Arithmetic Discrete Hyperspheres and Separatingness

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Abstract. In the framework of arithmetic discrete geometry, a discrete object is provided with its own analytical definition corresponding to a discretization scheme. It can thus be considered as the equivalent, in a discrete space, of an euclidean object. Linear objects, namely lines and hyperplanes, have been widely studied under this assumption and are now deeply understood. This is not the case for discrete circles and hyperspheres for which no satisfactory definition exists. In the present paper, we try to fill this gap. Our main result is the characterization of the k-minimal discrete hypersphere thanks to an arithmetic definition based on a non-constant thickness function. To reach such a topological property, we link adjacency and separability with norms.

1 Introduction

Discrete geometry attempts to provide an analogue of the euclidean geometry for the discrete space $\mathbb{Z}^d$. Such an investigation is not only theoretical, but has also practical applications since digital images can be seen as arrays of pixels, in other words, as subsets of $\mathbb{Z}^2$ or $\mathbb{Z}^3$.

Discrete lines, namely the analogue of euclidean lines in the discrete space $\mathbb{Z}^2$, have been widely studied. At first, J. Bresenham [1], H. Freeman [2] and A. Rosenfeld [3] have followed an algorithmic approach and have defined them as digitizations of euclidean lines. They have provided tools for drawing and recognition [1–3]. Later, J.-P. Reveillès has initiated the arithmetic discrete geometry [4] and has introduced the notion of arithmetic discrete line as the solution of a system of diophantine inequalities, that is, as the subset of $\mathbb{Z}^2$ contained in a band. Such an approach enhances the knowledge of discrete lines. In addition to give new drawing [4] and recognition [5] algorithms, it directly links topological and geometrical properties of an arithmetic discrete line with its definition. For instance, its connectedness is entirely characterized by the width of the band, that is, its arithmetic thickness. In d-dimensional discrete spaces, the notion of arithmetic discrete hyperplane [6] is a natural generalization of the arithmetic discrete line [4].

Similarly, first investigations into discrete circles have been algorithmic ones [7–9]. Discrete circles were only considered as digitizations of euclidean circles. It is thus natural to ask whether or not J.-P. Reveillès' arithmetic approach is extendable to discrete circles and can supply an arithmetic discrete definition of circles independent of euclidean circles. Such an extension has been proposed by É. Andres [10]. He has defined the discrete analytical hypersphere as the solutions of a system of diophantine equations, the subsets of $\mathbb{Z}^d$ contained in a ring of width its arithmetic thickness. Concentric discrete analytical hyperspheres tile the discrete space, but one does not retrieve topological properties as strict k-connectedness or k-minimality. Recently, an arithmetic definition implying a non-constant thickness function was proposed in [11]. It provides discrete circles of integer parameters with 0-minimality or 1-minimality and gives an arithmetic characterization of the well-known Bresenham's circle [7].

In the present paper, we focus on discrete hyperspheres and on the arithmetic approach to generalize results of [11]. We deeper study the notion of thickness. In particular, we characterize the k-minimal arithmetic discrete hypersphere by relating the thickness function to a particular norm, the k-minimality one, and to local discrete variations of the hypersphere.

The paper is organized as follows. First, we begin with some recalls on discrete geometry useful to fully understand the matter. Second, works already done on the discrete analytical hypersphere [10] and the arithmetic discrete circle [11] are presented. A general definition is then proposed to unify the both approaches. Next, we focus on the topological properties of d-dimensional discrete objects to characterize the k-minimal arithmetic discrete hypersphere.

2 Basic Notions

The aim of this section is to introduce the basic notions of discrete geometry used throughout the present paper. Let d be an integer at least equal to 2 and let $(\mathbf{e_1}, \ldots, \mathbf{e_d})$ denote the canonical basis of the euclidean vector space $\mathbb{R}^d$. Let us call *discrete set* any subset of the *discrete space* $\mathbb{Z}^d$. The point $\mathbf{x} = \sum_{i=1}^d x_i \mathbf{e_i} \in \mathbb{R}^d$, with $x_i \in \mathbb{R}$ for each $i \in \{1, \ldots, d\}$, is represented by $(x_1, \ldots, x_d)$. A point $\mathbf{v} \in \mathbb{Z}^d$ is a *voxel* in a d-dimensional space or a *pixel* in a 2-dimensional space.

Definition 1 (k-Adjacency or k-Neighborhood). *Let d be the dimension of the discrete space and $k \in \mathbb{N}$ such that $k < d$. Two voxels $\mathbf{v} = (v_1, \ldots, v_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ are* k-neighbors *or* k-adjacent *if and only if:*

$$\|\mathbf{v}-\mathbf{w}\|_\infty = \max\{|v_1-w_1|, \ldots, |v_d-w_d|\} = 1 \text{ and } \|\mathbf{v}-\mathbf{w}\|_1 = \sum_{i=1}^{d} |v_i-w_i| \leq d-k.$$

Let $k \in \{0, \ldots, d-1\}$. A discrete set E is said to be *k-connected* if for each pair of voxels $(\mathbf{v}, \mathbf{w}) \in \mathrm{E}^2$, there exists a finite sequence of voxels $(\mathbf{s_1}, \ldots, \mathbf{s_p}) \in \mathrm{E}^p$ such that $\mathbf{v} = \mathbf{s_1}$, $\mathbf{w} = \mathbf{s_p}$ and the voxels $\mathbf{s_j}$ and $\mathbf{s_{j+1}}$ are k-neighbors, for each $j \in \{1, \ldots, p-1\}$. For the sake of clarity, a 0-connected discrete set is simply said to be *connected.*

Let E be a discrete set, $\mathbf{v}$ be a voxel of E and $k \in \{0, \ldots, d-1\}$. The *k-connected component* of $\mathbf{v}$ in E is the maximal k-connected subset of E (with respect to set inclusion) containing $\mathbf{v}$.

Definition 2 (k-Separability). *A discrete set* E *is* k-separating *in a discrete set* F *if its complement in* F, $\overline{\mathrm{E}} = \mathrm{F} \backslash \mathrm{E}$, *has two distinct k-connected components.* E *is called a* separator *of* F.

Definition 3 (k-Simple Points, k-Minimality). *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. Let also* F *and* E *be two discrete sets such that* E *is k-separating in F. A voxel $\mathbf{v} \in \mathrm{E}$ is said to be* k-simple *if* $\mathrm{E} \setminus \{\mathbf{v}\}$ *remains k-separating in* F. *Moreover, a k-separating discrete set in* F *without k-simple points is said to be* k-minimal *in* F.

3 The Arithmetic Discrete Hypersphere

Initiated by J.-P. Reveillès with the definition of arithmetic discrete lines [4], the arithmetic discrete geometry has led to a wide literature, mainly about linear objects. Discrete lines [4], planes and hyperplanes [6] are now well characterized and only one parameter, the *arithmetic thickness* ω, controls their topological properties. As far as we know, no satisfactory generalization to non-linear objects exists. É. Andres has defined the *discrete analytical hypersphere*[10] so as to obtain concentric hyperspheres tiling the space.

Definition 4 (Discrete Analytical Hypersphere [10]). *Let d be the dimension of the space. Let $r \in \mathbb{R}_+^\star$, $\mathbf{o} = (o_1, \ldots, o_d) \in R^d$ and $\omega \in \mathbb{R}_+^\star$. The* discrete analytical hypersphere $\mathbb{S}(\mathbf{o}, r, \omega)$ *of* center $\mathbf{o}$, *of* radius r *and of* arithmetic thickness ω, *is the subset of $\mathbb{Z}^d$ defined by:*

$$\mathbb{S}(\mathbf{o}, r, \omega) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid \left(r - \frac{\omega}{2}\right)^2 \leq \sum_{i=1}^{d} (v_i - o_i)^2 < \left(r + \frac{\omega}{2}\right)^2 \right\}. \tag{1}$$

However, no topological characterization is possible with this definition and it is far from the hyperplane one, then in [11], the notion of arithmetic discrete circle was proposed. Its analytical description is not based on the usual constant arithmetic thickness ω, but on a *thickness function* $\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}_+^\star$. The importance of keeping apart the analytical expression of the considered curve and the approximation induced by the discrete space, that is, the thickness, was also highlighted. Finally, those considerations lead to partial topological results on discrete circles. The *naive* and *standard* ones with integer parameters were characterized and Bresenham's circle [7] was provided with an arithmetic definition. For that, the thickness function is regarded as a measurement, by the usual discrete norms, $\| \cdot \|_1$ and $\| \cdot \|_\infty$, of the local behaviour of the curve .

Definition 5 (Arithmetic Discrete Circle [11]). *Let* $\mathbf{o} = (o_1, o_2) \in \mathbb{R}^2$, $r \in \mathbb{R}_+$. *Let $\omega : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a map. The* arithmetic discrete circle $\mathrm{C}(\mathbf{o}, r, \omega)$ *of* center $\mathbf{o}$, radius r *and* thickness function ω *is the following set:*

$$\mathrm{C}(\mathbf{o}, r, \omega_{\mathbf{o}}) = \left\{ \mathbf{v} \in \mathbb{Z}^2 \mid -\frac{\omega(\mathbf{v})}{2} \leq (v_1 - o_1)^2 + (v_2 - o_2)^2 - r^2 < \frac{\omega(\mathbf{v})}{2} \right\}. \quad (2)$$

In the present paper, we improve this last definition and generalize it to discrete hyperspheres, the d-dimensional case.

Definition 6 (Arithmetic Discrete Hypersphere). *Let d be the dimension of the space, $r \in \mathbb{R}_+^*$ and $\mathbf{o} = (o_1, \ldots, o_d) \in \mathbb{R}^d$. Let $\omega_1 : \mathbb{R}^d \longrightarrow \mathbb{R}_-$ and $\omega_2 : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be maps. The* arithmetic discrete hypersphere $\mathrm{S}(\mathbf{o}, r, (\omega_1, \omega_2))$ *with* center $\mathbf{o}$, radius r *and* thickness functions ω_1 *and* ω_2 *is:*

$$\mathrm{S}(\mathbf{o}, r, (\omega_1, \omega_2)) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid \omega_1(\mathbf{v}) \leq \sum_{i=1}^{d} (v_i - o_i)^2 - r^2 < \omega_2(\mathbf{v}) \right\}. \quad (3)$$

Contrary to Definition 5, Definition 6 includes two distinct thickness functions ω_1 and ω_2. That way, the global thickness can be distributed more or less inside or outside the hypersphere. This feature is interesting when one consider non-linear analytical expression and when thickness should not be the same inside or outside the curvature. Moreover such a definition includes all the previous attempts to define arithmetically discrete hyperspheres. In particular, the set of discrete analytical hyperspheres (see Definition 4) introduced by É. Andres [10] is included in the set of arithmetic discrete hyperspheres. They are defined using two thickness functions ω_1 and ω_2 such that $\omega_1 \neq -\omega_2$.

Remark 1 (Discrete Analytical Hypersphere [10] and Arithmetic Discrete Hypersphere). The discrete analytical hypersphere $\mathbb{S}(\mathbf{o}, r, \omega)$ is the arithmetic discrete hypersphere $\mathrm{S}(\mathbf{o}, r, (\omega_1, \omega_2))$ such that:

$$\begin{aligned} \omega_1 : \mathbb{R}^d &\longrightarrow \mathbb{R}_- \\ \mathbf{x} &\longmapsto -r\omega + \frac{\omega^2}{4} \end{aligned} \qquad \begin{aligned} \omega_2 : \mathbb{R}^d &\longrightarrow \mathbb{R}_+ \\ \mathbf{x} &\longmapsto r\omega + \frac{\omega^2}{4} \end{aligned}$$

In the same way, the arithmetic discrete circle introduced in [11] is an arithmetic discrete hypersphere.

Remark 2 (Arithmetic Discrete circles [11] and Arithmetic Discrete Hyperspheres). An arithmetic discrete circle $\mathrm{C}(\mathbf{o}, r, \omega)$ is an arithmetic discrete hypersphere $\mathrm{S}(\mathbf{o}, r, (\omega_1, \omega_2))$ such that:

$$\begin{aligned} \omega_1 : \mathbb{R}^d &\longrightarrow \mathbb{R}_- \\ \mathbf{x} &\longmapsto -\frac{\omega(\mathbf{x})}{2} \end{aligned} \qquad \begin{aligned} \omega_2 : \mathbb{R}^d &\longrightarrow \mathbb{R}_+ \\ \mathbf{x} &\longmapsto \frac{\omega(\mathbf{x})}{2} \end{aligned}$$

Definition 6 allows to build a considerable amount of discrete objects and some do not look like what is expected from an hypersphere. Consequently, as in the case of the arithmetic discrete circle [11], suitable thickness functions are needed to characterize discrete hyperspheres with basic topological properties.

4 Separability of d-Dimensional Spaces

Before defining suitable thickness functions, we have to determine which topological properties are meaningful for hyperspheres. For that, we focus on the more general case of d-dimensional discrete sets. The most studied one is the arithmetic discrete plane [6] and the minimality (or the separability) is its best characterized topological property [6]. It seems anyway to be the most evident one for all d-dimensional discrete objects since it intuitively refers to objects without holes.

4.1 The k-Adjacency Norm

The k-separability, and so the k-minimality, are reached when none of the voxels on one side of the separator is k-adjacent with a voxel on the other side. In Section 2, we give a definition of k-adjacency. It is not easy to handle, so we propose an equivalent and more formal expression. Indeed, two voxels are k-adjacent if they have at least k coordinates in common and if the maximal difference between the others is equal to 1.

Theorem 1. *Let d be the dimension of the space. Let $k \in \mathbb{N}$ such that $k < d$. Two voxels $\mathbf{v} = (v_1, \ldots, v_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ are* k-neighbor *or* k-adjacent *if and only if:*

$$|v_{\sigma(d)} - w_{\sigma(d)}| + \sum_{i=1}^{k} |v_{\sigma(i)} - w_{\sigma(i)}| = 1, \tag{4}$$

with σ the permutation of the set $\{1, \ldots, d\}$ such that, for all $i \in \{1, \ldots, d-1\}$, $|v_{\sigma(i)} - w_{\sigma(i)}| \leq |v_{\sigma(i+1)} - w_{\sigma(i+1)}|$.

Proof.

$$\begin{aligned}
\mathbf{v} \text{ and } \mathbf{w} \text{ are } k\text{-adjacent.} &\Leftrightarrow \max_{1 \leq i \leq d} \{|v_i - w_i|\} = 1 \text{ and } \sum_{i=1}^{d} |v_i - w_i| \leq d - k. \\
&\Leftrightarrow |v_{\sigma(d)} - w_{\sigma(d)}| = 1 \text{ and } \sum_{i=1}^{k} |v_{\sigma(i)} - w_{\sigma(i)}| = 0. \\
&\Leftrightarrow |v_{\sigma(d)} - w_{\sigma(d)}| + \sum_{i=1}^{k} |v_{\sigma(i)} - w_{\sigma(i)}| = 1.
\end{aligned}$$

In fact, the k-adjacency can be characterized by a norm that we call *k-adjacency norm.*

Proposition 1. *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. The map $[\cdot]_k$ defined by:*

$$\begin{aligned}
[\cdot]_k : \mathbb{R}^d &\longrightarrow \mathbb{R} \\
\mathbf{x} &\longmapsto |x_{\sigma(d)}| + \sum_{i=1}^{k} |x_{\sigma(i)}|,
\end{aligned} \tag{5}$$

with σ the permutation of the set $\{1,\ldots,d\}$ such that, for all $i \in \{1,\ldots,d-1\}$, $|x_{\sigma(i)}| \leq |x_{\sigma(i+1)}|$, is a norm.

Proof. Here, *positivity*, *scalability* and *triangle inequality* are evident properties.

Definition 7 (The k-Adjacency Norm). *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. We call* k-adjacency norm, *the norm $[\cdot]_k$.*

We notice that the k-adjacency norm is related to usual discrete norms, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

Remark 3. Let d be the dimension of the space and $\mathbf{v} \in \mathbb{R}^d$. Then:

$$\begin{cases} [\mathbf{v}]_0 &= \|\mathbf{v}\|_\infty, \\ [\mathbf{v}]_{(d-1)} &= \|\mathbf{v}\|_1. \end{cases}$$

4.2 (λ, k)-Adjacency, (λ, k)-Separability and (λ, k)-Hull of a set of voxels

In Definition 6, we propose to distribute the thickness inside and outside the hypersphere. From a practical point of view, we can then consider the arithmetic discrete hypersphere as the union of two discrete hyperspheres, an outer one and an inner one. Since we are interesting in k-minimal discrete hyperspheres, inner and outer constituting hyperspheres can be thinner than k-minimal ones. So we need a notion more general than k-separability to define them. Since we characterize k-adjacency with a norm, we can extend it from the discrete space to the continuous one and define the (λ, k)-adjacency where distance between adjacent points can be a real number and not only an integer. From here, (λ, k)-separability is a generalization of k-separability. We thus control precisely the thickness of the set of voxels and answer the case of discrete hyperspheres thinner than k-minimal ones.

Definition 8 ((λ, k)-Adjacency). *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. Let $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}^d$. The d-dimensional points $\mathbf{x}$ and $\mathbf{y}$ are (λ, k)-adjacent if and only if $[\mathbf{x}-\mathbf{y}]_k \leq \lambda$.*

Separability and adjacency are related. It is thus natural to define the (λ, k)-separability from the (λ, k)-adjacency.

Definition 9 ((λ, k)-Separability). *A set E (λ, k)-separates a connected set U in subsets $\mathrm{U_1}, \ldots, \mathrm{U_p}$ if and only if:*

$$\forall \mathbf{x} \in \mathrm{U_i}, i \in \{1, \ldots d\}, \nexists \mathbf{y} \in \left(\bigcup_{i \neq j, j=1}^{p} \mathrm{U_j} \right) \mid [\mathbf{x}-\mathbf{y}]_k \leq \lambda. \tag{6}$$

With Definition 9, we can distinguish two parts in a set of voxels $\mathrm{E} \subset \mathrm{U}$, its discrete (λ, k)-hull, which contains its voxels (λ, k)-adjacent with the exterior, and its (λ, k)-interior, (λ, k)-separated from the exterior.

Definition 10 (**(λ, k)-Hull**). *Let d be the dimension of the space, $k \in \mathbb{N}$ such that $k < d$ and $\lambda \in \mathbb{R}$. Let also* O *be a set of voxels. The (λ, k)-hull,* $\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O})$, *of* O *based on the* normal thickness λ *relatively to the k-adjacency norm $[\cdot]_k$ is defined as follows:*

$$\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O}) = \left\{ \mathbf{v} \in \mathrm{O} \mid \exists \mathbf{w} \in \mathbb{Z}^d \backslash \mathrm{O}, [\mathbf{v} - \mathbf{w}]_k \leq \lambda \right\}. \tag{7}$$

The (λ, k)-interior of O *is then* $\mathrm{I}_{(\lambda,\mathrm{k})}(\mathrm{O}) = \mathrm{O} \backslash \mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O})$.

We focus on the particular case of set a voxels described by an analytical expression. So we now take care of the discrete hull of a discrete set defined in such a way. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a map. Then we define the discret set $\mathrm{O}(f, +)$ such that:

$$\mathrm{O}(f, +) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid f(\mathbf{v}) \geq 0 \right\}.$$

Taking into account the map f, the (λ, k)-hull of the discrete object $\mathrm{O}(f, +)$ can be rewritten as follows:

$$\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O}(f, +)) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid \exists \mathbf{x} \in \mathbb{R}^d, f(\mathbf{x}) = 0 \wedge f(\mathbf{v}) \geq 0 \wedge [\mathbf{v} - \mathbf{w}]_k < \lambda \right\}. \tag{8}$$

According to Equation (8), a (λ, k)-hull can be seen as a discrete object, based on a map f, (λ, k)-separating the discrete space. Unfortunately the set of voxels it contains is difficult to determine since the definition brings into play two different measurements, namely the norm $[\cdot]_k$ and the function f.

5 The Separating Arithmetic Discrete Hypersphere

The notion of (λ, k)-hull allows to define discrete d-dimensional hyperspheres of center $\mathbf{o} = (o_1, \ldots, o_d) \in \mathbb{Z}^d$ and radius $r \in \mathbb{R}_+^*$ with the particular map s defined by:

$$\begin{aligned} s : \mathbb{R}^d & \longrightarrow \mathbb{R} \\ \mathbf{x} = (x_1, \ldots, x_d) & \longmapsto \textstyle\sum_{i=1}^d (x_i - o_i)^2 - r^2. \end{aligned}$$

To go further, we consider the restriction of s, $s_E : E \longrightarrow \mathbb{R}$, to the subspace $E \subset \mathbb{R}^d$ such that:

$$E = \left\{ \mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \forall i \in \{1, \ldots, d\}, x_i - o_i \geq 0 \right\}.$$

The (λ, k)-hull $\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O}(s_E, +))$ of the object $\mathrm{O}(s_E, +)$ in E is:

$$\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O}(s_E, +)) = \left\{ \mathbf{v} \in (\mathbb{Z}^d \cap E) \mid \exists x \in E, s_E(\mathbf{x}) = 0 \wedge s_E(\mathbf{v}) \geq 0 \wedge [\mathbf{v} - \mathbf{w}]_k < \lambda \right\}.$$

Since $s_E(\mathbf{x}) = 0$ and $s_E(\mathbf{v}) \geq 0$, we have the following equalities:

$$\begin{aligned} s_E(\mathbf{v}) &= \left| s_E(\mathbf{v}) - s_E(\mathbf{x}) \right|, \\ &= \left| \sum_{i=1}^d 2(v_i - o_i)(v_i - x_i) - (v_i - x_i)^2 \right|. \end{aligned}$$

Since $\forall i \in \{1, \dots, d\}, v_i - o_i \geq 0$, the absolute values are not required:

$$s_E(\mathbf{v}) = \sum_{i=1}^{d} 2(v_i - o_i)(v_i - x_i) - (v_i - x_i)^2.$$

With the last condition on $\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O}(s_E,+))$, $[\mathbf{v} - \mathbf{x}]_k < \lambda$, we give an upper bound on $s_E(\mathbf{v})$ and:

$$\mathrm{H}_{(\lambda,\mathrm{k})}(\mathrm{O}(s_E,+)) = \left\{ \mathbf{v} \in (\mathbb{Z}^d \cap E) \mid 0 \leq s_E(\mathbf{v}) < \sum_{i=d-k+1}^{d} 2\lambda(v_{\sigma(i)} - o_{\sigma(i)}) - \lambda^2 \right\},$$

with σ the permutation of the set $\{1, \dots, d\}$ such that, for all $i \in \{1, \dots, d-1\}$, $|v_{\sigma(i)} - o_{\sigma(i)})| \leq |v_{\sigma(i+1)} - o_{\sigma(i+1)})|$. Finally, thanks to the symmetries of s, we extend this last result to $\mathrm{H}_{(\lambda,k)}(\mathrm{O}(s,+))$:

$$\mathrm{H}_{(\lambda,k)}(\mathrm{O}(s,+)) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid 0 \leq s(\mathbf{v}) < \sum_{i=d-k+1}^{d} \left| 2\lambda |v_{\sigma(i)} - o_{\sigma(i)}| - \lambda^2 \right| \right\}. \quad (9)$$

According to Definition 6 and intrinsic separability properties, the discrete set $\mathrm{H}_{(\lambda,k)}(\mathrm{O}(s,+))$ is a good candidate for being an arithmetic discrete hypersphere. Before introducing such a definition, we propose notations to express the upper bound of the inequation in expression (9). Indeed, one can see it as a norm, depending of k, applied on a vector, depending on the local behaviour of s.

Proposition 2. *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. The map $]\cdot[_k$ defined by:*

$$\begin{aligned}]\cdot[_k : \mathbb{R}^d &\longrightarrow \mathbb{R} \\ \mathbf{x} &\longmapsto \sum_{i=d-k+1}^{d} |x_{\sigma(i)}|, \end{aligned} \quad (10)$$

with σ the permutation of the set $\{1, \dots, d\}$ such that, for all $i \in \{1, \dots, d-1\}$, $|x_{\sigma(i)}| \leq |x_{\sigma(i+1)}|$, is a norm.

Proof. Here, *positivity*, *scalability* and *triangle inequality* are evident properties.

Definition 11 (The k-Minimality Norm). *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k < d$. We call* k-minimality norm*, the norm $]\cdot[_k$.*

Similarly as the k-adjacency norm, the k-minimality norm is related to usual discrete norms, $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

Remark 4. Let d be the dimension of the space and $\mathbf{v} \in \mathbb{R}^d$. Then:

$$\begin{cases}]\mathbf{v}[_{(d-1)} = \|\mathbf{v}\|_\infty, \\ ]\mathbf{v}[_0 = \|\mathbf{v}\|_1. \end{cases}$$

The upper bound in Equation (9) can be considered as the k-minimality norm of a particular vector.

To achieve our goal to obtain a thickness depending on the local behaviour of s, and more generally of the curve, we propose to define the *Discrete Variation Map* of a function, according to a thickness parameter:

Definition 12 (Discrete Variation Map). *Let d be the dimension of the space. Let $\lambda \in \mathbb{R}$ and $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function. The* discrete variation map *$\Delta_\lambda f$ of* normal thickness λ *related to the function f is:*

$$\begin{aligned}\Delta_\lambda f : \mathbb{Z}^d &\longrightarrow \mathbb{R}^d \\ \mathbf{v} &\longmapsto \left(f\left(\mathbf{v} + \frac{\partial_1 f(\mathbf{v})}{|\partial_1 f(\mathbf{v})|}\lambda \mathbf{e_1}\right) - f(\mathbf{v}), \ldots, f\left(\mathbf{v} + \frac{\partial_d f(\mathbf{v})}{|\partial_d f(\mathbf{v})|}\lambda \mathbf{e_d}\right) - f(\mathbf{v})\right).\end{aligned} \tag{11}$$

Now, we can define the *separating arithmetic discrete hypersphere* according to the Definition 6 of an arithmetic discrete hypersphere.

Definition 13 (Separating Discrete Hypersphere). *Let d be the dimension of the space and $k \in \mathbb{N}$ such that $k \leq d$. Let $\mathbf{o} = (o_1, \ldots, o_d)$ and $r \in \mathbb{R}_+^\star$. Let $\lambda_1 \in \mathbb{R}_-$ and $\lambda_2 \in \mathbb{R}_+$. The* arithmetic discrete hypersphere $\mathrm{S}(\mathbf{o}, r, k, \lambda_1, \lambda_2)$ *with center $\mathbf{o}$, radius r,* normal thickness $\lambda = \lambda_2 - \lambda_1$ *and related to the k-minimality norm is defined by:*

$$\mathrm{S}(\mathbf{o}, r, k, \lambda_1, \lambda_2) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid \omega_{(k,\lambda_1)}(\mathbf{v}) \leq s(\mathbf{v}) < \omega_{(k,\lambda_2)}(\mathbf{v}) \right\}. \tag{12}$$

with $\omega_{(k,\lambda_1)}(\mathbf{v}) = -]\Delta_{\lambda_1} s(\mathbf{v})[_k$ and $\omega_{(k,\lambda_2)}(\mathbf{v}) =]\Delta_{\lambda_2} s(\mathbf{v})[_k$.

Remark 5. In Definition 13, we arbitrarily decide to consider a large inequality outside the sphere and a strict one inside it. We also can do the opposite choice without changing the properties enounced below.

Due to the underlying notion of (λ, k)-hull, the arithmetic discrete hypersphere has separability properties.

Theorem 2 (Separating Discrete Hyperspheres and k-Separability). *The arithmetic discrete hypersphere $\mathrm{S}(\mathbf{o}, r, k, \lambda_1, \lambda_2)$ with normal thickness $\lambda = \lambda_2 - \lambda_1$, such that $\lambda \in [1, +\infty[$, is (λ, k)-separating in $\mathbb{Z}^d$.*

Proof (Sketch). The (λ, k)-hull of a discrete object (λ, k)-separates its interior and the remaining discrete space. So the union of the (λ_1, k)-hull of an object and of the (λ_2, k)-hull of its complement is $(\lambda_2 - \lambda_1, k)$-separating.

In particular, we characterize k-minimal arithmetic discrete hyperspheres. For instance, part of a 1-minimal sphere is drawn in Figure 1. We notice that voxels of one component of the exterior are 0-adjacent with voxels of the other component as expected.

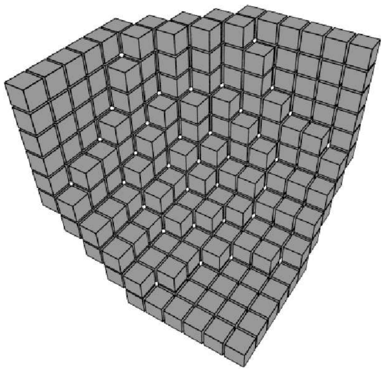


Fig. 1. Eighth of a 1-minimal discrete sphere with radius 9.

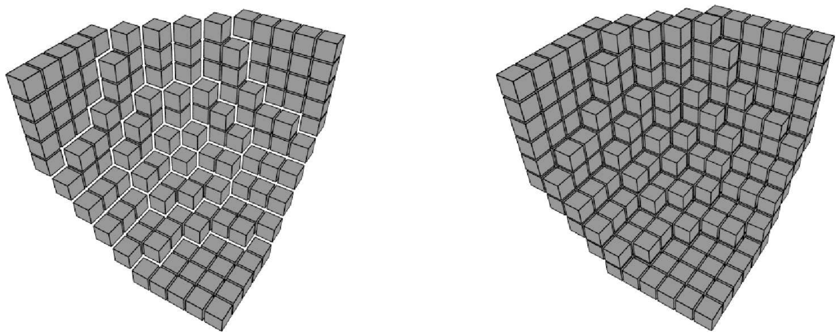
Theorem 3 (Arithmetic Discrete Hyperspheres and k-Minimality). *The arithmetic discrete hypersphere* $\mathrm{S}(\mathbf{o}, r, k, 0, 1)$ *is* k*-minimal in* $\mathbb{Z}^d$ *for* $k \in \{0, \ldots, d-1\}$*. Arithmetic discrete hyperspheres* $\mathrm{S}(\mathbf{o}, r, k, \epsilon - 1, \epsilon)$ *are also* k*-minimal for* $\epsilon \in [0, 1[$ *and* $k \in \{1, \ldots, d-1\}$*.*

Proof (Sketch). In a discrete hypersphere $\mathrm{S}(\mathbf{o}, r, k, \epsilon, 1-\epsilon))$, a k-simple point is a voxel which is just $(k-1)$-connected with one of both k-connected components of the exterior. Due to the symmetry of the hypersphere, if such a configuration does not appear in the neighborhood of the voxels $\mathbf{v}$ such that $|v_1 - o_1| \simeq \cdots \simeq |v_d - o_d| \simeq (r/\sqrt{d})$, it then appears nowhere and the discrete hypersphere is k-minimal. Finally, we verify that $(1, k)$-separating discrete hyperspheres $\mathrm{S}(\mathbf{o}, r, k, 0, 1)$ with $k \in \{0, \ldots, d-1\}$ and $\mathrm{S}(\mathbf{o}, r, k, \epsilon - 1, \epsilon)$ with $k \in \{1, \ldots, d-1\}$ are k-minimal in $\mathbb{Z}^d$.

Now we define naive and standard arithmetic discrete hyperspheres as already done for arithmetic discrete hyperplanes. In Figure 2(a), we notice, as expected, that some voxels inside the naive discrete hypersphere are 1-adjacent with voxels outside it. On the contrary, the standard discrete sphere in Figure 2(b) does not have any hole.

Definition 14 (Naive and Standard Discrete Hyperspheres). *A* naive *(respectively* standard*) discrete hypersphere is a* $(n-1)$*-minimal (respectively* 0*-minimal) one.*

In the d-dimensional discrete space, the most studied object is the discrete hyperplane [6]. It is a generalization of the arithmetic discrete line [4]. Provided with a particular arithmetic thickness, the arithmetic discrete hyperplane presents basic topological properties, that is, the k-minimality [6]. Such athickness can be defined by combining the k-minimality norm and the discrete variation map associated to the hyperplane. In Figure 3, k-minimal discrete planes have the same topological properties as k-minimal discrete sphere previously drawn.



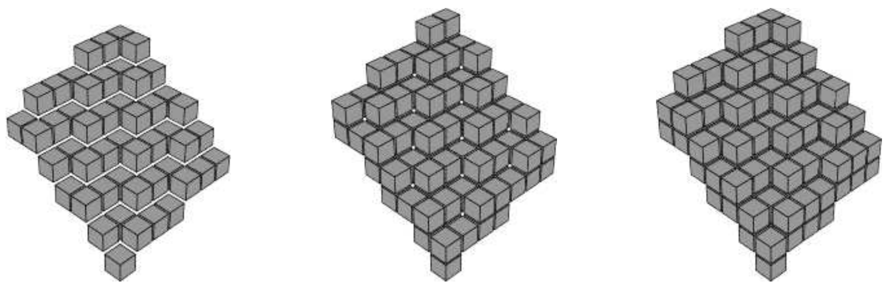
(a) Eighth of the naive discrete sphere. (b) Eighth of a standard discrete sphere.

Fig. 2. Naive and standard discrete spheres with radius 9.

Proposition 3. *Let $\mu \in \mathbb{R}$, $\mathbf{n} \in \mathbb{R}^d$. Let also $p : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a map such that: $\forall \mathbf{x} \in \mathbb{R}^d, p(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + \mu$. The k-minimal arithmetic discrete hyperplane $\mathrm{P}(\mathbf{n}, \mu, \omega_k)$ of normal vector $\mathbf{n}$ and translation parameter μ is defined as:*

$$\mathrm{P}(\mathbf{n}, \mu, \omega_k) = \left\{ \mathbf{v} \in \mathbb{Z}^d \mid 0 \le p(\mathbf{v}) < \left]\Delta_1 p(\mathbf{v})\right[_k \right\} . \tag{13}$$

Consequently, our characterization is not specific to hyperpshere but rather a generalization of what we already known about the discrete hyperplane.



(a) 0-minimal discrete plane. (b) 1-minimal discrete plane. (c) 2-minimal discrete plane.

Fig. 3. k-minimal discrete planes with normal vector $\mathbf{n} = (1, 2, 3)$.

6 Conclusion and Further Research

In the present paper, we have proposed an arithmetic definition of the discrete hypersphere largely drawn from the one of the discrete hyperplane, except that it takes into account the non-linearity of hyperspheres. We replace the usual constant arithmetic thickness by a thickness function. In particular, we characterize the k-minimal arithmetic discrete hypersphere thanks to a thickness function consisting on the application of the k-separability norm on a vector related to the local behaviour of the hypersphere. This characterization could be helpful to infer drawing and recognition algorithms. In the case of the discrete line, the arithmetic definition has improved the understanding of such algorithms. Why the same situation does not apply to the hypersphere or at least to the circle?

Beyond the particular case of the hypersphere, we go further in the understanding of the arithmetic discrete geometry. Focusing on simple object, namely lines or circles, one reduce the number of parameters which appear in the definition and one allows to better study them. In this point of view, we think that the present paper highlights two essential points: the use of norms and the study of the local discrete behaviour of the object. As far as we know, remarkable arithmetic discrete object have always been characterized by measuring, with norms, their local variations. Both have to be deeply investigated.

Lines or hyperplanes are linear objects and their normal vector is constant in magnitude and direction. For circles and hyperspheres, the normal vector is only constant in magnitude. The next natural step would be the arithmetic study of hypersurfaces for which the normal vector would have neither a constant magnitude nor a constant direction, in other words, the general case of hypersurfaces based on polynomials.

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