A 4-approximation for the line-column paths colouring problem in bi-directed meshes networks

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Abstract : *We study the row-column chain coloring problem in directed meshes (each directed chain is of one out of eight possible types). The decision problem is known to be NP-complete, and an 8-approximation algorithm has been provided for the associated optimization problem [KT03]. We improve on this result by providing a 4-approximation algorithm, thus catching up with the best non directed result known to us [BCP06].*

1. Introduction

Motivated by all-optical networks applications [KT03, BCP06], the input to the *directed paths colouring* decision problem is a digraph *G*, a collection $\mathcal{P}$ of directed paths of *G* (called a routing of *G*) and a positive integer *k*. The answer is *yes* when *k* colours suffice to colour each directed path so that no two paths using a common directed edge bear the same colour.

This problem is NP-complete [GJMP80] and, unless P = NP, there are no polynomial algorithms to solve the associated optimization problem, namely the *minimum directed paths colouring* problem, within a constant-ratio approximation (a proof is given when the problem is restrected to paths in meshes [Pal02]). Thus, attention has been paid to subproblems such as, for instance, restricting to paths in trees [EJ01] or row-column paths in meshes [Pal02, KT03].

Given some digraph *G*, *V(G)* and *E(G)* denote its set of vertices and its set of directed edges respectively. The *conflicts graph* $H(G,\mathcal{P})$ of a collection $\mathcal{P}$ of directed paths of a given digrah *G* is the graph whose vertices are the directed paths of $\mathcal{P}$, two directed paths being adjacent in $H(G,\mathcal{P})$ iff they use a common directed edge of *G*. Thus, an instance of the *directed paths coloring* decision problem has a *yes* answer iff $k \geq X(H(G, P))$, where $X(H(G, P))$ denotes the chromatic number of $H(G,\mathcal{P})$.

Given some collection $\mathcal{P}$ of directed paths in a digraph *G*, the *load of a directed edge e* of *G* is the number of paths of $\mathcal{P}$ which *e* belongs to. The *load* of the directed paths $\mathcal{P}$ is then the load of one of its orientd edge of maximal load. If *L* is the load of $\mathcal{P}$ and $\mathcal{X}$ is the chromatic numbre of the conflicts graph, then clearly $X \geq L$. Different *k*-approximating algorithms have taken advantage from the fact that they use $k \times L$ colours at most.

In this paper, we focus on so-called row-column directed paths in symmetrically directed meshes (see below). To our knowledge, the best approximation algorithms are 8-approximation algorithms [KT03]. We here provide an algorithm which uses $4L$ colours at most, therefore a 4-approximation algorithm.

Given some integer i, $\overrightarrow{P_{[i]}}$ is the digraph where $V(\overrightarrow{P_{[i]}}) = \{0, 1, ..., i-1, i\}$ and $E(\overrightarrow{P_{[i]}}) = \{(0, 1), (1, 2), ..., (i-1, i)\}$. Any digraph isomorphic to $\overrightarrow{P_{[i]}}$ is called an *directed path*. The *first vertex* (resp. *last vertx*) of a directed path is its vertex of null in-going degree (resp. out-going degree). The directed edge incident to its first vertex (resp. last vertex) is its *first directed edge* (resp. *last directed edge*). A *symmetrical path* is a graph isomorphic to $\overleftrightarrow{P_{[i]}}$ where $V(\overleftrightarrow{P_{[i]}}) = V(\overrightarrow{P_{[i]}})$ and where $E(\overleftrightarrow{P_{[i]}}) = E(\overrightarrow{P_{[i]}}) \cup \{(i, i-1), ..., (2, 1), (1, 0)\}$.

The *cartesian product* $G \times G'$ of two digraphs G and G' is the digraph whose vertices are directed pairs (x,x') where x is a vertex of G and x' a vertex of G' and such that $((x, x'), (y, y'))$ is a directed edge iff $x = y$ and (x', y') is a directed edge of G', or $x' = y'$ and (x, y) is a directed edge of G.

Given two positive integers i and j, $\overleftrightarrow{M_{[i,j]}}$ is the cartesian product of the directed path $\overrightarrow{P_{[i]}}$ with the directed path $\overrightarrow{P_{[j]}}$. Every digraph isomorphic to $\overleftrightarrow{M_{[i,j]}}$ is a *symmetrically directed mesh*, or simply a *mesh* in order to make it short in the sequel. A directed edge of $\overleftrightarrow{M_{[i,j]}}$ is a *R-edge* if it is of the form $((i, j), (i, j+1))$, a *L-edge* if it is of the form $((i, j), (i, j-1))$, a *U-edge* if it is of the form ((i,j),(i-1,j)) and a *D-edge* if it is of the form ((i,j),(i+1,j)).

A *R-path* (resp. *L-path*, *U-path*, *D-path*) is a directed path whose all edges are R-edges (resp. L-edges, U-edges, D-edges).

A *RU-path* is a directed path which is the union of a R-path and a U-path. The *corner* of a RU-path is its vertex common to its R-path and its U-path. The R-edge incident to the corner of a RU-path is its *corner-in-going* edge and the U-edge incident to its corner is its *corner-out-going*) edge. *RD-paths*, *LU-paths*, *LD-paths*, *UR-paths*, *UL-paths*, *DR-paths* and *DL-paths* are defined accordingly, with their corners, corner-in-going edges and corner-out-going edges. Paths of these eight different types are the *row-column* directed paths of a mesh.

2. The algorithm

In this section we provide a 4-approximate algorithm to solve the directed paths colouring problem when restricted to row-column paths in directed meshes. The row-column paths are partitionned in two classes, the UR, RD, UL and LD-paths on the one hand, and the DL, RU, DR and LU-paths on the other hand. The algorithm deals with each class at a time, colouring the paths of a class with at most $2L$ coulours, where L is the routing load. As colours are different from class to class, the algorithm uses at most $4L$ coulours on the whole, and from now on, we focus on one class only, namely the class of UR-paths, RD-paths, UL-paths and LD-paths.

For each of these four directed paths types, a particular vertex is elected to stand for its *anchor*:

- the anchor of an UR-path is its last vertex;
- the anchor of a RD-path is its corner;
- the anchor of an UL-path is its corner;
- the anchor of a LD-path is its first vertex.

Given some instance (*M*, *P*) of the directed paths colouring problem restricted to row-column paths in directed meshes, the *A* algorithm given below calls a procedure `Colour(P,i,j)` which greedily colours every directed paths, either a UR, RD, UL or a LD-path, whose anchor is vertex (i,j).

```
For i←0 to i=n
        For j←n to j=0
                Colour(P,i,j)
        EndFor
EndFor
```

Proposition :

Algorithm A is polynomial and uses 2L-1 coulours at most to colour every UR, RD, UL and LD-path of any given routing, where L is the routing load.

Proof :

The algorithm is clearly polynomial. We prove by induction on (i,j), increasing as specified in algorithm *A*, that no more than 2L coulours are needed to colour each of the directed paths (of the specified types) whose anchor is (i,j). When (i, j) is (0, n), one can check that L coulours suffice. Now let us assume that 2L-1 coulours at most have been used by algorithm *A* to colour every directed path whose anchor precedes vertex (i, j) and consider colouring all directed paths whose anchor is (i, j) : let us call this the current step, and let P be one of the directed paths to be coloured at current step.

Facts :

- if P is an UR-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an UR-path, a RD-path or an UL-path which uses P last or corner-in-going directed edge;
- if P is an RD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an LD-path, a RD-path or an UR-path which uses P corner-in-going or corner-out-going directed edge;
- if P is an UL-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an LD-path, a UR-path or an UL-path which uses P corner-in-going or corner-out-going directed edge;
- if P is an LD-path, every directed path which is already coloured or which is to be coloured at current step and which is in conflict with P must be an LD-path, a RD-path or an UL-path which uses P first or corner-out-going directed edge.

As in each case no more than L-1 paths use each of the two addressed directed edges, current step is performed using no more than 2L-1 colours. Which leads to conclusion.

Applying algorithm *A* successively to the UR, RD, UL and LD-paths on the one hand, and, using new colours, to the UD, RU, UR and LU-paths on the other hand straightforwardly leads to the following :

Theorem :

The directed paths colouring problem restricted to row-column paths in directed meshes is 4-approximable by an algorithm which solves the problem using $4L$-2 coulours at most.

3. Conclusion

In the directed case, the colouring problem for row-column directed paths in a symmetrically directed mesh is known to be NP-complete and 8-approximation algorithms have already been proposed. We have presented here a 4-approximate algorithm for this problem. To our knowledge, this result improves by a factor 2 previously known results, and catch on with unoriented case results where meshes and paths are unoriented ones and for which a 4-approximate result is known [BCP06].

Conveying our proof to the unoriented case leads to another proof of the 4-approximation result in the unoriented mesh. But conversely, the proof given in [BCP06] to a 4-approximation in the unoriented case, which generalizes to meshes of dimension *d*, has not yet seem to apply to the directed case. This could be investigated further more.

4. References

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