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# INITIAL POWERS OF STURMIAN SEQUENCES

VALÉRIE BERTHÉ, CHARLES HOLTON, AND LUCA Q. ZAMBONI

ABSTRACT. In this paper we investigate powers of prefixes of Sturmian sequences. We give an explicit formula for  $\text{ice}(\omega)$ , the initial critical exponent of a Sturmian sequence  $\omega$ , defined as the supremum of all real numbers  $p > 0$  for which there exist arbitrary long prefixes of  $\omega$  of the form  $u^p$ , in terms of its  $S$ -adic representation. This formula is based on Ostrowski's numeration system. Furthermore we characterize those irrational slopes  $\alpha$  of which there exists a Sturmian sequence  $\omega$  beginning in only finitely many powers of  $2 + \varepsilon$ , that is for which  $\text{ice}(\omega) = 2$ . In the process we recover the known results for the index (or critical exponent) of a Sturmian sequence. We also focus on the Fibonacci Sturmian shift and prove that the set of Sturmian sequences with  $\text{ice}$  strictly smaller than its everywhere value has Hausdorff dimension 1.

## 1. INTRODUCTION.

There are a number of recent papers on powers of words occurring in Sturmian sequences (see for instance [2, 3, 8, 9, 18, 17, 28, 34, 42, 45]). Quantities of interest include the supremum of powers of factors of a sequence (the *index* or *critical exponent* of the sequence), and the limit superior of powers of longer and longer factors of the sequence. It is well-known that these numbers are finite if and only if the partial quotients of the continued fraction expansion of the slope of the Sturmian sequence are bounded (see [33]). An explicit formula for the index of a Sturmian sequence was given by Vandeth (see Theorem 16 in [45]) in terms of the partial quotients of its slope.

This paper deals with powers of factors occurring at the beginning of Sturmian sequences, which we call initial powers. The work is motivated in part by a simple observation about the Fibonacci Sturmian shift, the shift space of all Sturmian sequences of slope  $\frac{1+\sqrt{5}}{2}$ . This space is infinite, minimal and uniquely ergodic; one might expect prefix powers to be somewhat uniform. Yet its characteristic sequence begins in no  $\frac{3+\sqrt{5}}{2} \approx 2.62$  power at all, while every sequence outside the shift orbit of the characteristic sequence begins in arbitrarily long words repeated 3 or more times. This example leads us to define the *initial critical exponent* of a sequence  $\omega$  over a finite alphabet, denoted  $\text{ice}(\omega)$ , as the supremum of all real numbers  $p > 0$  for which there exist arbitrarily long prefixes  $u$  of  $\omega$  such that  $u^p$  is also a prefix of  $\omega$ . We obtain an explicit formula for the initial critical exponent of a Sturmian sequence, in terms of a particular  $S$ -adic expansion. For characteristic Sturmian sequences, our formula for  $\text{ice}$  has probably been known since [36], though Hedlund and Morse did not address this question specifically. One can also obtain the formula for  $\text{ice}$  of a characteristic sequence using Cassaigne's formula for the recurrence quotient in [13]. See also [9, 46].

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Every Sturmian sequence  $\omega$  on the alphabet  $\{0, 1\}$  admits a unique  $S$ -adic representation as an infinite composition of the form

$$\omega = T^{c_1} \circ \tau_0^{a_1} \circ T^{c_2} \circ \tau_1^{a_2} \circ T^{c_3} \circ \tau_0^{a_3} \circ T^{c_4} \circ \tau_1^{a_4} \circ \dots,$$

where  $T$  denotes the one-sided shift map,  $\tau_0$  and  $\tau_1$  are the morphisms on  $\{0, 1\}^*$  defined by

$$\tau_0(0) = 0 \quad \tau_1(0) = 10,$$

$$\tau_0(1) = 01 \quad \tau_1(1) = 1,$$

$a_k \geq c_k \geq 0$  for all  $k$ ,  $a_k \geq 1$  for  $k \geq 2$ , and if  $c_k = a_k$  then  $c_{k-1} = 0$ . The sequence  $(a_k)_{k \geq 1}$  turns out to be the sequence of partial quotients of the slope (defined as the density of the symbol 1), while  $(c_k)_{k \geq 1}$  is the sequence of digits in the arithmetic *Ostrowski expansion* of the intercept of the Sturmian sequence (see for instance [19, 20, 29, 30, 27, 37, 43, 44] and the references in [10]). From this point of view, the *characteristic* (or *standard*) Sturmian sequence of a particular slope is the one having  $c_k = 0$  for all  $k$ . This expansion of  $\omega$  is just one of many possible expansions as an infinite composition of morphisms (see work of Arnoux [40], Arnoux-Fisher [4], Arnoux-Ferenczi-Hubert [6]). In each case these expansions are intimately linked to the Ostrowski numeration system.

In [3] it is shown that each Sturmian sequence begins in infinitely many squares (see also [18]), and hence  $\text{ice}(\omega) \geq 2$  for all Sturmian sequences  $\omega$ . We show that the value 2 is attainable, and give the following characterization of those slopes for which there is a Sturmian sequence with initial critical exponent equal to 2 :

**Theorem 1.1.** *Let  $\alpha = [0; a_1, a_2, a_3, \dots]$  be an irrational number and let  $X_\alpha$  be the set of all Sturmian sequences of slope  $\alpha$ . Then there is a Sturmian sequence  $\omega \in X_\alpha$  with  $\text{ice}(\omega) = 2$  if and only if for each pair of positive integers  $(s, t)$  with  $s > 1$  there are only finitely many indices  $k$  for which  $(a_k, a_{k+1}) = (s, t)$  or  $(a_k, a_{k+1}, a_{k+2}) = (1, 1, t)$ .*

We also show how to explicitly construct a Sturmian sequence  $\omega \in X_\alpha$  with  $\text{ice}(\omega) = 2$  in case one exists.

Write  $\text{ind}^*(\omega)$  for the limit superior of powers of longer and longer words appearing in a sequence  $\omega$ . We show

**Theorem 1.2.** *Let  $\omega$  be the characteristic Sturmian sequence of slope  $\alpha$ . Then*

$$\text{ind}^*(\alpha) = 1 + \text{ice}(\omega).$$

The paper is organized as follows. After first recalling some basic facts on Sturmian sequences and on  $\text{ice}$ , we introduce in Section 2 two  $S$ -adic representations of Sturmian sequences (additive and multiplicative versions) based on Ostrowski's numeration system, and conclude the section with a characterization of primitive substitutive Sturmian sequences. We derive an explicit formula for  $\text{ice}$  of Sturmian sequence in Section 3. We study general properties of  $\text{ice}$  in Section 4; special attention is given to the Fibonacci shift in Section 4.4: we study the topological properties of the set of values taken by  $\text{ice}$  on the Fibonacci Sturmian shift following [13] and prove that the Hausdorff dimension of the set of Sturmian sequences in the Fibonacci Sturmian shift with  $\text{ice}$  strictly smaller than its everywhere value (which is also its index) equals 1. We end with a proof of Theorem 1.1 in Section 5.

## 2. PRELIMINARIES.

**2.1. Definitions and notation.** Throughout the paper,  $\alpha$  denotes an irrational number in  $(0, 1)$ . Consider two two-interval exchange transformations,  $R_\alpha : [-\alpha, 1 - \alpha) \rightarrow [-\alpha, 1 - \alpha)$  and  $\tilde{R}_\alpha : (-\alpha, 1 - \alpha] \rightarrow (-\alpha, 1 - \alpha]$ , defined by

$$R_\alpha(z) = \begin{cases} z + \alpha & \text{if } z \in [-\alpha, 1 - 2\alpha) \\ z + \alpha - 1 & \text{if } z \in [1 - 2\alpha, 1 - \alpha) \end{cases}$$

and

$$\tilde{R}_\alpha(z) = \begin{cases} z + \alpha & \text{if } z \in (-\alpha, 1 - 2\alpha] \\ z + \alpha - 1 & \text{if } z \in (1 - 2\alpha, 1 - \alpha] \end{cases}.$$

Both can be considered as rotations of angle  $2\pi\alpha$ , since these are conjugate, after identification of points  $-\alpha$  and  $1 - \alpha$ , to a circle rotation. A Sturmian sequence  $\omega$  of *slope*  $\alpha$  is simply the forward itinerary (with respect to the natural partition) of a point  $x \in [-\alpha, 1 - \alpha]$  (called the *intercept*) under the action of one of these transformations, i.e., either

$$\forall k \in \mathbb{N} (\omega_k = 0 \iff R_\alpha^k(x) \in [-\alpha, 1 - 2\alpha))$$

or

$$\forall k \in \mathbb{N} (\omega_k = 0 \iff \tilde{R}_\alpha^k(x) \in (-\alpha, 1 - 2\alpha]).$$

It is clear from this interpretation that the slope of a Sturmian sequence is the density of the symbol 1.

**Notation.** In all that follows, the *coding of the orbit of the point  $y$  with respect to the partition  $(I, J)$  under the action of the two-interval exchange  $E$*  means the sequence  $v \in \{0, 1\}^{\mathbb{N}}$  defined by

$$\forall k \in \mathbb{N} (v_k = 0 \iff E^k(y) \in I).$$

The *complexity function*  $p : \mathbb{N} \rightarrow \mathbb{N}$  for a sequence  $\omega$  is given by

$$p(n) = \text{the number of distinct factors of } \omega \text{ of length } n.$$

Sturmian sequences are exactly those one-sided infinite sequences with complexity  $p(n) = n + 1$  for every  $n$  (see [36, 14]). The set  $X_\alpha$  of all Sturmian sequences of slope  $\alpha$  is an infinite, minimal, uniquely ergodic (one-sided) shift space. The characteristic sequence of slope  $\alpha$  is the unique *left-special* sequence in  $X_\alpha$ , i.e. the sequence having more than one  $T$ -preimage, where as before,  $T$  denotes the shift on  $X_\alpha$ ; this is the sequence with intercept 0 (it is the same for  $R_\alpha$  and  $\tilde{R}_\alpha$ ) and its two shift preimages code respectively the orbits of  $-\alpha$  under  $R_\alpha$  and  $1 - \alpha$  under  $\tilde{R}_\alpha$ . For more details on Sturmian sequences, see [31, 40].

We will use in Section 2.3 and 2.4 the notion of induction of a rotation. The *induced transformation* of the rotation  $R_\alpha$  (or similarly of  $\tilde{R}_\alpha$ ) on the interval  $I$  of  $[-\alpha, 1 - \alpha]$  is defined as follows. For  $x \in I$ , we call *the first return time* of  $x$  in  $I$  and denote by  $n_I(x)$  the smallest integer  $m > 0$  such that  $R_\alpha^m(x) \in I$  ( $m$  is finite since  $\alpha$  is irrational). The induced transformation of  $R_\alpha$  on  $I$  is the map  $x \mapsto R_\alpha^{n_I(x)}(x)$  on  $I$ .

A sequence is called *recurrent* if every factor appears infinitely many times, and *uniformly recurrent* if every factor appears with bounded gaps. A shift space  $(X, T)$  is said to be *linearly recurrent* if there exists a constant  $K$  such that for each clopen set  $U$  generated by a finite word  $u$ , the return time to  $U$  with respect to the shift  $T$  is bounded above by  $K|u|$ . For more details, see for instance [22].

If  $i \in \{0, 1\}$  we denote by  $\bar{i}$  the other symbol in  $\{0, 1\}$ . Thus  $\bar{i} = 1 - i$ ,  $\tau_i(i) = i$ , and  $\tau_i(\bar{i}) = i\bar{i}$ . Throughout the paper we write  $\theta$  for the golden mean,  $(1 + \sqrt{5})/2$ . We use Greek letters  $\omega$  and  $v$  for infinite sequences, and Roman letters  $u, v, w$  for finite words. The length of a word  $w$  over the alphabet  $\{0, 1\}$  is denoted by  $|w|$ . We write  $\mathbb{N}$  for the set of nonnegative integers ( $0 \in \mathbb{N}$ ) and  $\mathbb{N}^*$  for the set of positive integers.

**2.2. Initial critical exponent.** Positive integer powers of a finite word  $w$  are defined by

$$w^1 = w \quad \text{and} \quad w^n = w^{n-1}w \text{ for } n > 1,$$

and for arbitrary  $p \geq 0$ , the  $p$ th power of  $w$  is given by

$$w^p = w^{\lfloor p \rfloor} u$$

where  $u$  is the prefix of  $w$  of length  $\lceil (p - \lfloor p \rfloor) |w| \rceil$ . A word is called *primitive* if it is not an integer power of some shorter word. The *power* of a word  $w$  in a sequence  $\omega$  is the largest  $p$  (possibly  $\infty$ ) so that  $w^p$  is a factor of  $\omega$ . The *prefix power* of a word  $w$  in a sequence  $\omega$  is the largest  $p$  (possibly  $\infty$ ) so that  $w^p$  is a prefix of  $\omega$ . Define *the initial critical exponent* of  $\omega$ , denoted by  $\text{ice}(\omega)$ , as the limit superior of the prefix powers of the words  $\omega[0, n]$  in  $\omega$ . We similarly define  $\text{ind}^*(\omega)$  for a sequence  $\omega$  as the limit superior as  $n$  tends to  $\infty$  of the largest powers of the factors of length  $n$  appearing in  $\omega$ . For a minimal shift space  $X$ , we write  $\text{ind}^*(X)$  for the common value of  $\text{ind}^*$  on sequences of  $X$ . We prove some properties of  $\text{ice}$  and  $\text{ind}^*$ .

**Proposition 2.1.** *Let  $(X, T)$  be a (one-sided) shift space. Then*

- (1) *For any  $\omega \in X$  one has  $\text{ice}(\omega) \leq \text{ice}(T\omega)$ , and if the inequality is strict then  $T\omega$  is the shift image of at least two different members of  $X$ .*
- (2) *If  $(X, T)$  is minimal then  $\max_{\omega \in X} \text{ice}(\omega) = \text{ind}^*(X)$ .*
- (3) *If  $X$  is infinite and minimal then some  $\omega \in X$  has  $\text{ice}(\omega) \leq 1 + \theta = (3 + \sqrt{5})/2$ .*
- (4) *If  $(X, T)$  is minimal with sublinear complexity then  $\text{ice}$  is shift invariant off of the union of a finite set of orbits, hence  $\text{ice}$  is almost everywhere constant with respect to any ergodic Borel measure.*
- (5) *If  $(X, T)$  is linearly recurrent then  $\text{ice}$  is almost everywhere equal to  $\text{ind}^*(X)$  with respect to any invariant Borel measure.*

*Proof.* Let  $\omega \in X$ . If  $w$  is a prefix of  $\omega$  with prefix power  $p$  then the *first right conjugate* of  $w$ , i.e., the word  $v$  obtained from  $w$  by moving the first letter to the end, is a prefix of  $T\omega$  with prefix power  $p - \frac{1}{|w|}$ . The inequality in (1) follows.

Now suppose the inequality in (1) is strict. Then  $\text{ice}(T\omega) > 1$ . Let  $v_k$  be an increasing sequence of prefixes of  $T\omega$  whose corresponding prefix powers  $q_k$  converge to  $\text{ice}(T\omega)$ . Let  $a$  be the first letter of  $\omega$  and let  $b$  be a common last letter for infinitely many of the  $v_k$ . By passing to a subsequence we may assume that  $q_k > 1$  and  $v_k$  ends in  $b$  for all  $k$ . Note that  $a \neq b$ , since otherwise, for all  $k$ , the first left conjugate of  $v_k$  is a prefix of  $\omega$  with prefix power  $q_k + \frac{1}{|v_k|}$  and we obtain a contradiction:

$$\text{ice}(\omega) < \text{ice}(T\omega) = \lim_{k \rightarrow \infty} q_k = \lim_{k \rightarrow \infty} q_k + \frac{1}{|v_k|} \leq \text{ice}(\omega).$$

For each  $k$ ,  $T\omega$  begins in  $v_k v_k^{q_k-1}$  and  $\omega$  begins in  $av_k$ , hence  $av_k^{q_k-1}$  and  $bv_k^{q_k-1}$  are both factors of sequences of  $X$ . But  $|v_k^{q_k-1}| \rightarrow \infty$  and each  $v_k^{q_k-1}$  is a prefix of  $T\omega$ , hence  $aT\omega$  and  $bT\omega$  both belong to  $X$ .

To prove (2) we need the following:

(2') For every  $p \in (0, \text{ind}^*(X))$ , every word which appears in sequences of  $X$  is a prefix of some word whose  $p$ th power appears in sequences of  $X$ .

Proof of (2'). By minimality, if  $w$  appears in sequences of  $X$  then it appears in bounded gaps, i.e., there exists  $N = N(w)$  such that for all  $\omega$  in  $X$ , at least one of  $\omega, T\omega, \dots, T^{N(w)-1}\omega$  begins in  $w$ . Choose  $\eta > 0$  such that  $p + \eta < \text{ind}^*(X)$ , and let  $v$  be a word of length greater than  $N/\eta$  such that  $v^{p+\eta}$  appears in sequences of  $X$ . Then one of the first  $N - 1$  right conjugates of  $v$  has the required property.

Proof of (2). By (2') we can find a sequence  $w_k$  of words which appear in sequences of  $X$ , such that, for each  $k$ ,  $w_k^{p_k}$  is a prefix of  $w_{k+1}$ , where  $p_k \geq 1$  and  $p_k \rightarrow \text{ind}^*(X)$  and  $|w_k| \rightarrow \infty$  as  $k \rightarrow \infty$ . There is a unique  $\omega \in X$  having each  $w_k$  as a prefix, and the construction guarantees  $\text{ice}(\omega) \geq \text{ind}^*(X)$ . We always have  $\text{ice} \leq \text{ind}^*(X)$ , so this completes the proof.

Part (3) follows from [35].

To prove (4), we use Cassaigne's result from [12]: The first difference of the complexity function is bounded if complexity is sublinear. Let  $C > 0$  be an upper bound for the first difference of the complexity. By minimality, every word  $w$  in  $X$  of length  $n$  has at least one left extension, that is, a word  $aw$  occurring in  $X$  for some letter  $a$ ; hence there can be no more than  $C$  words of length  $n$  which have two or more left extensions and the set of sequences  $\omega$  in  $X$  that have more than one shift preimage has at most  $C$  elements.

It suffices to verify part (5) for ergodic measures. Choose a subsequence  $(n_k)_{k \geq 0}$  of the positive integers such that the sequence of maximal powers  $p_k$  of words of length  $n_k$  converges to  $\text{ind}^*(X)$ . Linear recurrence implies that  $(p_k)_{k \geq 0}$  is a bounded sequence, and for any  $\varepsilon > 0$ , the measure of the set of sequences beginning in a word of length  $n_k$  to power at least  $p_k - \varepsilon$  is bounded away from 0. This implies that the set of sequences with  $\text{ice} \geq \text{ind}^*(X) - \varepsilon$  is a set of positive measure. An application of (4) completes the proof.  $\square$

The focus of this paper is on the values of  $\text{ice}$  on the set  $X_\alpha$  of all Sturmian sequences of some fixed irrational slope  $\alpha$ . It follows from known results (see for instance [45]) that

$$\text{ind}^*(\alpha) := \text{ind}^*(X_\alpha) = 2 + \limsup_{k \rightarrow \infty} [a_k; a_{k-1}, \dots, a_1],$$

where

$$[r_0; r_1, r_2, \dots, r_n] = r_0 + \frac{1}{r_1 + \frac{1}{r_2 + \frac{1}{\dots + \frac{1}{r_n}}}}.$$

This implies in particular that any Sturmian sequence contains cubes (see also [9]) and that a Sturmian sequence has finite index if and only if its slope has bounded partial quotients (this last result was due to [34]). See [13] for a study of the topological structure of the set of values taken by the index.

**Lemma 2.2.** *The almost everywhere value of  $\text{ice}$  on  $X_\alpha$  is  $\text{ind}^*(\alpha)$ .*

*Proof.* Suppose first that  $\text{ind}^*(\alpha) = \infty$ . Let  $p > 2$  and  $N \geq 3$ . There is a primitive word  $u$  of length at least  $N$  and a power  $p' \geq Np + 1$  such that  $u^{p'}$  appears in  $X_\alpha$  and the exponent  $p'$  is maximal for words having the same length as  $u$ .

We claim that  $u^{p'-1}$  is *left special*, i.e., both  $0u^{p'-1}$  and  $1u^{p'-1}$  appear in  $X_\alpha$ . To see this, let  $a$  be the last letter of  $u$ . Since  $au^{p'}$  is the same as the first left conjugate of  $u$  to power  $p' + \frac{1}{|u|}$ , maximality of  $p'$  implies that this word does not appear in  $X_\alpha$ . One of the symbols  $b \in \{0, 1\}$  is such that  $bu^{p'}$  appears in  $X_\alpha$ , and we have just shown that  $b \neq a$ . Thus  $au^{p'-1}$  and  $bu^{p'-1}$  both appear in  $X_\alpha$ , the former as a suffix of  $u^{p'}$  and the latter as a prefix of  $bu^{p'}$ .

It is a property of Sturmian sequences that there is exactly one left special word of each length and every left special word  $v$  has two first return words the sum of whose lengths is  $|v| + 2$ . We know that  $u$  is a return word for  $u^{p'-1}$ , and it is a first return word because  $u$  is primitive and  $p' > 3$ . Therefore the other return word is the prefix of  $u^{p'-1}$  of length  $(p' - 2)|u| + 2$ . This implies that the set of points of  $X_\alpha$  beginning in a suffix of  $u^{p'-1}$  of length at least  $(p' - 1)|u|/N$  has measure at least

$$\frac{\left\lceil \frac{(N-1)(p'-1)|u|}{N} \right\rceil}{(p' - 2)|u| + 2} \geq \frac{N - 1}{N}$$

and such points begin in a word of length  $|u|$  to power  $p$ . The result follows easily from this.

In case  $\text{ind}^*(\alpha) < \infty$ , the partial quotients of  $\alpha$  are bounded and  $X_\alpha$  is linearly recurrent following [22]. Part (5) of Proposition 2.1 applies directly.  $\square$

Using the lemma and the formula for  $\text{ind}^*(\alpha)$  above, we see that the a.e. value of ice on  $X_\alpha$  is greater than 4 unless the partial quotients  $a_k$  are eventually 1. Furthermore, for Lebesgue almost every slope  $\alpha \in (0, 1)$  the partial quotients are unbounded, and thus ice is a.e. infinite on  $X_\alpha$ .

**2.3. An additive  $S$ -adic representation.** Let  $\omega \in \{0, 1\}^{\mathbb{N}}$  be a Sturmian sequence of slope  $\alpha$ . Exactly one of the words  $ii$  ( $i \in \{0, 1\}$ ) is a factor of  $\omega$  and there is a unique sequence  $\omega'$  such that  $\omega = T^b(\tau_i(\omega'))$ , where  $b = 0$  if  $\omega$  begins in  $i$  and  $b = 1$  otherwise. The map  $\omega \mapsto \omega'$  on  $X_\alpha$  is really just induction on the longer of the two intervals in the associated two-interval exchange. Specifically, suppose  $\omega$  codes the orbit of a point  $x$ ; if  $x$  is in the longer interval then  $\omega'$  codes the orbit of  $x$  in the induced interval exchange, and if  $x$  is in the other (shorter) interval then  $\omega'$  codes the orbit of the preimage of  $x$  (which is in the longer interval) in the induced interval exchange. With this interpretation it is clear that  $\omega'$  is also Sturmian. Thus we may iterate this “desubstitution” process to obtain our *additive  $S$ -adic expansion*:

**Proposition 2.3.** *Let  $\omega$  be a Sturmian sequence. There exist a sequence of Sturmian sequences  $(\omega^{(n)})_{n \geq 1}$  and two sequences  $(b_n)_{n \geq 1}, (i_n)_{n \geq 1}$  with values in  $\{0, 1\}$  such that*

- (1)  $\omega = T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(\omega^{(n)})$  for each  $n$ ,
- (2)  $(i_n)$  is not eventually constant,
- (3) if  $i_n = i_{n+1}$  and  $b_{n+1} = 0$  then  $b_n = 0$ ,
- (4) if  $i_n \neq i_{n+1}$  then  $b_n$  and  $b_{n+1}$  are not both 1.

*Proof.* The induction process described above gives us the three sequences satisfying assertion (1). If  $(i_n)$  were eventually constant, say  $i_n = i$  for all  $n \geq N$ , then  $\omega$  would contain arbitrary powers of  $\tau_{i_1} \circ \dots \circ \tau_{i_N}(i)$ , which is impossible since  $\omega$  is Sturmian.

Assertions (3) and (4) are easily deduced from the facts that  $\omega_0^{(n)}$  is the first letter of  $T^{b_{n+1}} \circ \tau_{i_{n+1}}$ , i.e.,

$$\omega_0^{(n)} = \begin{cases} i_{n+1} & \text{if } b_{n+1} = 0, \\ \bar{i}_{n+1} & \text{if } b_{n+1} = 1, \end{cases}$$

and

$$b_n = 1 \implies \omega_0^{(n)} = \bar{i}_n.$$

□

It is helpful to think of  $T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}$  as a composition of “inflations” (the  $\tau_{i_m}$ ) and “cuts” (the  $T^{b_m}$ ) where the amount cut after applying  $\tau_{i_m}$  to  $\omega^{(m)}$  is less than the inflated image of the first letter of  $\omega^{(m)}$ , i.e.,  $b_m < |\tau_{i_m}(\omega_0^{(m)})|$ . Extending this notion of  $T$  as the map which cuts off the first letter of a sequence, we shall abuse notation slightly and write  $Tw$  for the suffix of a word  $w$  obtained by deleting the first letter. Let us note that, by definition,  $|T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(\omega_0^{(n)})| \geq 1$  for all  $n$ , hence

$$T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(\omega^{(n)}) = T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(\omega_0^{(n)}) \star \tau_{i_1} \circ \dots \circ \tau_{i_n} \left( \left( \omega_k^{(n)} \right)_{k \geq 1} \right),$$

where, for clarity, we have written  $\star$  for concatenation. It is possible that

$$\left| T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(\omega_0^{(n)}) \right| = 1 \text{ for all } n;$$

This happens, for example, when  $i_n = b_n = n \pmod{2}$ .

The following useful lemma can be proved by straightforward induction.

**Lemma 2.4.** *If  $v$  and  $v'$  are sequences in  $\{0, 1\}$  beginning in different letters and  $\tau$  is any composition of the  $\tau_i$  then the longest common prefix of  $\tau(v)$  and  $\tau(v')$  has length  $|\tau(01)| - 2$ .*

We next show that what we have is indeed an additive  $S$ -adic expansion in the sense of [22, 24]. The important thing is that the sequences  $(i_n)$  and  $(b_n)$  entirely determine  $\omega$  – we do not need to keep track of the  $\omega^{(n)}$ .

**Proposition 2.5.** *Every pair of sequences  $(i_n)_{n \geq 1}, (b_n)_{n \geq 1}$  with values in  $\{0, 1\}$  satisfying (2)–(4) of Proposition 2.3 is the additive  $S$ -adic expansion of a unique Sturmian sequence.*

*Proof.* Suppose  $(i_n), (b_n)$  satisfies (2)–(4) of Proposition 2.3. If  $v, v' \in \{0, 1\}^{\mathbb{N}}$  then it follows from Lemma 2.4 and the previous remarks on cuts and inflations that  $T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(v)$  and  $T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(v')$  have a common prefix of length at least  $|\tau_{i_1} \circ \tau_{i_2} \circ \dots \circ \tau_{i_n}(i_n)| - 1$ , which tends to infinity. Thus  $\bigcap_{n=1}^{\infty} T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(\{0, 1\}^{\mathbb{N}})$  contains of a single point,  $\omega$ . We claim that  $\omega$  is Sturmian. Indeed, if  $v$  is any Sturmian sequence then

$$\omega = \lim_{n \rightarrow \infty} T^{b_1} \circ \tau_{i_1} \circ \dots \circ T^{b_n} \circ \tau_{i_n}(v).$$

The morphisms  $\tau_0$  and  $\tau_1$  are Sturmian (i.e., they take Sturmian sequences to Sturmian sequences, see [31]) and the complexity of a limit is less than or equal to the limit of the complexities, hence  $\omega$  has complexity  $p(n) \leq n + 1$  and is therefore either Sturmian or eventually periodic. It follows from the fact that  $(i_n)$  is not eventually constant that  $\omega$  is not eventually periodic, so  $p(n) \geq n + 1$  and  $\omega$  is Sturmian. One checks by induction that  $\omega$  has  $(i_n), (b_n)$  as its  $S$ -adic expansion. □



Such an expansion will be called the *additive Ostrowski  $S$ -adic expansion* associated with the sequence  $\omega$ . We will see below that Ostrowski expansions in the sense of [37] appear in a natural way when one considers a multiplicative version of these expansions.

**2.4. A multiplicative  $S$ -adic expansion.** A more compact version of the additive  $S$ -adic representation is desirable. As a sequence in  $\{0, 1\}$  we can write

$$i_1 i_2 \dots = 0^{a_1} 1^{a_2} 0^{a_3} 1^{a_4} \dots$$

with  $a_i \geq 1$  for  $i \geq 2$ . Let  $s_k = \sum_{j=1}^k a_j$  and  $c_k = \sum_{n=s_{k-1}+1}^{s_k} b_n$ . For all  $n \geq 1$  we have  $0 \leq c_n \leq a_n$  and if  $c_{n+1} = a_{n+1}$  then  $c_n = 0$ . We also have

$$b_1 b_2 \dots = 0^{a_1 - c_1} 1^{c_1} 0^{a_2 - c_2} 1^{c_2} \dots,$$

and for  $k > 0$

$$\omega = \tau_0^{a_1 - c_1} \circ (T \circ \tau_0)^{c_1} \circ \tau_1^{a_2 - c_2} \circ (T \circ \tau_1)^{c_2} \circ \dots \circ \tau_{k-1}^{a_k - c_k} \circ (T \circ \tau_{k-1 \bmod 2})^{c_k} (\omega^{(s_k)}).$$

To avoid cumbersome notation we shall henceforth write  $\tau_n$  for  $\tau_{n \bmod 2}$ . We can further simplify to obtain

$$\omega = T^{c_1} \tau_0^{a_1} \circ T^{c_2} \tau_1^{a_2} \circ T^{c_3} \tau_0^{a_3} \circ \dots \circ T^{c_k} \tau_{k-1}^{a_k} (\omega^{(s_k)}).$$

Let  $\alpha = [0; a_1 + 1, a_2, a_3, \dots]$ . Set

$$\begin{aligned} p_0 &= 0 & q_0 &= 1 \\ p_1 &= 1 & q_1 &= a_1 + 1 \end{aligned}$$

and for  $k \geq 2$ ,

$$p_k = a_k p_{k-1} + p_{k-2} \quad q_k = a_k q_{k-1} + q_{k-2}.$$

Set  $\delta_{-1} = 1 - \alpha$ ,  $\delta_0 = \alpha$ ,  $\delta_1 = 1 - (a_1 + 1)\alpha$ , and for  $k \geq 2$ ,  $\delta_k = |q_k \alpha - p_k| = (-1)^k (q_k \alpha - p_k)$ . One has

$$\forall k \in \mathbb{N}, \delta_{k-1} = a_{k+1} \delta_k + \delta_{k+1}.$$

The *continued fraction convergents* of  $\alpha$  are the rational numbers  $p_k/q_k$ , which, as the name suggests, converge to  $\alpha$ . The convergents are in a sense best possible rational approximations to  $\alpha$ . The following lemma can be proved by straightforward induction.

**Lemma 2.6.** *Write  $|w|_j$  for the number of occurrences of the letter  $j$  in word  $w$ . Then for  $i \in \{0, 1\}$*

$$(|\tau_0^{a_1} \circ \dots \circ \tau_{k-1}^{a_k}(i)|_0, |\tau_0^{a_1} \circ \dots \circ \tau_{k-1}^{a_k}(i)|_1) = \begin{cases} (q_k - p_k, p_k) & i = k \pmod{2} \\ (q_{k-1} - p_{k-1}, p_{k-1}) & i \neq k \pmod{2}. \end{cases}$$

It follows that the slope of  $\omega$  is equal to  $\lim p_k/q_k = \alpha$ . This means that the  $a_k$  and hence the sequence  $(i_n)$  are determined by the slope of  $\omega$ . Translating the condition on the sequences  $(i_n)$  and  $(b_n)$  to a condition on the  $c_k$ , we have shown how Sturmian sequences of slope  $\alpha = [0; a_1 + 1, a_2, \dots]$  are in one-to-one correspondence with sequences  $(c_k)$  such that  $0 \leq c_k \leq a_k$  and if  $c_{k+1} = a_{k+1}$  then  $c_k = 0$ . In fact we have the following:

**Proposition 2.7.** *Let  $\alpha = [0; a_1 + 1, a_2, a_3, \dots]$ . Let  $\omega$  be a Sturmian sequence which codes the orbit of the point  $x$  under the action of  $R_\alpha$  or  $\tilde{R}_\alpha$ . There exists a sequence of integers  $(c_n)_{n \in \mathbb{N}}$  where*

$$(1) \quad \forall n, \quad \begin{cases} 0 \leq c_n \leq a_n, \\ c_{n+1} = a_{n+1} \Rightarrow c_n = 0, \end{cases}$$

and a sequence of Sturmian sequences  $(v^{(k)})$  such that

$$(2) \quad \forall k, \quad \omega = T^{c_1} \tau_0^{a_1} \circ T^{c_2} \tau_1^{a_2} \circ T^{c_3} \tau_0^{a_3} \circ \dots \circ T^{c_k} \tau_{k-1}^{a_k}(v^{(k)}),$$

and

$$x = \sum_{k=1}^{+\infty} c_k (-1)^{k-1} \delta_{k-1} = \sum_{k=1}^{+\infty} c_k (q_{k-1} \alpha - p_{k-1}).$$

*Proof.* Let us suppose that  $\omega$  codes the orbit of  $x$  in  $[-\alpha, 1 - \alpha)$  under the rotation  $R_\alpha$  with respect to the partition  $([-\alpha, 1 - 2\alpha), [1 - 2\alpha, 1 - \alpha))$  (the  $\tilde{R}_\alpha$  case is similar). We define two-interval exchanges  $E^{(n)}$  for  $n \geq 0$  as follows:

If  $n$  is even then  $E^{(n)} : [-\delta_n, \delta_{n-1}) \rightarrow [-\delta_n, \delta_{n-1})$  is given by

$$E^{(n)}(z) = \begin{cases} z + \delta_n & \text{if } z \in [-\delta_n, -\delta_n + \delta_{n-1}) \\ z - \delta_{n-1} & \text{if } z \in [-\delta_n + \delta_{n-1}, \delta_{n-1}) \end{cases}.$$

If  $n$  is odd then  $E^{(n)} : [-\delta_{n-1}, \delta_n) \rightarrow [-\delta_{n-1}, \delta_n)$  is given by

$$E^{(n)}(z) = \begin{cases} z + \delta_{n-1} & \text{if } z \in [-\delta_{n-1}, -\delta_{n-1} + \delta_n) \\ z - \delta_n & \text{if } z \in [-\delta_{n-1} + \delta_n, \delta_n) \end{cases}.$$

Note that  $E^{(0)}$  equals  $R_\alpha$ . We also define inductively a sequence of points  $(x^{(n)})_{n \geq 0}$  where

$$x^{(n)} \in \begin{cases} [-\delta_n, \delta_{n-1}) & \text{if } n \text{ is even} \\ [-\delta_{n-1}, \delta_n) & \text{if } n \text{ is odd} \end{cases},$$

and a sequence of nonnegative integers  $(c_n)_{n \geq 1}$  by setting  $x^{(0)} = x$ , and for  $n > 0$ :

If  $n$  is even then

$$c_{n+1} = \begin{cases} 0 & \text{if } x^{(n)} \in [-\delta_n, \delta_{n+1}) \\ \left\lfloor \frac{x^{(n)} - \delta_{n+1}}{\delta_n} \right\rfloor + 1 & \text{if } x^{(n)} \in [\delta_{n+1}, \delta_{n-1}) \end{cases}$$

and

$$x^{(n)} = x^{(n)} - c_{n+1} \delta_n.$$

If  $n$  is odd then

$$c_{n+1} = \begin{cases} 0 & \text{if } x^{(n)} \in [-\delta_{n+1}, \delta_n) \\ \left\lfloor -\frac{x^{(n)} + \delta_{n+1}}{\delta_n} \right\rfloor & \text{if } x^{(n)} \in [-\delta_{n-1}, -\delta_{n+1}) \end{cases}$$

and

$$x^{(n)} = x^{(n)} - c_{n+1} \delta_n.$$

Let us check that the admissibility condition (1) holds. We have easily that  $c_n \leq a_n$  for all  $n \geq 1$ . If  $c_{2k+1} \neq 0$  then  $x^{(2k+1)} \in [\delta_{2k+1} - \delta_{2k}, \delta_{2k+1})$ , and thus  $c_{2k+2} \neq a_{2k+2}$ . If  $c_{2k+2} \neq 0$  then  $x^{(2k+2)} \in [-\delta_{2k+2}, -\delta_{2k+2} + \delta_{2k+1})$ , and thus  $c_{2k+3} \neq a_{2k+3}$ .

Furthermore, for all  $n \in \mathbb{N}$  we have  $x = x^{(n)} + \sum_{k=0}^{n-1} c_{k+1}(-1)^k \delta_k$  and thus

$$x = \sum_{k=0}^{+\infty} c_{k+1}(-1)^k \delta_k.$$

This last series converges, since  $\forall k \geq 1$ ,  $0 \leq c_k \delta_{k-1} \leq \frac{a_k}{q_k} \leq \frac{1}{q_{k-1}}$ .

We claim that if  $n$  is even then  $E^{(n+1)}$  is the induced transformation of  $E^{(n)}$  on the interval  $[-\delta_n, \delta_{n+1})$ . Let us check this. If  $z \in [-\delta_n, -\delta_n + \delta_{n+1})$  then

$$E^{(n)}(z) = z + \delta_n \in [0, \delta_{n+1})$$

and thus the induced transformation agrees with  $E^{(n+1)}$  on  $[-\delta_n, -\delta_n + \delta_{n+1})$ . If  $z \in [-\delta_n + \delta_{n+1}, \delta_{n+1})$  then

$$(E^{(n)})^k(z) = z + k\delta_n \geq \delta_{n+1} \text{ for } 1 \leq k \leq a_{n+1}$$

and

$$(E^{(n)})^{a_{n+1}+1}(z) = z + (a_{n+1})\delta_n - \delta_{n-1} = z - \delta_{n+1} \in [-\delta_n, 0),$$

as desired. One similarly checks that for  $n$  odd,  $E^{(n+1)}$  is the induced transformation on the interval  $[-\delta_{n+1}, \delta_n)$  of the map  $E^{(n)}$ .

For  $n \geq 1$  we let  $v^{(n)}$  be the Sturmian sequence coding the orbit of  $x^{(n)}$  in the two-interval exchange  $E^{(n)}$  with respect to natural partition. It follows that  $v^{(n)} = T^{c_{n+1}} \tau_{n+1}^{a_{n+1}}(v^{(n+1)})$  holds for every  $n$ .  $\square$

**Remarks.** Such an expansion will be called the (*multiplicative*) *Ostrowski  $S$ -adic expansion* associated with the sequence  $\omega$ . More generally, an expansion of the form

$$x = \sum_{k=0}^{+\infty} c_{k+1}(q_k \alpha - p_k),$$

where the sequence of integer digits  $(c_k)$  satisfies the admissibility condition (1) is called an *Ostrowski expansion* following [37] (see also [10, 19, 20, 29, 30, 27, 37, 43, 44]). Note that the characteristic sequence of slope  $\alpha$  corresponds to intercept  $x = 0$ , having all  $c_k$  equal to 0.

**2.5. The Ostrowski odometer.** Let  $\alpha = [0; a_1 + 1, a_2, \dots]$  and set

$$K_\alpha = \{(c_n)_{n \geq 1} \mid \forall n \geq 1 (c_n \in \mathbb{N}, 0 \leq c_n \leq a_n) \text{ and } (c_{n+1} = a_{n+1} \Rightarrow c_n = 0)\}.$$

It is easy to see that

$$K_\alpha = \{(c_n)_{n \geq 1} \mid \forall n \geq 1, c_n \in \mathbb{N}, c_1 q_0 + \dots + c_j q_{j-1} \leq q_j - 1\}.$$

Let  $c = (c_n)_{n \geq 1} \in K_\alpha$ , set

$$D(c) = \{j \geq 1; c_1 q_0 + \dots + c_j q_{j-1} = q_{j+1} - 1\},$$

and put  $m = \sup D(c)$  if  $D(c)$  is nonempty, and  $m = -1$  otherwise. Note that  $m = +\infty$  if and only if  $c$  is of the form

$$a_1 0 a_3 0 \dots \text{ or } 0 a_2 0 a_4 \dots,$$

and if  $m > 0$  then

$$c = \begin{cases} a_1 0 a_3 0 \dots a_{m-1} 0 c_{m+1} c_{m+2} \dots & \text{if } m \text{ is even} \\ 0 a_2 0 a_4 \dots 0 a_{m-1} 0 c_{m+1} c_{m+2} \dots & \text{if } m \text{ is odd.} \end{cases}$$

Following [25], one can define on the compact set  $K_\alpha$  (endowed with the product of the discrete topologies on the finite sets  $\{0 \leq c \leq a_n\}$ ) the addition  $\sigma$  by 1,

$$\sigma(c) = \begin{cases} 0^{m+1}(c_{m+1} + 1)c_{m+2} \dots & \text{if } m < \infty, \\ 0^\infty & \text{otherwise.} \end{cases}$$

The map  $\sigma$  is called the *Ostrowski  $\alpha$ -odometer*. The map  $\sigma : K_\alpha \rightarrow K_\alpha$  is onto and continuous, and  $(K_\alpha, \sigma)$  is minimal (for more details, see [25, 7]).

**Proposition 2.8.** *The dynamical systems  $(K_\alpha, \sigma)$  and  $(X_\alpha, T)$  are topologically conjugate.*

*Proof.* The sets  $X_\alpha$  and  $K_\alpha$  are in one-to-one correspondence via the map  $\Psi : X_\alpha \rightarrow K_\alpha$ ,  $\omega \mapsto (c_n)_{n \geq 1}$ , where  $(c_n)_{n \geq 1}$  is the Ostrowski  $S$ -adic expansion of Proposition 2.7.

Suppose  $\omega \in X_\alpha$  and  $\Psi(\omega) = c$  does not have a tail in common with  $a_1 0 a_3 0 \dots$  or  $0 a_2 0 a_4 \dots$ . Put  $m = \sup D(c)$  as before and let  $v^{(k)}$  be as in Proposition 2.7. Then  $c_{m+1} < a_{m+1}$  and

$$\begin{aligned} T(\omega) &= T(T^{c_1} \tau_0^{a_1} \circ \dots \circ T^{c_m} \tau_{m-1}^{a_m}(v^{(m)})) \\ &= \tau_0^{a_1} \circ \dots \circ \tau_{m-1}^{a_m}(T v^{(m)}) \\ &= \tau_0^{a_1} \circ \dots \circ \tau_{m-1}^{a_m} \circ T^{c_{m+1}+1} \tau_m^{a_{m+1}}(v^{(m+1)}), \end{aligned}$$

whence  $\Psi(T\omega) = \sigma(\Psi(\omega))$ . This holds for a dense set of  $\omega \in X_\alpha$ .  $\square$

**2.6. A characterization of primitive substitutive Sturmian sequences.** Let  $\mathcal{A}$  be a finite alphabet and  $\mathcal{A}^*$  denote the free monoid generated by  $\mathcal{A}$  for the concatenation, i.e.,  $\mathcal{A}^*$  is the set of finite words over the alphabet  $\mathcal{A}$ . A substitution is a non-erasing morphism of the free monoid  $\mathcal{A}^*$ . A substitution  $\tau$  is said to be *primitive* if there exists an integer  $k$  such that for all letters  $a, b$  in the alphabet  $\mathcal{A}$ ,  $a$  is a factor of  $\tau^k(b)$ . A sequence  $u$  is *primitive substitutive* if there exist a primitive substitution  $\tau$  over the alphabet  $\mathcal{B}$  and a letter-to-letter projection  $\varphi : \mathcal{B} \rightarrow \mathcal{A}$  such that  $u = \varphi(v)$ , where  $v = \tau(v)$  is a fixed point of  $\tau$ . The aim of this section is to characterize primitive substitutive Sturmian sequences. For characterizations of Sturmian sequences that are fixed points of substitutions, see for instance [15, 39, 47]. Let us recall the following fact on Ostrowski's numeration (see for instance [27]):

**Theorem 2.9.** *Let*

$$x = \sum_{k=1}^{+\infty} c_{k+1}(q_k \alpha - p_k),$$

*where the sequence  $(c_k)$  satisfies the admissibility conditions (1). Suppose  $\alpha$  quadratic. Then  $(c_n)$  is ultimately periodic if and only if  $x \in \mathbb{Q}(\alpha)$ .*

Let  $\omega$  be a *uniformly recurrent sequence*, i.e., a sequence in which every factor occurs infinitely many times with bounded gaps, and let  $h$  be a factor of  $\omega$ . A *return word* to  $h$  is a factor  $\omega[i, j]$ , where  $h$  occurs in  $\omega$  starting at the  $i$ th and  $j$ th places and nowhere between. Let  $\mathcal{A}_h$  be the set of return words to  $h$  in  $\omega$ . A sequence  $v$  with the same set of factors as  $\omega$  and having  $h$  as a prefix can be recoded over the alphabet  $\mathcal{A}_h$ . The recoded sequence, called a *derived sequence* of  $v$ , and is denoted by  $\mathcal{D}_h(v)$ . One can also associate a derived sequence with a sequence  $v$  not having  $h$  as a prefix as follows. Let  $p$  be a prefix of a return word in  $\mathcal{A}_h$  such that the sequence  $pv$  starts with  $h$  and has the same set of factors as  $\omega$ . We will

also call a derived sequence the sequence over  $\mathcal{A}_h$  obtained by coding the sequence  $pv$ . We will use the following result [21, 26, 23]:

**Theorem 2.10.** *A uniformly recurrent sequence is primitive substitutive if and only if the set of derived sequences (up to the alphabet) over all its factors is finite.*

Note that an expansion of the form

$$\omega = \tau_0^{a_1 - c_1} \circ (T \circ \tau_0)^{c_1} \circ \tau_1^{a_2 - c_2} \circ (T \circ \tau_1)^{c_2} \circ \dots \circ \tau_{k-1}^{a_k - c_k} \circ (T \circ \tau_{k-1})^{c_k} (\omega^{(s_k)}).$$

can explicitly be written in terms of a standard  $S$ -adic expansion, that is, as a limit of the composition of a finite number of substitutions following [24, 22], by introducing the morphisms  $\tau'_i$  for  $i \in \{0, 1\}$  defined by  $\tau'_i(i) = i$  and  $\tau'_i(j) = ji$ , for  $j \neq i$ . Indeed we have

$$\omega = \tau_0^{a_1 - c_1} \circ (\tau'_0)^{c_1} \circ \tau_1^{a_2 - c_2} \circ (\tau'_1)^{c_2} \circ \dots \circ \tau_{k-1}^{a_k - c_k} \circ (\tau'_{k-1})^{c_k} (\omega^{(s_k)}).$$

**Proposition 2.11.** *A Sturmian sequence  $\omega$  of slope  $\alpha$  which codes the orbit of  $x$  is primitive substitutive if and only if  $\alpha$  is a quadratic irrational and  $x \in \mathbb{Q}(\alpha)$ .*

*Proof.* If  $\alpha$  is quadratic and  $x \in \mathbb{Q}(\alpha)$ , then by using the  $S$ -adic representation on the four morphisms  $\tau_i$  and  $\tau'_i$ ,  $i \in \{0, 1\}$  one obtains that  $\omega$  is primitive substitutive.

Conversely, suppose  $\omega$  primitive substitutive. We will use the notation of Proposition 2.7. The the sequences  $v^{(k)}$  are derived sequences. More precisely,

$$v^{(n+1)} = \mathcal{D}_{(n \bmod 2)^{a_{n+1}}} ((n \bmod 2)^{c_{n+1}} v^{(n)}).$$

Indeed  $(n \bmod 2)^{c_{n+1}}$  has exactly two return words  $(n \bmod 2)^{c_{n+1}}1$  and  $(n \bmod 2)^{c_{n+1}+1}1$ , the second one corresponding to the interval of induction. The derived sequence of a derived sequence is again a derived sequence (up to the alphabet). Hence following Theorem 2.10, there are two sequences  $v^{(n)}$  and  $v^{(m)}$  which are equal, hence  $(a_n)$  and  $(c_n)$  are ultimately periodic.  $\square$

### 3. CALCULATING INITIAL POWERS.

The paradigm for our study is that large initial powers of  $\omega$  come from large initial powers of the  $\omega^{(n)}$ . Before giving a more precise statement let us prove a simpler fact. Let  $\omega$  be a Sturmian sequence and let  $i_n, b_n, \omega^{(n)}$  be defined as in the previous section. Recall that a word is primitive if it is not an integer power of a shorter word.

**Lemma 3.1.** *If  $\omega$  begins in a word  $w^r$  where  $r > 1$ ,  $|w| > 2$ , and  $w$  is primitive then there is a prefix  $w^{(1)}$  of  $\omega^{(1)}$  such that  $w$  is a cyclic permutation of  $\tau_{i_1}(w^{(1)})$ . Furthermore,  $|w^{(1)}| \geq 2$  and  $w^{(1)}$  is primitive.*

*Proof.* If  $b_1 = 0$  then  $\omega_0 = \omega_{|w|} = i_1$ . The only place that  $i_1$  occurs in the image of a letter under  $\tau_{i_1}$  is as the first letter. Thus the longest word of the form  $\tau_{i_1}(\omega_0^{(1)})\tau_{i_1}(\omega_1^{(1)})\dots\tau_{i_1}(\omega_j^{(1)})$  which is a prefix of  $w$  must in fact be  $w$ , so that  $w^{(1)} = \omega^{(1)}[0, j]$  does the job.

In the case  $b_1 = 1$ , we have  $\tau_{i_1}(\omega^{(1)}) = i_1\omega$ , and  $\omega_0 = \omega_{|w|} = \bar{i}_1$ . Since no sequence in the image of  $\tau_{i_1}$  can have  $\bar{i}_1\bar{i}_1$  as a factor, it must be that  $\omega_{|w|-1} = i_1$ . The same argument used in the first case produces a prefix  $w^{(1)}$  of  $\omega^{(1)}$  for which  $\tau_{i_1}(w^{(1)}) = i_1w[0, |w| - 2]$ , and  $i_1w[0, |w| - 2]$  is a cyclic permutation of  $w$ .

Now  $|\tau_{i_1}(u)| \leq 2|u|$  for any word  $u$ , and  $|\tau_{i_1}(w^{(1)})| = |w| > 2$ , so we must have  $|w^{(1)}| \geq 2$ , and if  $w^{(1)}$  were an integer power of some shorter word then  $w$  would be also, contrary to the hypothesis.  $\square$

We are now prepared to prove an important fact about initial powers. Let us recall that for all  $k > 0$ ,  $s_k = \sum_{j=1}^k a_j$ , and

$$\begin{aligned}\omega &= \tau_0^{a_1 - c_1} \circ (T \circ \tau_0)^{c_1} \circ \tau_1^{a_2 - c_2} \circ (T \circ \tau_1)^{c_2} \circ \cdots \circ \tau_{k-1}^{a_k - c_k} \circ (T \circ \tau_{k-1})^{c_k} (\omega^{(s_k)}) \\ &= T^{c_1} \tau_0^{a_1} \circ T^{c_2} \tau_1^{a_2} \circ T^{c_3} \tau_0^{a_3} \circ \cdots \circ T^{c_k} \tau_{k-1}^{a_k} (\omega^{(s_k)}).\end{aligned}$$

**Proposition 3.2.** *Suppose  $\omega$  begins in a word  $w$  to power  $r \geq 2$ , where  $|w| \geq 2$ , and  $w$  is primitive. Then there is a nonnegative integer  $m$  such that  $w$  is a cyclic permutation of  $\tau_{i_1} \circ \cdots \circ \tau_{i_m}(01)$ , and  $\omega^{(m)}$  begins in 01 or 10 to power  $> \lfloor r \rfloor - 1$ . Furthermore,  $m$  is one of the numbers  $s_k - 1$  or  $s_k - c_k - 1$ . If  $r \geq 3$  then  $m$  is one of the numbers  $s_k - 1$ .*

*Proof.* Let  $w^{(1)}$  be the prefix of  $\omega^{(1)}$  given by Lemma 3.1. If  $|w^{(1)}| > 2$  and the prefix power of  $w^{(1)}$  in  $\omega^{(1)}$  is  $> 1$  then we can apply the lemma again to get a prefix  $w^{(2)}$  of  $\omega^{(2)}$ . Continue in this way as long as possible, at the  $n$ th step obtaining a prefix  $w^{(n)}$  of  $\omega^{(n)}$  for which  $\tau_{i_n}(w^{(n)})$  is a cyclic permutation of  $w^{(n-1)}$ , stopping after  $m$  steps when either  $|w^{(m)}| = 2$  or the prefix power  $r'$  of  $w^{(m)}$  in  $\omega^{(m)}$  is 1. We claim that  $|w^{(m)}| = 2$ , from which it follows that  $w^{(m)}$  is 01 or 10 since  $w^{(m)}$  is primitive, hence  $w$  is a cyclic permutation of  $\tau_{i_1} \circ \cdots \circ \tau_{i_m}(01)$ , and  $r' > 1$ .

Let us prove that  $r' > 1$ . Write  $(w^{(m)})^\infty$  for the infinite periodic word  $w^{(m)}w^{(m)}w^{(m)} \dots$ . The longest common prefix shared by  $(w^{(m)})^\infty$  and  $\omega^{(m)}$  is  $(w^{(m)})^{r'}$ , so by Lemma 2.4 the longest common prefix of

$$\tau_{i_1} \circ \cdots \circ \tau_{i_m}((w^{(m)})^\infty)$$

and

$$\tau_{i_1} \circ \cdots \circ \tau_{i_m}(\omega^{(m)})$$

has length

$$|\tau_{i_1} \circ \cdots \circ \tau_{i_m}((w^{(m)})^{r'})| + |\tau_{i_1} \circ \cdots \circ \tau_{i_m}(01)| - 2 < |\tau_{i_1} \circ \cdots \circ \tau_{i_m}((w^{(m)})^{r'+1})|,$$

since  $w^{(m)}$  must contain both a 0 and a 1, by primitivity of  $w^{(m)}$ .

On the other hand,

$$T^{b_1} \circ \tau_{i_1} \circ \cdots \circ T^{b_m} \circ \tau_{i_m}((w^{(m)})^\infty)$$

and

$$T^{b_1} \circ \tau_{i_1} \circ \cdots \circ T^{b_m} \circ \tau_{i_m}(\omega^{(m)})$$

have  $w^r$  as their longest common prefix and thus

$$\tau_{i_1} \circ \cdots \circ \tau_{i_m}((w^{(m)})^\infty)$$

and

$$\tau_{i_1} \circ \cdots \circ \tau_{i_m}(\omega^{(m)})$$

have a common prefix of length  $\geq r|w|$ . Putting these inequalities together we have

$$|\tau_{i_1} \circ \cdots \circ \tau_{i_m}((w^{(m)})^{r'+1})| > r|\tau_{i_1} \circ \cdots \circ \tau_{i_m}(w^{(m)})|$$

from which we may deduce  $\lfloor r' \rfloor \geq \lfloor r \rfloor - 1$ , and if either  $r$  or  $r'$  is an integer then  $r' > r - 1$ . Thus  $r' > 1$  and hence  $|w^{(m)}| = 2$  as claimed. We thus have proved that  $\omega^{(m)}$  begins in 01 or 10 to power  $r' > \lfloor r \rfloor - 1$ .

Let us now examine  $m$  more closely. We know that  $\omega^{(m)}$  begins in 010 or 101; indeed  $w^{(m)} = 01$  or 10 and  $r' > 1$ . By symmetry we need only to consider former possibility.

Case 1:  $i_{m+1} = 0$ . Since  $\omega^{(m)}$  begins in 01, then  $b_{m+1} = 0$  and  $\omega^{(m+1)}$  must begin in 1. If  $i_{m+2} = 0$  then this means  $b_{m+2} = 1$ , i.e.,  $m$  is one of the numbers  $s_k - c_k - 1$ , where  $0 < c_k < a_k$ . Otherwise  $i_{m+2} = 1$  and  $m$  is one of the  $s_k - 1$ .

Case 2:  $i_{m+1} = 1$ . Then  $b_{m+1} = 1$  and  $\omega^{(m+1)}$  begins in 00, which means  $i_{m+2} = 0$ , and hence  $m$  is one of the  $s_k - 1$ .

From the first case we see that if  $m$  is one of the numbers  $s_k - c_k - 1$  ( $0 < c_k < a_k$ ) then  $\omega^{(m)} = \tau_0 \circ T \circ \tau_0(\omega^{(m+2)})$  and  $\omega^{(m+2)}$  begins in 10, which is enough to guarantee that  $\omega^{(m)}$  begins in 0100, i.e.,  $r' = 3/2$ . This cannot happen if  $r \geq 3$ , since  $\lfloor r' \rfloor \geq \lfloor r \rfloor - 1$ .  $\square$

Now that we know where prefix powers  $r \geq 2$  in  $\omega$  come from we can compute them exactly.

**Proposition 3.3.** *Let  $w$  and  $r$  be as in Proposition 3.2 and let  $m$ ,  $\omega^{(m)}$ , and  $w^{(m)}$  be as in its proof. Assume that  $r$  is the largest power of  $w$  which is a prefix of  $\omega$ . Then*

$$r = \begin{cases} \mathbf{1}_{a_{k+2}=c_{k+2}} + \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k} & \text{if } m = s_k - 1 \\ 1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_k - c_k q_{k-1}} & \text{if } m = s_k - c_k - 1 \text{ with } 0 < c_k < a_k \end{cases}$$

where  $\mathbf{1}_{a_{k+2}=c_{k+2}}$  is 1 if  $a_{k+2} = c_{k+2}$  and 0 otherwise.

Conversely, for each  $k$ ,  $\omega$  begins in a cyclic permutation of  $\tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k}(01)$  with prefix power  $\mathbf{1}_{a_{k+2}=c_{k+2}} + \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k}$  and for each  $k$  such that  $0 < c_k < a_k$ ,  $\omega$  begins in a cyclic permutation of the word  $\tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k - c_k - 1}(01)$  with prefix power  $1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_k - c_k q_{k-1}}$ .

Before proving the proposition let us state a lemma to be used in the calculation. It is proved easily by induction.

**Lemma 3.4.** *Let  $k > 0$  and set  $i = k \pmod{2}$ . Then*

$$\begin{aligned} |\tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k}(i\bar{i})| &= q_k + q_{k-1} = 2 + \sum_{j=1}^k a_j q_{j-1}, \\ |\tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k}(i)| &= q_k, \\ |T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k}(i)| &= q_k - \sum_{j=1}^k c_j q_{j-1}, \\ |T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k}(i\bar{i})| &= 2 + \sum_{j=1}^k (a_j - c_j) q_{j-1}. \end{aligned}$$

*Proof of Proposition 3.3.* First suppose  $m = s_k - 1$ . Set  $i = k \pmod{2}$ , that is  $i = i_{k+1}$ . The sequence  $\omega^{(s_k)}$  begins in  $i^{\mathbf{1}_{a_{k+2}=c_{k+2}} + a_{k+1} - c_{k+1}} \bar{i}$ . Indeed,  $\omega^{(s_{k+1})} = T^{c_{k+1}} \circ \tau_i^{a_{k+1}} \omega^{(s_{k+2})}$ ; if  $a_{k+2} \neq c_{k+2}$ , then  $\omega^{(s_{k+1})}$  begins in  $\bar{i}$ ; if  $a_{k+2} = c_{k+2}$ , then  $c_{k+1} = 0$ , and  $\omega^{(s_{k+1})}$  begins in  $i$  since  $\omega^{(s_{k+2})}$  begins in  $i$  and hence  $\omega^{(s_{k+1})}$  begins in  $i\bar{i}$ . The longest common prefix of

$$\omega = T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k}(\omega^{(s_k)})$$

and

$$T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k}(i^\infty)$$

has the following length from Lemma 2.4:

$$\begin{aligned}
& \left| T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k} (i^{\mathbf{1}_{a_{k+2}=c_{k+2}+a_{k+1}-c_{k+1}}}) \right| + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k} (i\bar{i}) \right| - 2 \\
&= (\mathbf{1}_{a_{k+2}=c_{k+2}} + a_{k+1} - c_{k+1} - 1) \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k} (i) \right| \\
&\quad + \left| T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k} (i) \right| + \left| \tau_0^{a_1} \circ \dots \circ \tau_{k-1}^{a_k} (i\bar{i}) \right| - 2 \\
&= (\mathbf{1}_{a_{k+2}=c_{k+2}} + a_{k+1} - c_{k+1}) \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k} (i) \right| \\
&\quad + \left| T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_k} \circ \tau_{k-1}^{a_k} (i\bar{i}) \right| - 2 \\
&= \sum_{j=1}^k (a_j - c_j) q_{j-1} + (\mathbf{1}_{a_{k+2}=c_{k+2}} + a_{k+1} - c_{k+1}) q_k \\
&= \sum_{j=1}^{k+1} (a_j - c_j) q_{j-1} + q_k (\mathbf{1}_{a_{k+2}=c_{k+2}}).
\end{aligned}$$

Thus  $\omega$  begins in a cyclic permutation of  $\tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k} (i)$  to power

$$\frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1} + q_k \mathbf{1}_{a_{k+2}=c_{k+2}}}{\left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-1}^{a_k} (i) \right|} = \mathbf{1}_{a_{k+2}=c_{k+2}} + \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k}.$$

Since  $\tau_{k-1}(i) = i\bar{i}$ , this power is exactly the value of  $r$ .

Next we consider the case  $m = s_k - c_k - 1$  with  $0 < c_k < a_k$ . Again, set  $i = k \bmod 2$ . From

$$\omega = \tau_0^{a_1 - c_1} \circ (T \circ \tau_0)^{c_1} \circ \tau_1^{a_2 - c_2} \circ (T \circ \tau_1)^{c_2} \circ \dots \circ \tau_{k-1}^{a_k - c_k} \circ (T \circ \tau_{k-1})^{c_k} (\omega^{(s_k)}),$$

it is easy to see that  $\omega^{(s_k - c_k)}$  begins in  $i\bar{i}$  and the longest common prefix of

$$\omega = T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_{k-1}} \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (\omega^{(s_k - c_k)})$$

and

$$T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_{k-1}} \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i^\infty)$$

has length

$$\begin{aligned}
& \left| T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_{k-1}} \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i) \right| \\
&\quad + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i\bar{i}) \right| - 2 \\
&= \left| T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_{k-1}} \circ \tau_{k-2}^{a_{k-1}} (\bar{i}^{a_k - c_k} i) \right| + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} (\bar{i}) \right| \\
&\quad + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i) \right| - 2 \\
&= \left| T^{c_1} \circ \tau_0^{a_1} \circ \dots \circ T^{c_{k-1}} \circ \tau_{k-2}^{a_{k-1}} (i\bar{i}) \right| + (a_k - c_k) \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} (\bar{i}) \right| \\
&\quad + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i) \right| - 2 \\
&= \sum_{j=1}^{k-1} (a_j - c_j) q_{j-1} + (a_k - c_k) q_{k-1} + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i) \right| \\
&= \sum_{j=1}^k (a_j - c_j) q_{j-1} + \left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i) \right|.
\end{aligned}$$

We also have

$$\left| \tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k} (i) \right| = q_k - c_k q_{k-1}$$



and thus  $\omega$  begins in a cyclic permutation of  $\tau_0^{a_1} \circ \tau_1^{a_2} \circ \dots \circ \tau_{k-2}^{a_{k-1}} \circ \tau_{k-1}^{a_k - c_k}(i)$  to power

$$1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_k - c_k q_{k-1}}.$$

As in the first case, this is exactly the value of  $r$ .

To prove the “conversely” part of the proposition, simply note that the formulas for the lengths above do not depend on  $r$  or  $m$  at all.  $\square$

**Corollary 3.5.**

$$\text{ice}(\omega) = \limsup_{k \rightarrow \infty} \max \left( \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k}, 1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_k - c_k q_{k-1}} \right).$$

*Proof.* Set  $x(k) = \mathbf{1}_{a_{k+2}=c_{k+2}} + \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k}$  and  $y(k) = 1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_k - c_k q_{k-1}}$ . One has from Proposition 3.3,

$$\text{ice}(\omega) = \max(\limsup_{k \rightarrow \infty} x(k), \limsup_{k \rightarrow \infty, 0 < c_k < a_k} y(k)).$$

Observe that

- If  $c_k = a_k$  then  $y(k) = x(k - 2)$ . Thus, if  $c_{k+2} = a_{k+2}$  then  $x(k) = y(k + 2)$ .
- If  $c_k = 0$  and  $c_{k+1} = a_{k+1}$  then  $y(k) < y(k + 1) = x(k - 1)$ .
- If  $c_k = 0$  and  $c_{k+1} < a_{k+1}$  then  $y(k) \leq x(k)$ .

The conclusion follows from these observations.  $\square$

## 4. SOME GENERAL PROPERTIES OF ICE.

4.1. **Notation.** In all that follows,

$$\begin{aligned} x(k) &= \mathbf{1}_{a_{k+2}=c_{k+2}} + \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k}, \\ x'(k) &= \frac{\sum_{j=1}^{k+1} (a_j - c_j) q_{j-1}}{q_k}, \\ y(k) &= 1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_k - c_k q_{k-1}}. \end{aligned}$$

One has

$$\text{ice}(\omega) = \limsup_{k \rightarrow \infty} \max(x(k), y(k)) = \limsup_{k \rightarrow \infty} \max(x'(k), y(k)).$$

4.2. **Characteristic sequence.** Recall that the characteristic sequence  $\omega$  of slope  $\alpha$  is the sequence obtained by setting all of the  $c_j$  equal to 0. We can easily compute  $\text{ice}(\omega)$  from

Corollary 3.5:

$$\begin{aligned}
\text{ice}(\omega) &= \limsup_{k \rightarrow \infty} \max \left( \frac{\sum_{j=1}^{k+1} a_j q_{j-1}}{q_k}, 1 + \frac{\sum_{j=1}^k a_j q_{j-1}}{q_k} \right) \\
&= \limsup_{k \rightarrow \infty} \frac{\sum_{j=1}^{k+1} a_j q_{j-1}}{q_k} \\
&= \limsup_{k \rightarrow \infty} \frac{q_{k+1} + q_k - 2}{q_k} \\
&= \limsup_{k \rightarrow \infty} 1 + a_{k+1} + \frac{q_{k-1}}{q_k} \\
&= 1 + \limsup_{k \rightarrow \infty} [a_k; a_{k-1}, \dots, a_1] \\
&= \text{ind}^*(\alpha) - 1.
\end{aligned}$$

This quantity is finite if and only if the  $a_k$  are bounded. One has  $\text{ice}(\omega) \leq 3$  if and only if all but finitely many of the  $a_k$  are equal to 1, in which case  $\alpha \in \mathbb{Q}(\theta)$  and  $\text{ice}(\omega) = 1 + \theta$ .

We can recover the shift invariance of  $\text{ice}$  off the orbit of  $\omega$  as follows. Let  $\omega(-\alpha)$  be the Sturmian sequence of slope  $\alpha$  coding the orbit of  $-\alpha$  under  $R_\alpha$ , and let  $\omega(1-\alpha)$  be the Sturmian sequence of slope  $\alpha$  coding the orbit of  $1-\alpha$  under  $\tilde{R}_\alpha$ . These sequences are the two shift preimages of the characteristic sequence  $\omega_*$ , i.e.,

$$\omega(-\alpha) = 0\omega \quad \text{and} \quad \omega(1-\alpha) = 1\omega.$$

Since  $\sigma(0a_20a_4\dots) = \sigma(a_10a_30\dots) = 0000\dots = \Psi(\omega)$ , it follows from Proposition 2.8 that

$$\Psi(\omega(-\alpha)) = 0a_20a_4\dots \quad \text{and} \quad \Psi(\omega(1-\alpha)) = a_10a_30\dots$$

Corollary 3.5 shows that for  $c \in K_\alpha$ ,  $\text{ice}(\Psi^{-1}(c))$  depends only on the tail of  $c$ , which is by definition the same as that of  $\sigma(c)$  unless  $c \in \{a_10a_30\dots, 0a_20a_4\dots\}$ . Thus  $\text{ice} = \text{ice} \circ T$  on  $X_\alpha \setminus \{\omega(-\alpha), \omega(1-\alpha)\}$ .

By Corollary 3.5,

$$\text{ice}(\omega(-\alpha)) = \limsup_{k \rightarrow \infty} \max \left( a_{2k+1} + \frac{q_{2k-1}}{q_{2k}}, 1 + a_{2k-1} + \frac{q_{2k-3}}{q_{2k-2}} \right),$$

and

$$\text{ice}(\omega(1-\alpha)) = \limsup_{k \rightarrow \infty} \max \left( a_{2k+2} + \frac{q_{2k}}{q_{2k+1}}, 1 + a_{2k} + \frac{q_{2k-2}}{q_{2k-1}} \right).$$

This implies  $\text{ice}(\omega(-\alpha)) \leq \text{ice}(\omega)$  and  $\text{ice}(\omega(1-\alpha)) \leq \text{ice}(\omega)$ . One may have equality as in the Fibonacci case ( $\alpha = \theta = [1; 1, 1, \dots]$ ), as well as a strict inequality as for instance for  $\alpha = [0; 3, 1, 3, 1, \dots]$ .

**4.3. The “keep one” sequence.** The aim of this section is to prove that there exists a Sturmian sequence of slope  $\alpha$  with very little repetition at the beginning, even if  $\alpha$  has unbounded partial quotients (and thus  $X_\alpha$  has arbitrarily large powers in its language).

**Proposition 4.1.** *For every irrational slope  $\alpha$  there exists a Sturmian sequence  $\omega \in X_\alpha$  such that  $\text{ice}(\omega) \leq 1 + \theta$ .*

*Proof.* This is a special case of (3) of Proposition 2.1, but we find it interesting to specifically give the  $S$ -adic expansion of such a point  $\omega$ . Set  $c_k = a_k - 1$  for all  $k$  and let  $\omega \in X_\alpha$  be the corresponding Sturmian sequence. We claim that  $\text{ice}(\omega) \leq \theta + 1$ . By Corollary 3.5,

$$\begin{aligned} \text{ice}(\omega) &= \limsup_{k \rightarrow \infty} \max \left( \frac{\sum_{j=1}^{k+1} q_{j-1}}{q_k}, 1 + \frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} \right) \\ &= \limsup_{k \rightarrow \infty} \max \left( 1 + \frac{\sum_{j=1}^k q_{j-1}}{q_k}, 1 + \frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} \right) \\ &= 1 + \limsup_{k \rightarrow \infty} \frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}}. \end{aligned}$$

Our next lemma completes the proof. □

**Lemma 4.2.** *The continued fraction convergents  $q_j$  satisfy*

$$\frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} < \theta.$$

*Proof.* Our proof is far from elegant and requires consideration of several cases. Let  $f_n$  be the Fibonacci sequence  $f_0 = 0, f_1 = 1$  and  $f_{n+1} = f_n + f_{n-1}$ . Also, set  $a'_1 = a_1 + 1$  and  $a'_n = a_n$  for  $n \geq 2$ .

If all of the  $a'_j$ ,  $j = 1, \dots, k-1$ , are equal to 1 then  $q_j = f_{j+1}$  for  $0 \leq j \leq k$  and

$$\frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} = \frac{f_{k+2} - 1}{f_{k+1}} < \theta,$$

since  $f_{k+2}/f_{k+1}$  is one of the continued fraction convergents for  $\theta$ .

Otherwise we let  $\ell \in \{1, 2, \dots, k-2\}$  be the greatest index for which  $a'_\ell \neq 1$ , or we set  $\ell = 1$  if  $a'_1 = \dots = a'_{k-2} = 1$  (and thus  $a'_{k-1} > 1$ ). We have

$$q_r = f_{r-\ell+1}q_\ell + f_{r-\ell}q_{\ell-1} \quad \text{for } \ell \leq r \leq k-2,$$

and from the recursive definitions,

$$\begin{aligned} \frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} &= \frac{(f_{k-\ell+2} - 1)q_\ell + (f_{k-\ell+1} - 1)q_{\ell-1} + (a'_{k-1} - 1)q_{k-2} + \sum_{j=1}^{\ell} q_{j-1}}{f_{k-\ell+1}q_\ell + f_{k-\ell}q_{\ell-1} + (a'_{k-1} - 1)q_{k-2}} \\ &= \frac{(f_{k-\ell+2} - 1 + (a'_{k-1} - 1)f_{k-\ell-1})q_\ell + (f_{k-\ell+1} + (a'_{k-1} - 1)f_{k-\ell-2})q_{\ell-1} + \sum_{j=1}^{\ell-1} q_{j-1}}{(f_{k-\ell+1} + (a'_{k-1} - 1)f_{k-\ell-1})q_\ell + (f_{k-\ell} + (a'_{k-1} - 1)f_{k-\ell-2})q_{\ell-1}} \\ &= \frac{(f_{k-\ell+2} + (a'_{k-1} - 1)f_{k-\ell-1})q_\ell + (f_{k-\ell+1} - a'_\ell + (a'_{k-1} - 1)f_{k-\ell-2})q_{\ell-1} + \sum_{j=1}^{\ell-2} q_{j-1}}{(f_{k-\ell+1} + (a'_{k-1} - 1)f_{k-\ell-1})q_\ell + (f_{k-\ell} + (a'_{k-1} - 1)f_{k-\ell-2})q_{\ell-1}} \\ &\leq \frac{(f_{k-\ell+2} + (a'_{k-1} - 1)f_{k-\ell-1})q_\ell + (f_{k-\ell+1} - (a'_\ell - 1) + (a'_{k-1} - 1)f_{k-\ell-2})q_{\ell-1}}{(f_{k-\ell+1} + (a'_{k-1} - 1)f_{k-\ell-1})q_\ell + (f_{k-\ell} + (a'_{k-1} - 1)f_{k-\ell-2})q_{\ell-1}}, \end{aligned}$$

since  $q_0 + \dots + q_{\ell-3} < q_{\ell-1}$ . We shall use the fact that  $\frac{a+b}{c+d}$  is between  $\frac{a}{b}$  and  $\frac{c}{d}$  for any positive real numbers  $a, b, c, d$ .

If  $a'_{k-1} > 1$  then, since  $\frac{f_{n+1}+m}{f_n+m} < \theta$  for any positive integers  $m, n$ ,

$$\frac{f_{k-\ell+2} + (a'_{k-1} - 1)f_{k-\ell-1}}{f_{k-\ell+1} + (a'_{k-1} - 1)f_{k-\ell-1}} < \theta$$

and

$$\frac{f_{k-\ell+1} - (a'_\ell - 1) + (a'_{k-1} - 1)f_{k-\ell-2}}{f_{k-\ell} + (a'_{k-1} - 1)f_{k-\ell-2}} < \theta,$$

and the desired inequality follows.

We are left to consider the possibility that  $a'_{k-1} = 1$  and  $a'_\ell > 1$ . The inequality above simplifies to

$$\frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} \leq \frac{f_{k-\ell+2}q_\ell + (f_{k-\ell+1} - (a'_\ell - 1))q_{\ell-1}}{f_{k-\ell+1}q_\ell + f_{k-\ell}q_{\ell-1}}.$$

If  $k - \ell$  is even then  $\frac{f_{k-\ell+2}}{f_{k-\ell+1}} < \theta$  and  $\frac{f_{k-\ell+1} - (a'_\ell - 1)}{f_{k-\ell}} \leq \frac{f_{k-\ell+1} - 1}{f_{k-\ell}} < \theta$ , and the desired inequality follows. In case  $k - \ell$  is odd, we have  $k - \ell \geq 3$  and  $\frac{f_{k-\ell+1}}{f_{k-\ell}} < \theta$ . Since  $(a'_\ell - 1)q_{\ell-1} > \frac{a'_\ell - 1}{a'_\ell + 1}q_\ell \geq \frac{1}{3}q_\ell$ , we have

$$\frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} \leq \frac{(f_{k-\ell+2} - \frac{1}{3})q_\ell + f_{k-\ell+1}q_{\ell-1}}{f_{k-\ell+1}q_\ell + f_{k-\ell}q_{\ell-1}}$$

and the observation that  $\frac{f_n - \frac{1}{3}}{f_{n-1}} < \theta$  for  $n \geq 5$  completes the proof.  $\square$

**Remarks.** By Proposition 3.3, all prefix powers  $r \geq 2$  in the “keep one” Sturmian sequence of slope  $\alpha$  are of the form

$$1 + \frac{\sum_{j=1}^k q_{j-1}}{q_k} \quad \text{or} \quad 1 + \frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}}.$$

Thus, by Lemma 4.2, the Sturmian sequence obtained this way begins in no  $1 + \theta$  power at all. It is easy to show that  $\limsup_{k \rightarrow \infty} 1 + \frac{\sum_{j=1}^k q_{j-1}}{q_{k-1} + q_{k-2}} < 1 + \theta$  unless  $(a_n)$  has arbitrarily long strings of consecutive ones. It follows that  $\text{ice}(\text{“keep one”}) \leq 1 + \theta$  with equality if and only if every sequence of slope  $\alpha$  has  $\text{ice} \geq 1 + \theta$ .

**4.4. The Fibonacci case.** We prove some characteristic properties of the Fibonacci Sturmian shift  $X_{\frac{1}{\theta}}$ , which we henceforth denote by  $X_\theta$ .

**Proposition 4.3.** (1) *The function  $\text{ice}$  is shift invariant on  $X_\theta$  and  $\text{ice}(\omega_*) = 1 + \theta$ , where  $\omega_*$  denotes the characteristic sequence.*

(2) *Every  $\omega$  in  $X_\theta$  begins in arbitrarily large cubes except those  $\omega$  in the  $\mathbb{Z}$ -orbit of the characteristic sequence  $\omega_*$  (see also [9]).*

(3) *One has*

$$\begin{aligned} \text{ice}(X_\theta) &= \{2 + \theta - \sum_{i \geq 1} \gamma_i \theta^{-i}; \forall i \gamma_i \in \{0, 1\}, \gamma_i \gamma_{i+1} = 0; \\ &\forall k \in \mathbb{N}, \sum_{i \geq 1} \gamma_i \theta^{-i} \leq \sum_{i \geq 1} \gamma_{i+k} \theta^{-i}\}. \end{aligned}$$

*The set  $\text{ice}(X_\theta)$  is a compact subset of  $[1 + \theta, 2 + \theta]$ , with empty interior; it is uncountable. The set  $\mathbb{Q}(\theta) \cap [1 + \theta, 2 + \theta]$  is dense in  $\text{ice}(X_\theta)$ .*

(4) *Let  $\omega$  be a Sturmian sequence and  $(c_k)$  be its Ostrowski expansion. One has  $\text{ice}(\omega) < 2 + \theta$  if and only if  $(c_k)_{k \geq 1}$  does not contain arbitrarily long strings of consecutive 0s. Let  $\omega(x)$  be a Sturmian sequence of intercept  $x$ . More precisely,*

$$\dim_H \{x \in [1 - \theta, 2 - \theta]; \text{ice}(\omega(x)) < 2 + \theta\} = 1.$$

**Remarks.** One easily checks that if  $\gamma \in \{0, 1\}^{\mathbb{N}^*}$  with  $\gamma_i \gamma_{i+1} = 0$  for all  $i$ , then for any given  $k$

$$\sum_{i \geq 1} \gamma_i \theta^{-i} \leq \sum_{i \geq 1} \gamma_{i+k} \theta^{-i} \iff (\gamma_i)_{i \geq 1} \leq_{\text{lex}} (\gamma_{i+k})_{i \geq 1} = T^k(\gamma),$$

$\leq_{\text{lex}}$  denoting the lexicographic order.

Following the third assertion, an element  $\text{ice}(\omega)$  of  $\text{ice}(X_\theta)$  is of the form  $2 + \theta - \sum_{i \geq 1} \gamma_i \theta^{-i}$ . Furthermore, there exists a unique sequence  $\gamma$  in  $\{0, 1\}^{\mathbb{N}^*}$  with  $\gamma_i \gamma_{i+1} = 0$  for all  $i$ , and  $\gamma \leq_{\text{lex}} T^k(\gamma)$  for all  $k$ , such that  $\text{ice}(\omega) = 2 + \theta - \sum_{i \geq 1} \gamma_i \theta^{-i}$ . Indeed if  $\gamma$  does not ultimately end in  $0101\dots$ , then  $(\gamma_i)_{i \geq 1}$  is the  $\theta$ -expansion of  $2 + \theta - \text{ice}(\omega)$  in the sense of [38, 41] and its  $\theta$ -expansion is not finite since  $\gamma \leq_{\text{lex}} T^k(\gamma)$  for all  $k$ ; otherwise, if the  $\theta$ -expansion of  $2 + \theta - \text{ice}(\omega)$  is finite, and say, equals  $\sum_{i=1}^l \varepsilon_i \theta^{-i}$ , with  $\varepsilon_l = 1$ , then  $\gamma$  equals

$$\varepsilon_1 \varepsilon_2 \dots \varepsilon_{l-1} 01010101 \dots$$

Let us note that the set of sequences  $\gamma'$  with values in  $\{0, 1\}$  that satisfy  $\gamma' \leq_{\text{lex}} T^k(\gamma')$  for all  $k$  has been proved in [1] to be the set of kneading sequences for one parameter families of maps of the interval, piecewise increasing with a single discontinuity. The set of sequences  $\gamma$  that we consider here is a subset consisting of the sequences  $\gamma$  with values in  $\{0, 1\}$  such that

$$\forall k \in \mathbb{N}, \gamma \leq_{\text{lex}} T^k(\gamma) \leq_{\text{lex}} (10)^\infty.$$

Let  $\varphi_\theta : \{0, 1\}^{\mathbb{N}^*} \rightarrow \mathbb{R}$ ,  $\gamma = (\gamma_i)_{i \geq 1} \mapsto \sum_{i \geq 1} \gamma_i \theta^{-i}$ . In other words,

$$\text{ice}(X_\theta) = 2 + \theta - \varphi_\theta(\{\gamma \in \{0, 1\}^{\mathbb{N}^*}; \forall k \in \mathbb{N}, \gamma \leq_{\text{lex}} T^k(\gamma) \leq_{\text{lex}} (10)^\infty\}).$$

*Proof.* The proof of Assertion 3 is directly inspired from [13]. The computation of the Hausdorff dimension is due to A. Rémondière (private communication).

We have proved the first assertion in Section 4.2.

Let us prove the second assertion. Let  $\omega$  be a Sturmian sequence of slope  $\theta$  not belonging to the  $\mathbb{Z}$ -orbit under the action of the shift  $T$  of the characteristic sequence  $\omega_*$ . Let  $(c_k) \in \{0, 1\}^{\mathbb{N}}$  be its Ostrowski expansion following Proposition 2.7. By assumption, the pattern  $001$  appears infinitely often in the sequence  $(c_k)$ . Fix an integer  $k$  for which  $c_k = 1$ ,  $c_{k-1} = 0$ ,  $c_{k-2} = 0$ . One has

$$y(k) = 1 + \frac{q_{k-2} + q_{k-3} + \sum_{j=1}^{k-3} (1 - c_j) q_j}{q_{k-2}}.$$

One easily proves by induction that for any positive integer  $l$

$$\sum_{j=1}^l c_j q_{j-1} \leq q_l - 1, \quad \sum_{j=1}^l (1 - c_j) q_{j-1} \geq q_{l-1} - 1,$$

hence

$$y(k) \geq 2 + \frac{q_{k-3} + q_{k-4}}{q_{k-2}} = 3.$$

Let us prove the third assertion (which follows [13] in a similar situation). Let us first observe that for any Sturmian sequence  $\omega$  in  $X_\theta$ , then

$$\text{ice}(\omega) = \limsup_{k \rightarrow \infty} y(k).$$

It is a direct consequence of the following:

- if  $a_{k+1} = c_{k+1} = 1$ , then  $c_k = 0$ , and  $y(k) = x'(k) + 1$ ;
- if  $c_{k+1} = c_k = 0$ , then  $y(k) = x'(k)$ ;
- if  $c_{k+1} = 0$  and  $c_k = 1$ , then  $c_{k-1} = 0$ , and  $y(k) = 1 + x'(k - 2)$ .

Furthermore,

$$\text{ice}(\omega) = \limsup_{k \rightarrow \infty} 2 + \frac{\sum_{j=1}^{k-2} (a_j - c_j) q_{j-1}}{q_{k-2}} = 2 + \theta - \liminf_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-2} c_j q_{j-1}}{q_{k-2}}.$$

Indeed, one has:

- if  $c_k = 0$ , then  $y(k) \leq 1 + \frac{q_k + q_{k-1}}{q_k} \leq 1 + \theta$ ;
- if  $c_k = 1$ , then

$$\begin{aligned} y(k) &= 2 + \frac{\sum_{j=1}^{k-2} (a_j - c_j) q_{j-1}}{q_{k-2}} \\ &\geq 2 + \frac{q_{k-2} + q_{k-3} - \sum_{j=1}^{k-2} c_j q_{j-1}}{q_{k-2}} \\ &\geq 2 + \theta - \frac{\sum_{j=1}^{k-2} c_j q_{j-1}}{q_{k-2}} \geq 1 + \theta. \end{aligned}$$

Let us prove that

$$\begin{aligned} \text{ice}(X_\theta) = \mathcal{S} &:= \{2 + \theta - \sum_{i \geq 1} \gamma_i \theta^{-i}; \forall i \ \gamma_i \in \{0, 1\}, \gamma_i \gamma_{i+1} = 0; \\ &\forall k \in \mathbb{N}, \sum_{i \geq 1} \gamma_i \theta^{-i} \leq \sum_{i \geq 1} \gamma_{i+k} \theta^{-i}\}. \end{aligned}$$

Let  $(k_i)$  be an increasing sequence of indices with  $c_{k_i} = 1$  such that

$$\text{ice}(\omega) = 2 + \theta - \lim_{i \rightarrow \infty} \frac{\sum_{j=1}^{k_i-2} c_j q_{j-1}}{q_{k_i-2}}.$$

By compactness (König's lemma, see for instance [31]), the sequence of words  $(c_{k_i-2} \dots c_1)_{i \in \mathbb{N}}$  admits a limit point in  $\{0, 1\}^{\mathbb{N}}$  that we denote by  $(\gamma_i)_{i \geq 1}$ .

One checks by normal convergence that

$$\text{ice}(\omega) = 2 + \theta - \sum_{i \geq 1} \frac{\gamma_i}{\theta^i},$$

and that the sequence  $(\gamma_i)_{i \geq 1}$  satisfies the following: for all  $i \geq 1$ ,  $\gamma_i = 1$  implies  $\gamma_{i+1} = 0$ . Furthermore, one has

$$\forall k \in \mathbb{N}, \sum_{i \geq 1} \gamma_i \theta^{-i} \leq \sum_{i \geq 1} \gamma_{i+k} \theta^{-i}.$$

Indeed for  $k \in \mathbb{N}$

$$\text{ice}(\omega) \geq 2 + \theta - \lim_{i \rightarrow \infty} \frac{\sum_{j=1}^{k_i-k-2} c_j q_{j-1}}{q_{k_i-k-2}} = 2 + \theta - \sum_{i \geq 1} \gamma_{i+k} \theta^{-i}.$$

Furthermore the set of factors of  $(\gamma_i)$  is included in the mirror image of the set of factors of the sequence  $(c_k)$  (the mirror image of a factor  $w_1 w_2 \dots w_n$  is  $w_n w_{n-1} \dots w_1$ ).

Conversely, let  $(\gamma_i)_{i \geq 1} \in \{0, 1\}^{\mathbb{N}^*}$  be a sequence such that  $\gamma_i \gamma_{i+1} = 0$  for all  $i$ , and  $\forall k \in \mathbb{N}$ ,  $\sum_{i \geq 1} \gamma_i \theta^{-i} \leq \sum_{i \geq 1} \gamma_{i+k} \theta^{-i}$ .

- Let us assume that  $\gamma$  is a recurrent sequence, that is, every factor of  $\gamma$  appears infinitely often. Let  $(w_n)_{n \geq 1}$  be a sequence of factors of  $\gamma$ , such that for all  $n$ ,  $w_n$  contains as a factor all the factors of length  $n$ , and  $w_n$  is a suffix of  $w_{n+1}$ ; such a sequence of words can be constructed since  $\gamma$  is recurrent. The sequence of words  $(\tilde{w}_n)$  admits a limit point  $(c_k) \in \{0, 1\}^{\mathbb{N}^*}$ ; by construction, the set of factors of  $(c_k)$  is exactly the mirror image of the set of factors of  $\gamma$ . Let  $\omega$  be the Sturmian sequence associated with the sequence  $(c_k)_{k \geq 1}$  according to Proposition 2.7. There exists a strictly increasing sequence of integers  $(k_i)$  such that  $(c_{k_i-2} \dots c_1)$  converges towards  $\gamma$ . Then one has

$$\text{ice}(\omega) \geq 2 + \theta - \lim_{i \rightarrow \infty} \frac{\sum_{j=1}^{k_i-2} c_j q_{j-1}}{q_{k_i-2}} = 2 + \theta - \sum_{i \geq 1} \gamma_i / \theta^i.$$

Furthermore  $\text{ice}(\omega)$  is obtained for a sequence  $\gamma'$  that has the same set of factors as  $\gamma$ , i.e., there exists a strictly increasing sequence of integers  $(n_k)$  such that  $\gamma' = \lim_{k \rightarrow \infty} T^{n_k}(\gamma)$ ; one has  $\sum_{i \geq 1} \gamma'_i / \theta^i \geq \sum_{i \geq 1} \gamma_i / \theta^i$ , since  $\forall k \in \mathbb{N}$ ,  $\sum_{i \geq 1} \gamma_i \theta^{-i} \leq \sum_{i \geq 1} \gamma_{i+k} \theta^{-i}$ , and hence

$$\text{ice}(\omega) \leq 2 + \theta - \sum_{i \geq 1} \gamma_i / \theta^i.$$

- Suppose now that  $\gamma$  is not recurrent. Let  $u$  be the longest prefix of  $\gamma$  such that  $u$  appears infinitely often in  $\gamma$ . Such a word exists, otherwise  $\gamma$  equals  $10000\dots$  or  $01111\dots$ , and both sequences are excluded by the conditions on  $\gamma$ . Let  $(n_i)_{i \in \mathbb{N}}$  be the increasing sequence of indices of successive occurrences of  $u$ ; set  $v_i = \gamma_1 \gamma_2 \dots \gamma_{n_i-1}$ . Let us define the sequence  $c = (c_k)_{k \in \mathbb{N}^*}$  as  $c = \tilde{v}_1 \tilde{v}_2 \dots$ ; one easily checks that  $(c_k)$  contains no 11; let  $\omega$  be the corresponding Sturmian sequence. Let  $k_i = n_1 + \dots + n_i - i$ , for  $i \in \mathbb{N}$ . One has  $\lim_{i \rightarrow \infty} \frac{\sum_{k=1}^{k_i-2} c_j q_{j-1}}{q_{k_i-2}} = \sum_{i \geq 1} \gamma_i \theta^{-i}$ , and thus  $\text{ice}(\omega) \geq 2 + \theta - \sum_{i \geq 1} \gamma_i / \theta^i$ . One also has  $\text{ice}(\omega) \leq 2 + \theta - \sum_{i \geq 1} \gamma_i / \theta^i$ . Indeed, let  $\gamma'$  (with  $\forall k$ ,  $\gamma' \leq_{\text{lex}} T^k(\gamma') \leq_{\text{lex}} (10)^\infty$ ) be the (unique) sequence that satisfies  $\text{ice}(\omega) = 2 + \theta - \sum_{i \geq 1} \gamma'_i \theta^{-i}$ . It remains to prove that the set of factors of  $\gamma'$  is included in the set of factors of  $\gamma$ . Let  $w$  be a factor of  $\gamma'$ ;  $\tilde{w}$  appears infinitely often in  $(c_k)$ ; by definition of  $u$ , the occurrences of  $\tilde{w}$  are ultimately included in words  $\tilde{v}_k$ .

Now it is easy to deduce the topological properties of  $\mathcal{S}$ . The set  $\mathcal{S}$  is easily seen to be a closed set. Indeed,  $\mathcal{S}_k := \varphi_\theta(\{\gamma \in \{0, 1\}^{\mathbb{N}^*}; \gamma \leq_{\text{lex}} T^k(\gamma) \leq_{\text{lex}} (10)^\infty\})$  is a closed set, and so does  $\mathcal{S}$  as  $\mathcal{S} = 2 + \theta - \bigcap_k \mathcal{S}_k$ .

The set  $\mathcal{S}$  is uncountable. Take for the sequences  $\gamma$  sequences which start with 0001 and which then do not contain any more the pattern 0001, that is, which are built over the patterns 01 and 001.

Let  $2 + \theta - \sum_{i \geq 1} \gamma_i \theta^{-i} \in \mathcal{S}$ . Any interval centered at this point will contain a point of the form  $2 + \theta - \sum_{i \geq 1} \gamma'_i \theta^{-i}$  such that there exists an integer  $k$  with  $\gamma'_i \theta^{-i} > \gamma'_{i+k} \theta^{-i}$ ; if the pattern 00 occurs infinitely often in  $\gamma$ , exchange it for a sufficiently large occurrence by the pattern 10; otherwise,  $\gamma$  ends in  $0101\dots$ , and exchange this ending by  $10000\dots$ .

Any periodic sequence with period a prefix of a sequence  $\gamma$  produces an element of  $\mathbb{Q}(\theta)$ , hence the set  $\mathbb{Q}(\theta) \cap [1 + \theta, 2 + \theta]$  is dense in  $\text{ice}(X_\theta)$ .

Consider now the fourth assertion. Let us first prove that  $\text{ice}(\omega) < 2 + \theta$  if and only if  $(c_k)_{k \geq 1}$  does not contain arbitrarily long strings of consecutive 0s. Assume that  $(c_k)_{k \geq 1}$  contains arbitrarily long strings of consecutive 0s, then  $\liminf_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-2} c_j q_{j-1}}{q_{k-2}} = 0$ , and  $\text{ice}(\omega) = 2 + \theta$ . Conversely, if  $\text{ice}(\omega) = 2 + \theta$ , then  $\gamma = 0$  and  $(c_k)$  contains arbitrarily long strings of consecutive 0s, since  $\gamma$  is a limit point of the sequence of words  $(c_n c_{n-1} \dots c_1)_{n \geq 1}$ .

With the notation of Section 2.4, if  $\alpha = \theta - 1 = [0; 1, \dots, 1, \dots]$ , then  $\delta_n = 1/\theta^{n+1}$ , and hence

$$\sum_{k \geq 1} c_k (-1)^{k-1} \delta_{k-1} = \sum_{k \geq 1} c_k (-1)^{k-1} \theta^k.$$

Let  $\tilde{\varphi}_\theta : \{0, 1\}^{\mathbb{N}^*} \rightarrow \mathbb{R}$ ,  $\gamma = (\gamma_i)_{i \geq 1} \mapsto \sum_{i \geq 1} \gamma_i (-1)^{i-1} \theta^{-i}$ . We are thus considering the Hausdorff dimension of the set

$$\{x \in [1 - \theta, 2 - \theta]; \exists (c_k) \in \{0, 1\}^{\mathbb{N}^*}, x = \sum_{k \geq 1} c_k (-1)^{k-1} \theta^k,$$

$$\forall k, c_k c_{k+1} = 0, (c_k) \text{ contains bounded strings of consecutive 0s}\}$$

$$= \tilde{\varphi}_\theta(\{(c_k) \in \{0, 1\}^{\mathbb{N}^*}; \forall k, c_k c_{k+1} = 0, (c_k) \text{ contains bounded strings of consecutive 0s}\}).$$

For  $p \geq 3$ , let

$$\mathcal{C}_p = \tilde{\varphi}_\theta(\{(c_k) \in \{0, 1\}^{\mathbb{N}^*}; \forall k, c_k c_{k+1} = 0, (c_k) \text{ does not contain } 0^p\}).$$

These sets are closed and

$$\{x \in [1 - \theta, 2 - \theta]; \text{ice}(\omega) < 2 + \theta\} = \cup_{p \geq 3} \mathcal{C}_p.$$

Let us first note that the sequence  $(\dim_H \mathcal{C}_p)$  is non-decreasing, since for all  $p \geq 3$ ,  $\mathcal{C}_p \subset \mathcal{C}_{p+1}$ . Hence one has  $\dim_H(\cup_{p \geq 3} \mathcal{C}_p) = \lim_{p \rightarrow \infty} \dim_H \mathcal{C}_p$ . Indeed  $\dim_H(\cup_{p \geq 3} \mathcal{C}_p) \geq \dim_H \mathcal{C}_p$ , for any  $p \geq 3$ . Conversely, if  $d > \lim_{p \rightarrow \infty} \dim_H \mathcal{C}_p$ , then the  $d$ -dimensional Hausdorff measure of  $\mathcal{C}_p$  equals zero, and so does the measure of  $\cup_{p \geq 3} \mathcal{C}_p$ , hence  $\dim_H(\cup_{p \geq 3} \mathcal{C}_p) \leq \lim_{p \rightarrow \infty} \dim_H \mathcal{C}_p$ .

For  $i = 0, 1$ , let

$$\mathcal{C}_p^i = \{x \in [1 - \theta, 2 - \theta]; \exists (c_k) \in \{0, 1\}^{\mathbb{N}^*},$$

$$x = \sum_{k \geq 1} c_k (-1)^{k-1} \theta^k; \forall k, c_k c_{k+1} = 0, (c_k) \text{ does not contain } 0^p, c_1 = i\}.$$

One has

$$\mathcal{C}_p = \mathcal{C}_p^0 \cup \mathcal{C}_p^1,$$

$$\mathcal{C}_p^1 = 1/\theta - 1/\theta \mathcal{C}_p^0,$$

$$\mathcal{C}_p^0 = \cup_{1 \leq k \leq p-1} (-1)^k / \theta^k \mathcal{C}_1^0,$$

$$\mathcal{C}_p^0 = \cup_{1 \leq k \leq p-1} (-1)^k / \theta^{k+1} + (-1)^{k+1} / \theta^{k+1} \mathcal{C}_p^0.$$

For  $1 \leq k \leq p-1$ , let  $s_k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $t \mapsto (-1)^k / \theta^{k+1} + (-1)^{k+1} / \theta^{k+1} t$ , be the similarity of ratio  $1/\theta^{k+1}$ ; the set of similarities  $s_1, \dots, s_{p-1}$  satisfies the open set condition [32] (take as open set  $]1 - \theta, 2 - \theta[ = ] - 1/\theta, 1/\theta^2[$ ). Fix  $p \geq 3$ ; if  $d_p$  denotes the Hausdorff dimension of  $\mathcal{C}_p$ , then  $\sum_{k=2}^p (1/\theta^{d_p})^k = 1$ ; in particular  $d_p > 0$ ; furthermore if  $u_p = 1/\theta^{d_p}$ , then  $u_p^{p+1} - u_p^2 - u_p + 1 = 0$ .



Let  $d_\infty$  denote the limit of the non-decreasing sequence  $(d_p)$ ;  $d_\infty > 0$ ; let  $u_\infty = 1/\theta^{d_\infty}$ ;  $1 > u_\infty \geq 1/\theta$ ; The sequence  $(u_p)$  tends to 0, since  $(d_p)$  tends to  $d_\infty > 0$ . Hence  $-u_\infty^2 - u_\infty + 1 = 0$ , and  $u_\infty = 1/\theta$ . □

## 5. SMALLEST PREFIX POWERS.

Now we turn our attention to minimizing ice over  $X_\alpha$  and proving Theorem 1.1, which we recall below.

**Theorem 1.1** *Let  $\alpha = [0; a_1, a_2, a_3, \dots]$  be an irrational number and  $X_\alpha$  be the set of all Sturmian sequences of slope  $\alpha$ . Then there is a Sturmian sequence  $\omega \in X_\alpha$  with  $\text{ice}(\omega) = 2$  if and only if for each pair of positive integers  $(s, t)$  with  $s > 1$  there are only finitely many  $k$  for which  $(a_k, a_{k+1}) = (s, t)$  or  $(a_k, a_{k+1}, a_{k+2}) = (1, 1, t)$ .*

We note that if  $\min(\text{ice}(X_\alpha)) = 2$  then  $\alpha$  has unbounded partial quotients and only finitely many strings of more than two consecutive 1s in the sequence of partial quotients  $(a_k)_{k \geq 1}$ . Furthermore the set of  $\alpha$  satisfying the assumptions of the theorem has zero measure. In particular, no Sturmian shift with a quadratic slope can contain a sequence of ice equal to 2, and by Proposition 2.11, there are no substitutive Sturmian sequences  $\omega$  with  $\text{ice}(\omega) = 2$ .

**5.1. Some first restrictions.** Given the partial quotients  $a_k$  of  $\alpha$  we must choose the  $c_k$  (satisfying the admissibility condition (1)) so as to minimize the lim sup in Corollary 3.5. A couple of observations will help narrow the playing field:

- If  $a_k - c_k > 2$  for infinitely many  $k$  then  $\text{ice}(\omega) \geq 3$ . Indeed if  $a_k - c_k \geq 3$ , then  $x'(k-1) \geq \frac{(a_k - c_k)q_{k-1}}{q_{k-1}} \geq 3$ .
- Given a sequence  $(c_k)$  we can define a new sequence  $c'_k$  by setting

$$c'_k = \begin{cases} c_k & \text{if } a_k = c_k \text{ or } a_{k+1} = c_{k+1} \\ a_k - 1 & \text{otherwise.} \end{cases}$$

The sequence  $c'_k$  also satisfies the admissibility condition (1) and determines a Sturmian sequence of slope  $\alpha$ , and the only quantities in the formula of Corollary 3.5 which are increased by substituting the  $c'_k$  for the  $c_k$  are the ones of the form  $y(k) = 1 + \frac{\sum_{j=1}^k (a_j - c_j)q_{j-1}}{q_k - c_k q_{k-1}}$  where  $k$  is an index for which  $c'_k \neq c_k$ , in which case  $c'_k = a_k - 1 > c_k$  and

$$\begin{aligned} 1 + \frac{\sum_{j=1}^k (a_j - c'_j)q_{j-1}}{q_k - c'_k q_{k-1}} &= 1 + \frac{\sum_{j=1}^k (a_j - c'_j)q_{j-1}}{q_{k-1} + q_{k-2}} \\ &< 1 + \frac{\sum_{j=1}^k (a_j - c'_j)q_{j-1}}{q_{k-1}} \\ &\leq \frac{\sum_{j=1}^k (a_j - c_j)q_{j-1}}{q_{k-1}} \end{aligned}$$

so that ice of the new sequence is no greater than that of the given sequence.

Consequently, in our quest to minimize ice over  $X_\alpha$  we need only consider sequences where for each  $k$

- $c_k \in \{0, a_k - 1, a_k\}$ ,

- if  $c_k = 0$  then  $a_k = 1$  or  $c_{k+1} = a_{k+1}$ ,
- if  $a_k \geq 2$  then  $c_k > 0$  (and hence  $c_{k+1} < a_{k+1}$ ).

**5.2. Special slopes.** We describe those slopes  $\alpha$  for which  $X_\alpha$  has a sequence with ice equal to 2. First we rule out some of the noncontenders. As before,  $\alpha = [0; a_1 + 1, a_2, a_3, \dots]$ ,  $x(k) = \mathbf{1}_{a_{k+2}=c_{k+2}} + \frac{\sum_{j=1}^{k+1}(a_j - c_j)q_{j-1}}{q_k}$ ,  $x'(k) = \frac{\sum_{j=1}^{k+1}(a_j - c_j)q_{j-1}}{q_k}$  and  $y(k) = 1 + \frac{\sum_{j=1}^k(a_j - c_j)q_{j-1}}{q_k - c_k q_{k-1}}$ .

**Proposition 5.1.** *If  $(s, t)$  is a pair of integers with  $s > 1$  such that  $(a_k, a_{k+1}) = (s, t)$  for infinitely many  $k$  then every  $\omega \in X_\alpha$  has  $\text{ice}(\omega) \geq 2 + \frac{1}{2(s+1)(t+1)+1}$ .*

*Proof.* Fix an index  $k$  for which  $a_k > 1$ . There are four cases to consider:

- (1) Suppose  $a_{k+1} = c_{k+1}$ . We have  $c_k = 0$  and

$$\begin{aligned} y(k+1) &= 1 + \frac{\sum_{j=1}^k (a_j - c_j)q_{j-1}}{q_{k-1}} \\ &\geq 1 + a_k. \end{aligned}$$

- (2) Suppose  $a_{k+2} = c_{k+2}$ . We have

$$\begin{aligned} y(k+2) &= 1 + \frac{\sum_{j=1}^{k+1} (a_j - c_j)q_{j-1}}{q_k} \\ &\geq 1 + a_{k+1} + \frac{a_{k-1}q_{k-2}}{q_k} \\ &\geq 1 + a_{k+1} + \frac{1}{2a_k + 1}. \end{aligned}$$

- (3) Suppose  $a_{k+2} - c_{k+2} \geq 2$ . We have

$$\begin{aligned} x(k+1) &\geq a_{k+2} - c_{k+2} + \frac{\sum_{j=1}^{k+1} (a_j - c_j)q_{j-1}}{q_{k+1}} \\ &\geq a_{k+2} - c_{k+2} + \frac{a_k q_{k-1}}{q_{k+1}} \\ &\geq a_{k+2} - c_{k+2} + \frac{1}{2a_{k+1} + 1}. \end{aligned}$$

- (4) Suppose  $c_{k+2} = a_{k+2} - 1$  and  $c_{k+1} < a_{k+1}$ . We have

$$\begin{aligned} y(k+2) &= 1 + \frac{q_{k+1} + \sum_{j=1}^{k+1} (a_j - c_j)q_{j-1}}{q_{k+1} + q_k} \\ &\geq 1 + \frac{q_{k+1} + q_k + \sum_{j=1}^k (a_j - c_j)q_{j-1}}{q_{k+1} + q_k} \\ &\geq 2 + \frac{a_{k-1}q_{k-2}}{q_{k+1} + q_k} \\ &\geq 2 + \frac{1}{2(a_k + 1)(a_{k+1} + 1) + 1}. \end{aligned}$$

In every case, one of  $x(k+1)$ ,  $y(k+1)$  and  $y(k+2)$  is at least  $2 + \frac{1}{2(a_k+1)(a_{k+1}+1)+1}$ . The result follows from this fact and Proposition 3.3.  $\square$

**Proposition 5.2.** *If  $t$  is an integer such that  $(a_k, a_{k+1}, a_{k+2}) = (1, 1, t)$  for infinitely many  $k$  then every  $\omega \in X_\alpha$  has  $\text{ice}(\omega) \geq 2 + \frac{1}{8t+1}$ .*

*Proof.* Fix an index  $k$  for which  $a_k = a_{k+1} = 1$ . We can save ourselves some labor by noting that in our proof of Proposition 5.1 the assumption  $a_k > 1$  was used only in the first of the four cases; in each of the last three cases the same estimates are valid and we see that one of  $x(k+1)$  and  $y(k+2)$  is at least  $2 + 1/9 \geq 1/(8a_{k+2} + 1) = 1/(8t + 1)$ . In the case that  $c_{k+1} = a_{k+1}$  we must have  $c_{k+2} < a_{k+2}$ ; if we replace  $k$  with  $k+1$  in our proof of Proposition 5.1 (Case 2, 3 and 4 applied on  $c_{k+3}$ ), the argument shows that one of  $x(k+2)$  and  $y(k+3)$  is at least  $2 + 1/(4(a_{k+2} + 1) + 1) \geq 1/(8a_{k+2} + 1) = 1/(8t + 1)$ .  $\square$

Finally, we can prove the main theorem.

*Proof of Theorem 1.1.* One direction follows from the preceding propositions. Let us prove the converse. Let  $\alpha$  be as in the statement of the theorem. We shall define the sequence  $(c_k)$  and check that the Sturmian sequence it represents has  $\text{ice}$  equal to 2. Since  $\text{ice}$  does not depend on the first values of  $c_k$ , we will define  $(c_k)$  for  $k$  large enough such that the pattern 111 no longer appears in  $a_k, a_{k+1}, \dots$ . We just require that the first values of  $(c_k)$  satisfy the admissibility condition (1). Here it is:

$$c_k = \begin{cases} a_k - 1 & \text{if } a_k > 1, a_{k-1} > 1, \\ a_k - 1 & \text{if } a_k > 1, a_{k-1} = a_{k-2} = 1, \\ a_k & \text{if } a_k > 1, a_{k-1} = 1, a_{k-2} > 1, \\ 0 & \text{if } a_k = 1 \text{ and } a_{k-1} > 1 \\ a_k & \text{if } a_k = 1 \text{ and } a_{k-1} = 1 \end{cases}$$

We verify the admissibility condition: If  $c_k = a_k$  then either  $a_k > 1, a_{k-1} = 1$  and  $a_{k-2} > 1$  or  $a_k = 1, a_{k-1} = 1$  and thus  $a_{k-2} > 1$ ; in both cases we have  $c_{k-1} = 0$ .

Note that  $a_k - c_k \in \{0, 1\}$  for all  $k \geq 1$ , hence  $x'(k) \leq y(k)$  for every  $k \geq 1$ . Assume that  $\text{ice}(\omega) > 2$ . Then there exist  $\varepsilon > 0$  such that one of the following four possibilities holds for infinitely many integers  $k$ :

- A:**  $a_k > 1$  and  $c_k = a_k - 1$ ;
- B:**  $a_k > 1$  and  $c_k = a_k$ ;
- C:**  $a_k = a_{k-1} = 1$ ;
- D:**  $a_k = 1$  and  $a_{k-1} > 1$ .

Case **A**: Suppose  $a_k > 1$  and  $c_k = a_k - 1$ . Then either  $a_{k-1} > 1$  or  $a_{k-1} = a_{k-2} = 1$ . Then

$$2 + \varepsilon \leq y(k) \leq 1 + \frac{\sum_{j=1}^k (a_j - c_j) q_{j-1}}{q_{k-1} + q_{k-2}},$$

therefore

$$(1 + \varepsilon)(q_{k-1} + q_{k-2}) \leq \sum_{j=1}^k q_{j-1} \leq q_{k-1} + q_{k-2} + q_{k-3} + \sum_{j=1}^{k-3} q_{j-1}.$$

Since

$$\sum_{j=1}^{k-3} q_{j-1} \leq \sum_{j=1}^{k-3} a_j q_{j-1} \leq q_{k-3} + q_{k-4},$$

we have

$$\varepsilon(q_{k-1} + q_{k-2}) \leq 2q_{k-3} + q_{k-4},$$

hence

$$\varepsilon(a_{k-1}q_{k-2} + a_{k-2}q_{k-3}) \leq 3q_{k-3} \leq 3q_{k-2}.$$

In particular,  $\varepsilon(a_{k-1}q_{k-2}) \leq 3q_{k-2}$  and  $\varepsilon(a_{k-2}q_{k-3}) \leq 3q_{k-3}$  hold for infinitely many  $k$ , therefore there exists a pair of integers  $(s, t)$  such that  $(a_{k-2}, a_{k-1}) = (s, t)$  for infinitely many  $k$ . It follows from our assumption on  $\alpha$  that  $s = 1$ . There are two cases to consider:

- $s = t = 1$ , and thus for infinitely many  $k$ ,

$$\begin{aligned} a_k > 1, \quad a_{k-1} = a_{k-2} = 1, \quad c_k = a_k - 1, \quad c_{k-1} = a_{k-1}, \\ a_{k-3} > 1, \quad c_{k-2} = 0, \quad c_{k-3} \geq a_{k-3} - 1, \end{aligned}$$

and

$$\begin{aligned} 2 + \varepsilon \leq y(k) &\leq 1 + \frac{q_{k-1} + q_{k-3} + q_{k-4} + \sum_{j=1}^{k-4} (a_j - c_j)q_{j-1}}{q_{k-1} + q_{k-2}} \\ &= 1 + \frac{q_{k-1} + q_{k-2} + \sum_{j=1}^{k-4} (a_j - c_j)q_{j-1}}{q_{k-1} + q_{k-2}}, \end{aligned}$$

and thus

$$\varepsilon(q_{k-1} + q_{k-2}) \leq 2q_{k-4}.$$

As

$$q_{k-1} + q_{k-2} = q_{k-2} + q_{k-3} + q_{k-3} + q_{k-4} = 3q_{k-3} + 2q_{k-4} \geq (3a_{k-3} + 2)q_{k-4},$$

we see that

$$\varepsilon(3a_{k-3} + 2)q_{k-4} \leq 2q_{k-4}.$$

Since this inequality holds for infinitely many  $k$ , there exists an integer  $s > 1$  such that  $a_{k-3} = s$  and  $a_{k-2} = 1$  for infinitely many  $k$ , a contradiction.

- $s = 1$  and  $t > 1$ . We thus have  $a_k > 1$ ,  $a_{k-1} = t > 1$ ,  $a_{k-2} = 1$ , and  $c_k = a_k - 1$ . One can assume  $a_{k-3} > 1$ , by assumption on  $\alpha$  (the pattern  $11t$  appears only finitely many times). Hence  $c_{k-1} = a_{k-1}$  and  $c_{k-2} = 0$ . We thus obtain

$$\begin{aligned} 2 + \varepsilon \leq y(k) &\leq 1 + \frac{q_{k-1} + q_{k-3} + q_{k-4} + \sum_{j=1}^{k-4} (a_j - c_j)q_{j-1}}{q_{k-1} + q_{k-2}} \\ &= 1 + \frac{q_{k-1} + q_{k-2} + \sum_{j=1}^{k-4} (a_j - c_j)q_{j-1}}{q_{k-1} + q_{k-2}}, \end{aligned}$$

that is,

$$\varepsilon(q_{k-1} + q_{k-2}) \leq 2q_{k-4}.$$

As

$$q_{k-1} + q_{k-2} = tq_{k-2} + q_{k-3} + q_{k-3} + q_{k-4} = (t+2)q_{k-3} + (t+1)q_{k-4},$$

one gets

$$\varepsilon((t+2)a_{k-3} + (t+1))q_{k-4} \leq 2q_{k-4}.$$

Since this inequality holds for infinitely many  $k$ , there exists an integer  $t'$  such that  $a_{k-3} = t'$ ,  $a_{k-2} = 1$  and  $a_{k-1} = t$  for infinitely many  $k$ , a contradiction.

Case **B**: Suppose  $a_k > 1$ ,  $a_{k-1} = 1$  and  $a_{k-2} > 1$ . Then  $c_k = a_k$  and  $c_{k-1} = 0$ . We have

$$2 + \varepsilon \leq y(k) = 1 + \frac{q_{k-2} + \sum_{j=1}^{k-2} (a_j - c_j)q_{j-1}}{q_{k-2}} = 2 + \frac{\sum_{j=1}^{k-2} (a_j - c_j)q_{j-1}}{q_{k-2}},$$

hence

$$\varepsilon q_{k-2} \leq q_{k-3} + \sum_{j=1}^{k-3} (a_j - c_j)q_{j-1} \leq 3q_{k-3},$$

and

$$\varepsilon a_{k-2} q_{k-3} \leq 3q_{k-3}.$$

Since this inequality holds for infinitely many  $k$ , there exists an integer  $s > 1$  such that  $a_{k-1} = 1$  and  $a_{k-2} = s$  for infinitely many  $k$ , a contradiction.

Case **C**: Suppose  $a_k = 1$  and  $a_{k-1} > 1$ . Then  $c_k = 0$ . One has

$$2 + \varepsilon \leq y(k) \leq 1 + \frac{q_{k-1} + q_{k-2} + \sum_{j=1}^{k-2} (a_j - c_j)q_{j-1}}{q_k} = 1 + \frac{q_k + \sum_{j=1}^{k-2} (a_j - c_j)q_{j-1}}{q_k},$$

hence

$$\varepsilon q_k \leq q_{k-2} + q_{k-3},$$

and

$$\varepsilon (a_{k-1} + 1)q_{k-2} \leq \varepsilon (q_{k-1} + q_{k-2}) \leq 2q_{k-2}.$$

This last inequality holds for infinitely many  $k$ . It follows that for some  $s > 1$  we have  $a_k = 1$  and  $a_{k-1} = s$  for infinitely many  $k$ , a contradiction.

Case **D**: Suppose  $a_k = 1$  and  $a_{k-1} = 1$ . Then, by hypothesis,  $a_{k-2} > 1$ ,  $c_k = 1$  and  $c_{k-1} = 0$ . We have

$$2 + \varepsilon \leq y(k) = 1 + \frac{q_{k-2} + q_{k-3} + \sum_{j=1}^{k-3} (a_j - c_j)q_{j-1}}{q_{k-2}},$$

that is,

$$\varepsilon q_{k-2} \leq 3q_{k-3},$$

and

$$\varepsilon a_{k-2} q_{k-3} \leq 3q_{k-3}.$$

Since this inequality holds for infinitely many  $k$ , there once again exists an integer  $s > 1$  such that for  $a_{k-1} = 1$  and  $a_{k-2} = s$  for infinitely many  $k$ , contrary to hypothesis.  $\square$

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