, j), (i-1, j))$  and a *D-edge* if it is of the form  $((i, j), (i+1, j))$ .

A *R-path* (resp. *L-path*, *U-path*, *D-path*) is a directed path whose all edges are R-edges (resp. L-edges, U-edges, D-edges).

A *RU-path* is a directed path which is the union of a R-path and a U-path. The *corner* of a RU-path is its vertex common to its R-path and its U-path. The R-edge incident to the corner of a RU-path is its *corner-in-going* edge and the U-edge incident to its corner is its *corner-out-going* edge. *RD-paths*, *LU-paths*, *LD-paths*, *UR-paths*, *UL-paths*, *DR-paths* and *DL-paths* are defined accordingly, with their corners, corner-in-going edges and corner-out-going edges. Paths of these eight different types are the *row-column* directed paths of a mesh.

## 2. The algorithm

In this section we provide a 4-approximate algorithm to solve the directed paths colouring problem when restricted to row-column paths in directed meshes. The row-column paths are partitionned in two classes, the UR, RD, UL and LD-paths on the one hand, and the DL, RU, DR and LU-paths on the other hand. The algorithm deals with each class at a time, colouring the paths of a class with at most  $2L$  colours, where  $L$  is the routing load. As colours are different from class to class, the algorithm uses at most  $4L$  colours on the whole, and from now on, we focus on one class only, namely the class of UR-paths, RD-paths, UL-paths and LD-paths.

For each of these four directed paths types, a particular vertex is elected to stand for its *anchor*:

- the anchor of an UR-path is its last vertex;
- the anchor of a RD-path is its corner;
- the anchor of an UL-path is its corner;
- the anchor of a LD-path is its first vertex.

Given some instance  $(M, P)$  of the directed paths colouring problem restricted to row-column paths in directed meshes, the *A* algorithm given below calls a procedure  $\text{colour}(P, i, j)$  which greedily colours every directed paths, either a UR, RD, UL or a LD-path, whose anchor is vertex  $(i, j)$ .

```

For i←0 to i=n
  For j←n to j=0
    Colour(P, i, j)
  EndFor
EndFor

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**Proposition :**

Algorithm *A* is polynomial and uses  $2L-1$  colours at most to colour every UR, RD, UL and LD-path of any given routing, where  $L$  is the routing load.

**Proof :**

The algorithm is clearly polynomial. We prove by induction on  $(i, j)$ , increasing as specified in algorithm *A*, that no more than  $2L$  colours are needed to colour each of the directed paths (of the specified types) whose anchor is  $(i, j)$ . When  $(i, j)$  is  $(0, n)$ , one can check that  $L$  colours suffice. Now let us assume that  $2L-1$  colours at most have been used by algorithm *A* to colour every directed path whose anchor precedes vertex  $(i, j)$  and consider colouring all directed paths whose anchor is  $(i, j)$  : let us call this the current step, and let  $P$  be one of the directed paths to be coloured at current step.

Facts :

- if  $P$  is an UR-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with  $P$  must be an UR-path, a RD-path or an UL-path which uses  $P$  last or corner-in-going directed edge;
- if  $P$  is an RD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with  $P$  must be an LD-path, a RD-path or an UR-path which uses  $P$  corner-in-going or corner-out-going directed edge;
- if  $P$  is an UL-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with  $P$  must be an LD-path, a UR-path or an UL-path which uses  $P$  corner-in-going or corner-out-going directed edge;
- if  $P$  is an LD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with  $P$  must be an LD-path, a RD-path or an UL-path which uses  $P$  first or corner-out-going directed edge.

As in each case no more than  $L-1$  paths use each of the two addressed directed edges, current step is performed using no more than  $2L-1$  colours. Which leads to conclusion.

Applying algorithm  $A$  successively to the UR, RD, UL and LD-paths on the one hand, and, using new colours, to the UD, RU, UR and LU-paths on the other hand straightforwardly leads to the following :

**Theorem :**

The directed paths colouring problem restricted to row-column paths in directed meshes is 4-approximable by an algorithm which solves the problem using  $4L-2$  colours at most.

## 3. Conclusion

In the directed case, the colouring problem for row-column directed paths in a symmetrically directed mesh is known to be NP-complete and 8-approximation algorithms have already been proposed. We have presented here a 4-approximate algorithm for this problem. To our knowledge, this result improves by a factor 2 previously known results, and catch on with unoriented case results where meshes and paths are unoriented ones and for which a 4-approximate result is known [BCP06].

Conveying our proof to the unoriented case leads to another proof of the 4-approximation result in the unoriented mesh. But conversely, the proof given in [BCP06] to a 4-approximation in the unoriented case, which generalizes to meshes of dimension  $d$ , has not yet seem to apply to the directed case. This could be investigated further more.

## 4. References

[BCP06] - unpublished - *Guillaume Bagan, Olivier Cogis et Jérôme Palaysi. Approximable Row-column Routing Problems in All-Optical Mesh Networks. 2006*

[EJ01] - journal article - *Thomas Erlebach et Klaus Jansen. The complexity of path coloring and call scheduling. Theoretical Computer Science. 2006.*

[GJMP80] - journal article - *M. R. Garey, D. S. Johnson, G. L. Miller and C. H. Papadimitriou. The complexity of coloring circular arcs and chords. Journal on Algorithms and Discrete Methods. 1980.*

[KT03] - inproceedings - *Mustapha Kchikech et Olivier Togni. Paths Coloring Algorithms in Mesh Networks. Discrete Mathematics and Theoretical Computer Science : 4TH International Conference, DMTCS 2003 Dijon, France, July 7-12, 2003 Proceedings. 2003*

[Pal02] - inproceedings - *Jérôme Palaysi. Approximability of Paths Coloring Problem in Mesh and Torus Networks. Mathematics and Computer Science II : Algorithms, Trees, Combinatorics and Probabilities. 2002*