Odometers on Regular Languages

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ABSTRACT. Odometers or "adding machines" are usually introduced in the context of positional numeration systems built on a strictly increasing sequence of integers. We generalize this notion to systems defined on an arbitrary infinite regular language. In this latter situation, if $(A, <)$ is a totally ordered alphabet, then enumerating the words of a regular language L over A with respect to the induced genealogical ordering gives a one-to-one correspondence between $\mathbb{N}$ and L. In this general setting, the odometer is not defined on a set of sequences of digits but on a set of pairs of sequences where the first (resp. the second) component of the pair is an infinite word over A (resp. an infinite sequence of states of the minimal automaton of L). We study some properties of the odometer like continuity, injectivity, surjectivity, minimality,... We then study some particular cases: we show the equivalence of this new function with the classical odometer built upon a sequence of integers whenever the set of greedy representations of all the integers is a regular language; we also consider substitution numeration systems as well as the connection with β-numerations.

1. INTRODUCTION

To any infinite regular language L over a totally ordered alphabet $(A, <)$, an *abstract numeration system* $S = (L, A, <)$ is associated in the following way [22]. Enumerating the words of L by increasing genealogical order gives a one-to-one correspondence between $\mathbb{N}$ and L: the non-negative integer n being represented by the $(n+1)$-th word of the ordered language L. In particular, these systems generalize classical positional systems like the k-ary systems, the Fibonacci system or more generally the numeration systems built on a sequence of integers satisfying a linear recurrence relation whose characteristic polynomial is the minimal polynomial of a Pisot number [4].

In this framework of abstract numeration systems, the properties of the so-called S-recognizable sets of integers have been extensively studied (see for instance [22, 31, 32]). Moreover, these abstract systems have been extended to allow not only the representation of integers but also of real numbers [23]. In this latter situation, a real number is represented by an infinite word which is the limit of a converging sequence of words in L. Clearly, these systems lead to the generalization of various concepts related to the representation of integers like the automatic sequences or the summatory functions of additive functions [20, 33].

In this paper, we want to define and study the properties of odometers (also called adding machines) in the framework of abstract numeration systems built on an infinite regular language. In [19] odometers for positional numeration systems defined on a strictly increasing sequence $(U_n)_{n \in \mathbb{N}}$ of integers such that $U_0 = 1$ are investigated. In this latter situation, the odometer function is defined on the set $\mathcal{R}$ of right infinite words $\alpha_0 \alpha_1 \alpha_2 \ldots$ satisfying a *greedy property* [14], i.e., for all $\ell \geq 0$,

$$(1) \qquad \sum_{i=0}^{\ell} \alpha_i U_i < U_{\ell+1}.$$

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(We will also consider the greedy property (1) for finite words in the following.)
The least significant digits are written first: the prefix $\alpha_0 \cdots \alpha_k$ of a word in $\mathcal{R}$ has
value $\alpha_0 U_0 + \cdots + \alpha_k U_k$. The odometer is thus defined as the infinite extension
of the successor function acting on the finite words of the form $\alpha_0 \cdots \alpha_k$. As an
example, let us consider a finite word: the usual decimal representation of one
hundred seventy-two is the word "172"; but here, we will write representations
backward and so this number is written "271" and adding one leads to the word
"371". Moreover, the language of the finite words representing all the integers can
be embedded into $\mathcal{R}$ by concatenating 0^ω to the right of a greedy representation
starting with the least significant digit first. So, one hundred seventy-two gives
the element $271 (0)^\omega \in \mathcal{R}$. Adding one to an infinite word in $\mathcal{R}$ can produce a
carry propagating to the right. As an example, the application of the odometer to
$9992 (0)^\omega$ gives $0003 (0)^\omega$. In the case of the Fibonacci system where $U_0 = 1$, $U_1 = 2$
and $U_{n+2} = U_{n+1} + U_n$ applying the odometer to $010101 (0)^\omega$ gives $0000001 (0)^\omega$
(indeed, to be in $\mathcal{R}$ the greedy condition (1), i.e., the pattern "11" does not occur,
must be satisfied). For these reasons, we will consider mirror of representations all
along this paper.

There is an important literature devoted to the study of odometers. Let us briefly
quote [5] which continues the study of [19] from a combinatorial and topological
point of view, and [6] for a metrical approach. Odometers can also be defined
for two-sided dynamical systems as investigated in [15, 35], we refer to [37] for
the golden ratio case. See also [7] for an ergodic application of this notion in the
framework of unimodal maps and wild attractors. Lastly, let us mention [16] which
studies the sequential properties of the successor function for positional numeration
systems.

This paper is organized as follows. After recalling the basic notions required
in this paper, we define in Section 2 the set $\widetilde{\mathcal{K}}$ on which the odometer acts, and
state a few preliminary properties. Special focus is given on its extremal elements
in Section 3, which allows us to define the odometer in Section 4. Its first proper-
ties (continuity, injectivity, surjectivity and minimality) are then stated in Section
5. We illustrate this study by making explicit the connection with a few well-
known situations where the odometer is perfectly described: we consider the case
of positional number systems in Section 6, the case of substitution numeration
systems (with special focus on Pisot substitutions) in Section 7, and the case of
β-numeration in Section 8. We consider the possibility of getting a real represen-
tation of the odometer in Section 9 and end this paper by considering some special
cases in Section 10.

2. Preliminaries

Let $A = \{a_0 < a_1 < \cdots < a_k\}$ be a finite and totally ordered alphabet. In this
paper $L \subset A^*$ will always denote a regular language such that $a_0^* L \subseteq L$. In other
words, L is infinite and has the following property

$$(2) \qquad\qquad w \in L \Leftrightarrow \forall n \geq 0,\ a_0^n w \in L.$$

In some sense, property (2) can be related to the property of numeration systems
built on a sequence of integers $(U_n)_{n \in \mathbb{N}}$ such that if $w = w_0 \cdots w_k$ is the greedy
representation of an integer $w_0 U_0 + \cdots + w_k U_k$ (remember that we have taken
the convention to write the least significant digit first) then $w\,0^n$, $n \in \mathbb{N}$, still
satisfies the greedy condition (1) and represents the same integer. Here, since we
will consider the mirror of words in L, we will be able to write an arbitrary number
of a_0's on the right of the mirror of any word in L and still obtain mirror of words
belonging to L. Property (2) will therefore ensure the embedding of the finite words

of L representing the non-negative integers into some set of infinite sequences that will be precised later (Definition 1 below).

The minimal automaton of L is denoted $\mathcal{M}_L = (Q, q_0, A, \delta, F)$ where Q is the set of states, q_0 is the initial state, $F \subseteq Q$ is the set of final states and $\delta : Q \times A \to Q$ is the transition function. We assume that δ is *total* or equivalently that $\mathcal{M}_L$ is *complete*, i.e., δ is defined for all pairs $(q, a) \in Q \times A$ (notice that even with this assumption, $\mathcal{M}_L$ might contain a *sink*, i.e., a non-final state s such that for any $a \in A$, $\delta(s, a) = s$). As usual, δ can be extended to $Q \times A^*$. As a consequence of property (2), $\mathcal{M}_L$ has a loop in q_0 of label a_0. For the properties of the minimal automaton, see for instance [12].

For any state $q \in Q$, we denote by L_q the regular language accepted by $\mathcal{M}_L$ from state q,

$$L_q = \{w \in A^* \mid \delta(q, w) \in F\}$$

and by $\mathbf{u}_q(n)$ the number of words of length n in L_q. In particular, $L = L_{q_0}$.

Since A is totally ordered, we can order the words of A^* using the *genealogical ordering*. Let $u, v \in A^*$. We say that $u < v$ if $|u| < |v|$ or if $|u| = |v|$ and there exist $p, u', v' \in A^*$, $a, b \in A$, $a < b$ such that $u = pau'$ and $v = pbv'$. If M is a language over A, we define $Max(M)$ as the set of the greatest words of each length in M, i.e.,

$$Max(M) = \{u \in M \mid \forall v \in M, |u| = |v| \Rightarrow v \le u\}.$$

Observe that for all $n \ge 0$, $\#(Max(M) \cap A^n) \in \{0, 1\}$. In the same way, we can also define the set $Min(M)$ containing the smallest word of each length in M. It is well-known that if M is regular then $Max(M)$ and $Min(M)$ are also regular [36].

If $w = w_0 \cdots w_\ell$ is a word over A then the *reversal* (or *mirror*) of w is $w_\ell \cdots w_0$ and is denoted $\widetilde{w}$. If M is a language, then $\widetilde{M}$ is the language $\{\widetilde{w} \mid w \in M\}$. We also consider the non-deterministic finite automaton $\widetilde{\mathcal{M}_L} = (Q, F, A, \widetilde{\delta}, \{q_0\})$ having the same set of states as $\mathcal{M}_L$, F as set of initial states, the transition relation $\widetilde{\delta} \subseteq Q \times A \times Q$ is defined by

$$(q, a, r) \in \widetilde{\delta} \Leftrightarrow \delta(r, a) = q.$$

So, since q_0 is the only final state then a word w is accepted by $\widetilde{\mathcal{M}_L}$ if and only if $\widetilde{w} \in L$. Since $\mathcal{M}_L$ is accessible, in $\widetilde{\mathcal{M}_L}$ for any state q there exists at least one path from q to q_0. Moreover, we also have a loop in q_0 of label a_0. Let us already observe that in our later developments, the set of final states of $\widetilde{\mathcal{M}_L}$ will be irrelevant since we mainly work with limits of words recognized by $\widetilde{\mathcal{M}_L}$.

We denote by $\widetilde{\mathcal{L}}$ the set of infinite words over A which are the limits of the converging sequences of words belonging to $\widetilde{L}$. Otherwise stated, $x = x_0 x_1 x_2 \ldots$ belongs to $\widetilde{\mathcal{L}}$ if there exists a sequence $(w_n)_{n \in \mathbb{N}}$ of words in L such that for all $\ell > 0$ there exist $N_\ell > 0$ such that for all $n \ge N_\ell$, $\widetilde{w_n}$ and x have a common prefix of length at least ℓ. We use the topology induced by the infinite product topology on $A^{\mathbb{N}}$. Notice that if all the sates in $\mathcal{M}_L$ are final, then $\widetilde{\mathcal{L}}$ is the set of labels of infinite paths in $\widetilde{\mathcal{M}_L}$.

Due to the non-deterministic behavior of $\widetilde{\mathcal{M}_L}$, the reading of a word in this automaton can lead to more than one path. As an example, assume that in $\mathcal{M}_L$ we have three states p, q and r such that $\delta(p, a) = r$ and $\delta(q, a) = r$ then in $\widetilde{\mathcal{M}_L}$ from state r when reading a both states p and q could be reached and a non-deterministic choice has to be made. Therefore, we will not only consider words but also the extra information given by the sequence of reached states. This is the reason of the introduction of the set $\widetilde{\mathcal{K}}$ defined below.

Definition 1. We define the set $\widetilde{\mathcal{K}} \subseteq (A \times Q)^\omega$ by $(x, y) = (x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots)$ belongs to $\widetilde{\mathcal{K}}$ if and only if the following conditions hold

(1) x belongs to $\widetilde{\mathcal{L}}$,

(2) y_0 belongs to F,

(3) for all $i \geq 0$, (y_i, x_i, y_{i+1}) belongs to $\widetilde{\delta}$, i.e., $\delta(y_{i+1}, x_i) = y_i$.

Lemma 2. *If $(x, y) = (x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots)$ belongs to $\widetilde{\mathcal{K}}$ then for all $k \in \mathbb{N}$, $x_k \cdots x_0$ belongs to $L_{y_{k+1}}$.*

Proof. By definition of $\widetilde{\mathcal{K}}$, for $i = 0, \ldots, k$, $\delta(y_{i+1}, x_i) = y_i$. Therefore in $\mathcal{M}_L$ we have a path from y_{k+1} to y_0 of label $x_k \cdots x_0$. Since y_0 is a final state, this means that $x_k \cdots x_0$ is accepted from y_{k+1}. $\qquad\square$

Remark 3. Let $k \geq 0$. If $(x, y) = (x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots)$ belongs to $\widetilde{\mathcal{K}}$ then $y_0 \cdots y_k$ is completely determined by $x_0 \cdots x_k$ and y_{k+1}. This is due to the third condition in the definition of $\widetilde{\mathcal{K}}$ and because $\mathcal{M}_L$ is deterministic.

Definition 4. Let $j \geq 0$. A finite word $(x, y) = (x_0 x_1 \cdots x_k, y_0 y_1 \cdots y_k) \in (A \times Q)^{k+1}$, $k > j$, (resp. an infinite word $(x, y) = (x_0 x_1 \ldots, y_0 y_1 \ldots) \in (A \times Q)^\omega$) is said to have the property $\mathbf{max}_j$ and we write $(x, y) \in \mathbf{max}_j$ if $x_j \cdots x_0$ belongs to $Max(L_{y_{j+1}})$. In the same way, (x, y) has the property $\mathbf{min}_j$ if $x_j \cdots x_0 \in Min(L_{y_{j+1}})$.

Lemma 5. *Let $(x, y) = (x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots) \in \widetilde{\mathcal{K}}$ and $j \geq 0$. If (x, y) has property $\mathbf{max}_j$ (resp. $\mathbf{min}_j$) then for all $k < j$, (x, y) has also the property $\mathbf{max}_k$ (resp. $\mathbf{min}_k$).*

Proof. Assume that $(x, y) \in \mathbf{max}_j$ but $(x, y) \notin \mathbf{max_k}$, $k < j$. Therefore there exists $x'_k \cdots x'_0$ accepted from y_{k+1} and genealogically greater than $x_k \cdots x_0$. So $x_j \cdots x_{k+1} x'_k \cdots x'_0$ belongs to $L_{y_{j+1}}$ and is greater than $x_j \cdots x_0$. This is a contradiction. $\qquad\square$

Corollary 6. *Let $(x, y) = (x_0 x_1 x_2 \ldots, y_0 y_1 y_2 \ldots) \in \widetilde{\mathcal{K}}$ and $j \geq 0$. If $(x, y) \notin \mathbf{max}_j$ (resp. $(x, y) \notin \mathbf{min}_j$) then for all $i \geq j$, $(x, y) \notin \mathbf{max}_i$ (resp. $(x, y) \notin \mathbf{min}_i$).*

Let us now present some other properties of this set $\widetilde{\mathcal{K}}$.

Proposition 7. *For each $x \in \widetilde{\mathcal{L}}$, there exists $y \in Q^\omega$ such that (x, y) belongs to $\widetilde{\mathcal{K}}$.*

Proof. If $w = w_1 \cdots w_\ell$ is a word in L, we denote by $p(w)$ the word

$$p(w) := \delta(q_0, w_1)\, \delta(q_0, w_1 w_2) \cdots \delta(q_0, w_1 \cdots w_\ell) \in Q^\ell$$

which represents the path given by the states reached consecutively in $\mathcal{M}_L$ by reading w. Since x belongs to $\widetilde{\mathcal{L}}$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of words in $\widetilde{L}$ converging to x. For an infinite number of $n \in \mathbb{N}$, the last element of $p(\widetilde{x_n})$ is a same state in F. We take the corresponding subsequence $(x_{k_1(n)})_{n \in \mathbb{N}}$. For an infinite number of n, the words $p(\widetilde{x_{k_1(n)}})$ have the same suffix of length two. So we consider the corresponding subsequence $(x_{k_2(n)})_{n \in \mathbb{N}}$. If we iterate this process, $(x_{k_n(1)})_{n \in \mathbb{N}}$ is converging to x if n tends to infinity and the reversal of the $p(\widetilde{x_{k_n(1)}})$'s are converging to an infinite word y in Q^ω such that (x, y) belongs to $\widetilde{\mathcal{K}}$. $\qquad\square$

Example 8. In this example, we consider a regular language $L \subset \{a < b < c\}^*$ satisfying the hypothesis $a^* L \subset L$ and given by its minimal automaton depicted in Figure 1. We just present some elements belonging to $\widetilde{\mathcal{K}}$:

$$((bba)^\omega, (210)^\omega),\ \ (a)^\omega, 1(0)^\omega),\ \ (b(a)^\omega, 2(1)^\omega),$$

$$(ba(bbc)^\omega, 10(210102021)^\omega)\ \ \text{and}\ \ (ba(bbc)^\omega, 10(021210102)^\omega).$$
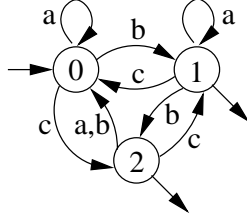
FIGURE 1. The minimal automaton of a language L.

As shown in the previous example, to one infinite word $x \in \widetilde{\mathcal{L}}$, it may correspond more than one sequence of states. If two such sequences give rise to elements in $\widetilde{\mathcal{K}}$ then they differ almost everywhere.

Lemma 9. *Let (x, y) and (x, y') be two elements of $\widetilde{\mathcal{K}}$ such that $y \neq y'$. Then there exists an index i such that $y_i \neq y'_i$ and for all $n \geq 0$, $y_{i+n} \neq y'_{i+n}$.*

Proof. This is a direct consequence of Remark 3. $\qquad\square$

The next proposition shows that to any finite word in L corresponds at least one element in $\widetilde{\mathcal{K}}$. The same kind of properties holds in the case of numeration systems built on a sequence of integers. If w is the greedy representation of an integer (least significant digit first), then $w(0)^\omega$ belongs to the set $\mathcal{R}$ of right infinite words satisfying the greedy property 1.

Proposition 10. *If $w = w_k \cdots w_1$ belongs to L then there exists $y_1 \cdots y_k \in Q^k$ such that $(w_1 \cdots w_k\, (a_0)^\omega, y_1 \cdots y_k\, (q_0)^\omega)$ belongs to $\widetilde{\mathcal{K}}$.*

Proof. By our assumption (2) on L, if w belongs to L then $a_0^n w$ also belongs to L, $n \geq 0$. Using the same notation as in the proof of Proposition 7, if $y_k \cdots y_1 = p(w_k \cdots w_1)$ then $q_0^n\, y_k \cdots y_1 = p(a_0^n\, w_k \cdots w_1)$. The result follows easily. $\qquad\square$

3. Properties of $Max(\widetilde{\mathcal{K}})$ and $Min(\widetilde{\mathcal{K}})$

For odometers defined upon classical positional systems related to a sequence $(U_n)_{n \in \mathbb{N}}$ of integers, some sequences of digits play a special role. Namely, they are the sequences for which the carry when adding one can propagate to infinity. A sequence $\alpha_0 \alpha_1 \alpha_2 \ldots$ is of this kind if

$$\sum_{i=0}^{\ell_j} \alpha_i U_i = U_{\ell_j + 1} - 1$$

for a strictly increasing infinite sequence $(\ell_j)_{j \in \mathbb{N}}$ of indices. In our framework, the corresponding elements in $\widetilde{\mathcal{K}}$ will be defined as the elements in $Max(\widetilde{\mathcal{K}})$. The elements which have the dual property will belong to $Min(\widetilde{\mathcal{K}})$.

Definition 11. Let us define two particular subsets of $\widetilde{\mathcal{K}}$,

$$Max(\widetilde{\mathcal{K}}) = \{(x, y) \in \widetilde{\mathcal{K}} \mid \forall i \geq 0, (x, y) \in \mathbf{max}_i\}$$

and

$$Min(\widetilde{\mathcal{K}}) = \{(x, y) \in \widetilde{\mathcal{K}} \mid \forall i \geq 0, (x, y) \in \mathbf{min}_i\}.$$

Let us observe that following Lemma 5, then it is sufficient in the definition of $Max(\widetilde{\mathcal{K}})$ (resp. $Min\widetilde{\mathcal{K}}$)) that there exist infinitely many i such that $(x, y) \in \mathbf{max}_i$ (resp. $(x, y) \in \mathbf{min}_i$). In this section, we concentrate on the structural properties of those sets $Max(\widetilde{\mathcal{K}})$ and $Min(\widetilde{\mathcal{K}})$. The following lemma is obvious.

Lemma 12. *Let L be a regular language satisfying our assumption* (2).

- *A word w belongs to $Min(L)$ if and only if for all $n \geq 0$, $a_0^n w$ belongs to $Min(L)$ (assuming that a_0 is the smallest letter in the ordered alphabet A).*
- *Let q be a state of $\mathcal{M}_L$. If vw belongs to $Max(L_q)$ then the word w belongs to $Max(L_{\delta(q,v)})$.*

Definition 13. Let w be the smallest word in $Min(L)$, i.e., w is the first word in the ordered language L. If $w = w_1 \cdots w_\ell$ is not the empty word (i.e., if $q_0 \notin F$) then we have a path in $\mathcal{M}_L$ of the form

$$q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} \cdots \xrightarrow{w_\ell} q_\ell \in F.$$

We set $\mathbf{0} = (\widetilde{w}(a_0)^\omega, q_\ell \cdots q_1 (q_0)^\omega)$. Otherwise $w = \varepsilon$ and we set $\mathbf{0} = ((a_0)^\omega, (q_0)^\omega)$.

Proposition 14. *The sets $Max(\widetilde{\mathcal{K}})$ and $Min(\widetilde{\mathcal{K}})$ are non-empty.*

Proof. As a consequence of Lemma 5 and Lemma 12, the element $\mathbf{0}$ given in Definition 13 belongs to $Min(\widetilde{\mathcal{K}})$.

We use the same idea as in the proof of Proposition 7. Let w_i be the i-th word of $Max(L)$ (clearly, $|w_i| < |w_{i+1}|$ for all $i \geq 1$). An infinite number of w_i's have the same last letter a_{k_1} and lead in $\mathcal{M}_L$ from q_0 to a same final state q_{k_1}. We therefore consider the corresponding subsequence $(w_{k_1(n)})_{n \in \mathbb{N}}$ built upon those w_i's. We iterate this process: an infinite number of words among the $w_{k_1(n)}$'s have the same suffix $a_{k_2} a_{k_1}$ and finally lead in $\mathcal{M}_L$ to the states q_{k_2} followed by q_{k_1}. Therefore we build a sequence converging to

$$(a_{k_1} a_{k_2} \cdots, q_{k_1} q_{k_2} \cdots).$$

Thanks to Lemma 12, this element belongs to $Max(\widetilde{\mathcal{K}})$. $\square$

Example 15. We consider the language and the automaton given in Example 8. It is easy to check that $(b(c)^\omega, 2(120)^\omega)$, $(c^\omega, (120)^\omega)$ and $(c^\omega, (201)^\omega)$ belong to $Max(\widetilde{\mathcal{K}})$. We also have $\mathbf{0} = (b(a)^\omega, 1(0)^\omega)$ and $(a^\omega, 1^\omega)$ as elements of $Min(\widetilde{\mathcal{K}})$. To show that these elements are the only ones, we will need some more results about the structure of $Max(L_q)$ and $Min(L_q)$.

In some particular cases, the structure of $Max(L_q)$ is easy to obtain.

Notation 16. Let q be a state in $\mathcal{M}_L$. If there exists $a \in A$ such that $\delta(q,a)$ is not the sink then we denote by $m(q)$ the largest letter having this property, otherwise we set $m(q) = \varepsilon$.

Recall that a state s is a sink if for any $a \in A$, $\delta(s,a) = s$ and s is not a final state. Let us introduce a small algorithm to detect what we will call the *maximal cycles* in $\mathcal{M}_L$.

Algorithm 17. Let $q \in Q$.

- Set $y_0 \leftarrow q$ and $i \leftarrow 0$.
- If $m(y_i) \neq \varepsilon$ then set $y_{i+1} \leftarrow \delta(y_i, m(y_i))$ and $i \leftarrow i + 1$. Otherwise stop the algorithm.
- If $y_0, \ldots, y_i$ are all different, repeat the previous step. Otherwise, a cycle is found and stop the algorithm.

After applying this algorithm to a state $q \in Q$ which is not the sink, we can have two kinds of situations. If we encounter some state y_k such that $m(y_k) = \varepsilon$ then we have obtained something like

$$y_0 \xrightarrow{m(y_0)} y_1 \xrightarrow{m(y_1)} \cdots \longrightarrow y_{k-1} \xrightarrow{m(y_{k-1})} y_k$$

where all the y_i's are different and y_k belongs necessarily to F (because $\mathcal{M}_L$ is minimal). Or we have the situation

$$y_0 \xrightarrow{m(y_0)} y_1 \xrightarrow{m(y_1)} \cdots \longrightarrow \mathbf{y_k} \xrightarrow{m(y_k)} \cdots \longrightarrow y_{k+n} \xrightarrow{m(y_{k+n})} \mathbf{y_k}$$

where $y_0, \ldots, y_{k+n}$ are all different; we say that $(y_k, m(y_k), \ldots, y_{k+n}, m(y_{k+n}), y_k)$ is a *maximal cycle* starting in y_k and the word $m(y_k) \cdots m(y_{k+n})$ is the *label* of this cycle. Notice that two maximal cycles have no state in common or share exactly the same states. In this latter case, the label of one of the two cycles is a cyclic permutation of the other one.

Example 18. Considering the automaton of Example 8, we have three maximal cycles: $(0, c, 2, c, 1, c, 0)$, $(2, c, 1, c, 0, c, 2)$ and $(1, c, 0, c, 2, c, 1)$ all having the same label ccc and sharing the same states.

Lemma 19. *If $\mathcal{C}$ is a maximal cycle of label w starting in q, then there exist an integer $k \leq |w|$ depending only on $\mathcal{C}$ and k words $u_1, \ldots, u_k$ of minimal length such that $|u_i| \not\equiv |u_j| \mod |w|$ if $i \neq j$ and*

$$Max(L_q) = w^* \{u_1, \ldots, u_k\}.$$

Proof. Let w be the label of a maximal cycle $\mathcal{C}$ starting in q. If v belongs to $Max(L_q)$ then by construction of the maximal cycle, wv also belongs to $Max(L_q)$. Assume now that $u, v \in Max(L_q)$ are such that $|u| \equiv |v| \mod |w|$ and $|u| < |v|$. Therefore, there exists i such that $w^i u$ belongs to $Max(L_q)$ and $|w^i u| = |v|$. But $Max(L_q)$ contains at most one word of each length, so $w^i u = v$. Consequently, if v belongs to $Max(L_q)$ then it is of the form $w^n u$ for some $n \geq 0$ and w is not a prefix of u. For each $j \in \{0, \ldots, k-1\}$ there is at most one u of this kind such that $|u| \equiv j \mod |w|$ (actually u is the smallest word of length $j + n|w|$ possibly belonging to $Max(L_q)$, $n \geq 0$). Notice that it does not mean that $|u| < |w|$. Clearly two states in the same maximal cycle give rise to the same kind of maximal set. $\square$

It is more difficult to express the form of $Max(L_q)$ when this set is infinite and q does not belong to a maximal cycle. But hopefully we have a more general result extending Lemma 19 which holds even if q does not belong to a maximal cycle. Indeed, since $\#(Max(L_q) \cap A^n) \leq 1$ for all $n \in \mathbb{N}$ then it is well-known (see [27] or [36]) that there exists a finite set R of words, an integer $k \geq 0$ and words $u_i, w_i \in A^*$, $v_i \in A^+$, $i = 0, \ldots, k$ such that

$$(3) \qquad Max(L_q) = \bigcup_{i=0}^{k} u_i\, v_i^*\, w_i \cup R$$

where the languages $u_i\, v_i^*\, w_i$ are pairwise disjoint and also disjoint from R. Otherwise stated, if $i \neq j$ then

$$\{|u_i w_i| + n|v_i| : n \in \mathbb{N}\} \cap \{|u_j w_j| + n|v_j| : n \in \mathbb{N}\} = \emptyset$$

and $\{|u_i w_i| + n|v_i| : n \in \mathbb{N}\} \cap |R| = \emptyset$, for all i ($|R|$ denotes the set of lengths of elements of R). One can observe that the form of $Max(L_q)$ given in Lemma 19 is a special case of (3).

Proposition 20. *Any element in $Max(\widetilde{\mathcal{K}})$ is ultimately periodic and $Max(\widetilde{\mathcal{K}})$ is finite.*

Proof. (a) The ideas of the first part of this proof are the same as in [23, Lemma 7]. Let q be such that $\#Max(L_q) = \infty$. If x is a word in $Max(L_q)$ of length large enough then thanks to (3) there exist unique words u, v, w (depending on x) such that $x = uv^n w$. Among

$$(4) \qquad \delta(q, u),\ \delta(q, u\, v), \ldots,\ \delta(q, u\, v^{\#Q})$$

a same state appears at least twice. Let t be the first state appearing twice in this list. Let $i < j$ be the smallest integers such that $\delta(q, u\,v^i) = \delta(q, u\,v^j) = t$. We set $P = (j - i)|v|$. We can already notice that P is bounded by $\#Q \cdot |v|$. In what follows, we write simply t, i, j, P assuming that the word x is understood from the context.

(b) If $z = z_0 \cdots z_\ell$ is a word over A and f is a state, we build the pair

$$p_f(z) := (z_0 \cdots z_\ell, \delta(f, z_0)\, \delta(f, z_0 z_1) \cdots \delta(f, z_0 \cdots z_\ell)).$$

Actually, $p_f(z)$ contains a word z and the sequence of states reached from f when reading z in $\mathcal{M}_L$ (since $\mathcal{M}_L$ is complete, $\delta(f, z_0 \cdots z_k)$ is defined for $0 \le k \le \ell$). Consider again the word $x = uv^n w \in Max(L_q)$ introduced in (a). For n large enough, $p_q(uv^n w)$ is a word over $A \times Q$ having

   i) a non-periodic prefix $p_q(u\,v^i)$ of length bounded by $|u| + \#Q \cdot |v|$;

   ii) a maximal periodic factor having a period of length P; actually the Euclidean division of $n - i$ by $P/|v|$ gives

$$n - i = m\,\frac{P}{|v|} + r \quad \text{with } r < P/|v|.$$

     The periodic factor corresponding to $v^{mP/|v|}$ is $p_t(v^{mP/|v|})$ and the period corresponding to $v^{P/|v|}$ is $p_t(v^{P/|v|})$ where t is as in (a) the first state appearing twice in the list (4);

   iii) a non-periodic suffix of length bounded by $|w| + P$, indeed this factor corresponds to $v^r w$ and is of the form $p_t(v^r w)$.

For a better understanding, the situation is sketched in Figure 2.
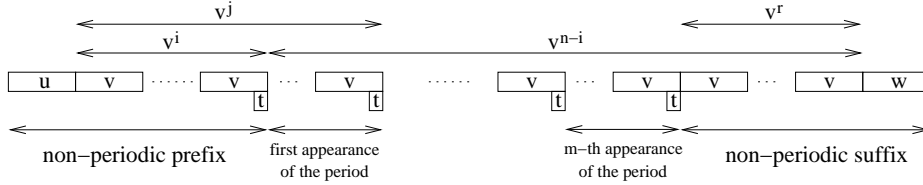

FIGURE 2. A schematic representation of $p_q(uv^n w)$.

(c) Let $n' > n$ and $x' = uv^{n'} w$. Then $p_q(uv^n w)$ and $p_q(uv^{n'} w)$ have the same prefix corresponding to $u\,v^i$. The periodic factors have the same period of length P but the number of repetitions could be larger for x'. Finally, if n and n' are not congruent modulo $P/|v|$ then the corresponding suffixes could be different, otherwise the suffixes are the same. Notice that there are only finitely many possible suffixes corresponding to the words of the form $v^r w$ for $r = 0, \ldots, P/|v| - 1$.

(d) From the previous observations, we can easily exhibit elements in $Max(\widetilde{\mathcal{K}})$. Let n_0 be large enough and set $x_m = p_q(u\,v^{n_0 + mP/|v|}\,w)$ for $m \ge 0$. From the previous point, $\widetilde{x_m}$ is converging to an ultimately periodic element in $\widetilde{\mathcal{K}}$. From Lemma 12, this element belongs to $Max(\widetilde{\mathcal{K}})$.

(e) Clearly, any element $(x, y) = (x_0 x_1 \ldots, y_1 y_2 \ldots)$ in $Max(\widetilde{\mathcal{K}})$ is ultimately periodic. Since Q is finite, a state q must appear infinitely often in y say in strictly increasing positions $k(n)$. For each n, $x_{k(n)} \cdots x_0$ belongs to $Max(L_q)$ and the words of this kind have a longer and longer common suffix when n is increasing. As a consequence of (a), (x, y) is ultimately periodic with x of the form $\widetilde{w}\,(\widetilde{v})^\omega$, for some finite words v and w.

(f) In (d), we have obtained elements of $Max(\widetilde{\mathcal{K}})$ of a special form but in (e) we have shown that any element in $Max(\widetilde{\mathcal{K}})$ is of this kind. To conclude, we have to

show that $Max(\widetilde{\mathcal{K}})$ is finite. First from (3), for each state q the number of words u_i, v_i, w_i used to obtain the structure of $Max(L_q)$ is finite. For each of these 3-tuples (u_i, v_i, w_i) of words, we can obtain ultimately periodic elements in $Max(\widetilde{\mathcal{K}})$ but the period of such an element is bounded by $\#Q \cdot |v_i|$ (see (a)) and the length of its prefix is bounded by $|w_i| + \#Q \cdot |v_i|$ (see (b)). In other words, we have a finite number of 3-tuples (u_i, v_i, w_i) each one giving at most a finite number of elements in $Max(\widetilde{\mathcal{K}})$. $\qquad\square$

This proof shows that the elements of $Max(\widetilde{\mathcal{K}})$ can be determined by the knowledge of the languages $Max(L_q)$. As we will see in the following example, obtaining the decomposition of the form (3) for the languages $Max(L_q)$ gives rise to all the elements in $Max(\widetilde{\mathcal{K}})$. Moreover, observe that these languages $Max(L_q)$ can be efficiently obtained from $\mathcal{M}_L$.

Naturally, Algorithm 17, Lemma 19 and Proposition 20 are easily adapted to the set $Min(\widetilde{\mathcal{K}})$. In this case, similarly as in Notation 16, if there exists $a \in A$ such that $\delta(q, a)$ is not the sink then we denote by $m(q)$ the smallest letter having this property.

Example 21. Continuing again Example 8. We are now able to show that $Max(\widetilde{\mathcal{K}})$ contains exactly the elements $(b(c)^\omega, 2(120)^\omega)$, $(c^\omega, (120)^\omega)$ and $(c^\omega, (201)^\omega)$. We have a maximal cycle of label ccc containing the three states of $\mathcal{M}_L$, so using Lemma 19 we obtain $Max(L_0) = (ccc)^*\{c, cc, ccb\}$, $Max(L_1) = (ccc)^*\{\varepsilon, b, cc\}$ and $Max(L_2) = (ccc)^*\{\varepsilon, c, cb\}$. Let us first see which elements in $Max(\widetilde{\mathcal{K}})$ come from the words in $Max(L_0)$. The word $(ccc)^n c$ read from the state 0 gives in $\mathcal{M}_L$ the path

$$0 \xrightarrow{c} 2 \xrightarrow{c} 1 \xrightarrow{c} 0 \cdots \xrightarrow{c} 2 \xrightarrow{c} 1 \xrightarrow{c} 0 \xrightarrow{c} 2.$$

With the notation of the proof of Proposition 20, we have

$$p_0((ccc)^n c) = ((ccc)^n c, (210)^n 2).$$

Reading this path from the right and letting n tends to infinity gives the element $(c^\omega, 2(012)^\omega) = (c^\omega, (201)^\omega)$. In the same way, the word $(ccc)^n cc$ gives $(c^\omega, (120)^\omega)$ and finally $(ccc)^n ccb$ gives $(b(c)^\omega, 2(120)^\omega)$. If we do the same for the words in $Max(L_i)$, $i = 1, 2$, then we consider paths starting in i and we obtain exactly the same three elements of $\widetilde{\mathcal{K}}$. It is clear that each set $Max(L_q)$ produces the same elements of $Max(\widetilde{\mathcal{K}})$ because all the states are in the same maximal cycle.

Let us now show that $Min(\widetilde{\mathcal{K}})$ contains exactly $\mathbf{0} = (b(a)^\omega, 1(0)^\omega)$ and $(a^\omega, 1^\omega)$. Here we have two minimal cycles: $(0, a, 0)$ and $(1, a, 1)$. So thanks to the analogue of Lemma 19, we have $Min(L_0) = a^*b$ and $Min(L_1) = a^*$. From the analogue of (3), one finds $Min(L_2) = a^*ab \cup \{\varepsilon, c\}$. For instance, starting in state 2 and reading $a^n ab$ gives the path

$$2 \xrightarrow{a} 0 \xrightarrow{a} 0 \cdots \xrightarrow{a} 0 \xrightarrow{a} 0 \xrightarrow{b} 1.$$

Reading this path from the right and letting n tends to infinity gives the element $\mathbf{0}$. Starting in 0 with $a^n b$ also leads to the same element $\mathbf{0}$. Finally starting in 1 with a^n gives $(a^\omega, 1^\omega)$. Obviously, if two states q and q' belong to two different minimal cycles then the sets $Min(L_q)$ and $Min(L_{q'})$ will never lead to a same element in $Min(\widetilde{\mathcal{K}})$ because the two cycles have no state in common.

4. Defining the odometer

In [19], if a sequence $\alpha_0\alpha_1\alpha_2\ldots$ of digits belonging to $\mathcal{R}$, i.e., satisfying the greedy condition (1), is such that there exists M such that for all $\ell \geq M$

$$[\alpha_0 \cdots \alpha_\ell] := \sum_{i=0}^{\ell} \alpha_i U_i < U_{\ell+1} - 1$$

then the odometer maps $\alpha_0\alpha_1\alpha_2\ldots$ onto $\alpha_0' \cdots \alpha_k' \alpha_{k+1}\alpha_{k+2}\ldots \in \mathcal{R}$ where $\alpha_0' \cdots \alpha_k'$ is the representation of $[\alpha_0 \cdots \alpha_k]+1$ computed through the greedy algorithm (and it is shown that the result is independent of the choice of the index $k \geq M$). Obviously, the representations of $[\alpha_0 \cdots \alpha_k]$ and $[\alpha_0 \cdots \alpha_k]+1$ have the same length. Otherwise, infinitely often the situation $[\alpha_0 \cdots \alpha_\ell] = U_{\ell+1} - 1$ occurs and then the odometer is defined to map $\alpha_0\alpha_1\alpha_2\ldots$ onto 0^ω.

Here we want to do the same in the context of abstract numeration systems and define a function $\tau_L : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}$, or simply τ if L is clearly understood, having the adding behavior awaited for an odometer. First we define τ on $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$. Assume that for $(x,y) = (x_0x_1\ldots, y_0y_1\ldots) \in \widetilde{\mathcal{K}}$ there exists $i \geq 0$ such that $(x,y) \notin \mathbf{max}_i$. For each state q of $\mathcal{M}_L$, we define the function

$$\mathrm{Succ}_q : L_q \to L_q$$

mapping the k-th word in the genealogically ordered language L_q to the $(k+1)$-th one in the same language (if L_q is finite, we decide that Succ_q maps the largest word in L_q onto the smallest one). Since $(x,y) \notin \mathbf{max}_i$, it is clear that $x_i \cdots x_0$ and $\mathrm{Succ}_{y_{i+1}}(x_i \cdots x_0)$ have the same length. Let us denote this latter word belonging to $L_{y_{i+1}}$ by $x_i' \cdots x_0'$. We set $y_i' = \delta(y_{i+1}, x_i')$ and $y_j' = \delta(y_{j+1}', x_j')$ for $j = i-1, \ldots, 0$. In other words, $y_i', \ldots, y_0'$ are the states reached in $\mathcal{M}_L$ when reading $x_i' \cdots x_0'$ from y_{i+1}. In particular, observe that y_0' belongs to F. Hence $(x_0' \cdots x_i' x_{i+1}\ldots, y_0' \cdots y_i' y_{i+1}\ldots)$ belong to $\widetilde{\mathcal{K}}$. The function τ is defined by

$$\tau(x_0 \cdots x_i\, x_{i+1}\ldots, y_0 \cdots y_i\, y_{i+1}\ldots) = (x_0' \cdots x_i'\, x_{i+1}\ldots, y_0' \cdots y_i'\, y_{i+1}\ldots).$$

We have to show that τ is well-defined. Assume that there exist $i < j$ such that $(x,y) \notin \mathbf{max}_i$ and $(x,y) \notin \mathbf{max}_j$. (Notice that in view of Corollary 6, if $(x,y) \notin \mathbf{max}_i$ then for all $k \geq i$, $(x,y) \notin \mathbf{max}_k$.) Then the previous construction does not depend on the choice of the index. Indeed, notice that by definition of $\widetilde{\mathcal{K}}$, $\delta(y_{j+1}, x_j \cdots x_{i+1}) = y_{i+1}$ and as a consequence of the genealogical ordering,

$$\mathrm{Succ}_{y_{j+1}}(x_j \cdots x_{i+1}x_i \cdots x_0) = x_j \cdots x_{i+1}\, \mathrm{Succ}_{y_{i+1}}(x_i \cdots x_0).$$

Therefore, as a consequence of Remark 3, the corresponding sequences of states are the same: if $y_j'' \cdots y_0''$ are the states reached in $\mathcal{M}_L$ when reading $\mathrm{Succ}_{y_{j+1}}(x_j \cdots x_0)$ from y_{j+1}, we have

$$y_j'' \cdots y_{i+1}'' y_i'' \cdots y_0'' = y_j \cdots y_{i+1} y_i' \cdots y_0'.$$

Thus, the value of τ does not depend on the index i such that $(x,y) \notin \mathbf{max}_i$.

Example 22. We still consider the language and the automaton given in Example 8. For instance, $(x,y) = (bbaccb(a)^\omega, 210201(0)^\omega)$ belongs to $\widetilde{\mathcal{K}}$. The word b belongs to $Max(L_1)$ so $(x,y) \in \mathbf{max}_1$ but bb belongs to $L_0 \setminus Max(L_0)$ so $(x,y) \notin \mathbf{max}_2$. It is easy to see that the next word accepted from 0 is cc and the path is $0 \overset{c}{\to} 2 \overset{c}{\to} 1$, thus

$$\tau(bb|accb(a)^\omega, 21|0201(0)^\omega) = (cc|accb(a)^\omega, 12|0201(0)^\omega).$$

If we had considered the word $ccabb$ accepted from state 1 (because $(x,y) \notin \mathbf{max}_5$), the next word in L_1 is $ccacc$ and this would have lead to the same element in $\widetilde{\mathcal{K}}$:

$$\tau(bbacc|b(a)^\omega, 21020|1(0)^\omega) = (ccacc|b(a)^\omega, 12020|1(0)^\omega).$$

In the next section, we will see that in general, the continuity of the odometer cannot be ensured on $Max(\widetilde{\mathcal{K}})$ whatever is the value taken by τ for the points in this set (see Example 29). Therefore, we decide that for all $(x, y) \in Max(\widetilde{\mathcal{K}})$, $\tau(x, y) = \mathbf{0}$, where $\mathbf{0}$ is the canonical element of $Min(\widetilde{\mathcal{K}})$ given in Definition 13.

Remark 23. We can as in [19] or [40] define a partial ordering on $\widetilde{\mathcal{K}}$, called *antipodal order*, in the following way. We have $(x, y) \prec (x', y')$ if $(x, y) = (x', y')$ or there exists some index k such that $x_k < x'_k$ and for all $j > k$, $(x_j, y_j) = (x'_j, y'_j)$. The elements in $Max(\widetilde{\mathcal{K}})$ are therefore the maximal elements in $(\widetilde{\mathcal{K}}, \prec)$. For any $(x, y) \notin Max(\widetilde{\mathcal{K}})$, then its image under τ is the smallest (with respect to $\prec$) of all the elements in $\widetilde{\mathcal{K}}$ which are larger than (x, y). Hence the map τ is a successor function which can be considered as an adic transformation following [40].

5. Properties of the odometers

Proposition 24. *The application τ is surjective onto $\widetilde{\mathcal{K}} \setminus Min(\widetilde{\mathcal{K}})$.*

Proof. The proof is immediate. Let $(x, y) = (x_0 x_1 \ldots, y_0 y_1 \ldots)$ be such that (x, y) is not in $\mathbf{min}_i$ for some i. Therefore, there exists a word $x' = x'_i \cdots x'_0$ of length $i + 1$ such that $\mathrm{Succ}_{y_{i+1}}(x'_i \cdots x'_0) = x_i \cdots x_0$. As usual, if $y' = y'_i \cdots y'_0$ is the path followed in $\mathcal{M}_L$ from y_{i+1} when reading x' then

$$\tau(\widetilde{x'} x_{i+1} x_{i+2} \ldots, \widetilde{y'} y_{i+1} y_{i+2} \ldots) = (x, y).$$

$\square$

Remark 25. A similar result holds in the framework of positional number systems: the odometer is proved to be surjective if and only if 0^ω admits an antecedent (see [19]).

Proposition 26. *The application τ is injective on $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$.*

Proof. Let $(x, y) = (x_0 x_1 \ldots, y_0 y_1 \ldots)$ and $(x', y') = (x'_0 x'_1 \ldots, y'_0 y'_1 \ldots)$ be in $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$ and such that $\tau(x, y) = \tau(x', y')$. Let i and i' be such that $x_i \cdots x_0 \notin Max(L_{y_{i+1}})$ and $x'_i \cdots x'_0 \notin Max(L_{y'_{i+1}})$. Pose $I = \sup\{i, i'\}$. Thanks to Lemma 5, (x, y) and (x', y') do not belong to $\mathbf{max}_I$ so the application of τ will at most affect their prefix of length $I + 1$. Since $\tau(x, y) = \tau(x', y')$, we have $x_j = x'_j$ and $y_j = y'_j$ for all $j > I$. Therefore, $x_I \cdots x_0$ and $x'_I \cdots x'_0$ belongs to $L_{y_{I+1}} = L_{y'_{I+1}}$ and have the same successor. So these two words are the same. The conclusion that $y_I \cdots y_0$ and $y'_I \cdots y'_0$ are the same comes from Remark 3. $\square$

Corollary 27. *The map τ is one-to-one from $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$ onto $\widetilde{\mathcal{K}} \setminus Min(\widetilde{\mathcal{K}})$.*

Proof. It is a direct consequence of the fact that $\tau(\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})) \subset \widetilde{\mathcal{K}} \setminus Min(\widetilde{\mathcal{K}})$. Indeed the restriction of τ on $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$ is surjective onto $\widetilde{\mathcal{K}} \setminus Min(\widetilde{\mathcal{K}})$ since the image of $Max(\widetilde{\mathcal{K}})$ equals $\{\mathbf{0}\} \subset Min(\widetilde{\mathcal{K}})$. $\square$

The topology on $(A \times Q)^\omega$ is as usual induced by the distance d defined by

$$d((x, y), (x', y')) = 2^{-k} \text{ where } k = \inf\{i \mid (x_i, y_i) \neq (x'_i, y'_i)\}.$$

Proposition 28. *The application τ is continuous on $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$.*

Proof. Let $(u, v) \in \widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$ and $\epsilon > 0$. We show that there exists $\eta > 0$ such that if $d((u, v), (x, y)) < \eta$ then $d(\tau(u, v), \tau(x, y)) < \epsilon$. Let j be the smallest index such that $(u, v) \notin \mathbf{max}_j$. If there exists $i > j$ such that (u, v) and (x, y) have the

same prefix of length $i+1$ then $\tau(u,v)$ and $\tau(x,y)$ also have the same prefix of length $i+1$. Clearly, one can take

$$\eta = 2^{-\sup\{1-\log_2 \epsilon, j+1\}}.$$

$\square$

The following example shows that τ is generally not continuous on the points of $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$.

Example 29. Consider the regular language L accepted by the automaton depicted in Figure 3 (where the sink is not represented). For instance, $(u,v) = (d^\omega, (12)^\omega)$
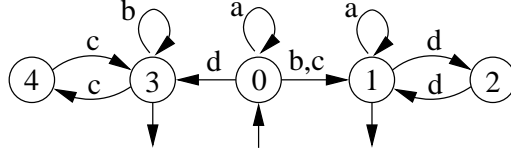

FIGURE 3. The minimal automaton of a language L.

belongs to $Max(\widetilde{\mathcal{K}})$. The points

$$((dd)^n b(a)^\omega, (12)^n 1(0)^\omega) \text{ and } ((dd)^n c(a)^\omega, (12)^n 1(0)^\omega)$$

can be chosen arbitrarily close of (u,v) for n large enough. Whatever is the value of $\tau(u,v)$, the application τ is not continuous at (u,v). Indeed,

$$\tau((dd)^n b(a)^\omega, (12)^n 1(0)^\omega) = ((aa)^n c(a)^\omega, (11)^n 1(0)^\omega)$$

but

$$\tau((dd)^n c(a)^\omega, (12)^n 1(0)^\omega) = ((bb)^n d(a)^\omega, (33)^n 3(0)^\omega).$$

So clearly, if a point (x,y) is close from an element in $Max(\widetilde{\mathcal{K}})$ then its image $\tau(x,y)$ is close from an element in $Min(\widetilde{\mathcal{K}})$ but nothing more can be said.

Remark 30. In the case of positional number systems, such a phenomenon cannot occur: the odometer is always continuous on the points which correspond to $Max(\widetilde{\mathcal{K}})$.

Proposition 31. *The set $\widetilde{\mathcal{K}}$ is a compact subset of $(A \times Q)^\omega$. If the odometer τ is continuous, then the dynamical system $(\widetilde{\mathcal{K}}, \tau)$ is minimal, that is, every non-empty closed subset of $\widetilde{\mathcal{K}}$ invariant under the action of τ is equal to $\widetilde{\mathcal{K}}$.*

Proof. We follow here the proof of [19] adapted to our situation. The compactness of $\widetilde{\mathcal{K}}$ is immediate as a closed subset of $(A \times Q)^\omega$.

We assume that τ is continuous. Let us prove that the closure of the orbit $\{\tau^n(x,y) \mid n \in \mathbb{N}\}$ of any point $(x,y) \in \widetilde{\mathcal{K}}$ is equal to $\widetilde{\mathcal{K}}$.

Let us first observe that the orbit $\{\tau^n(\mathbf{0}) \mid n \in \mathbb{N}\}$ of $\mathbf{0}$ is dense in $\widetilde{\mathcal{K}}$. Indeed, let $(x,y) \in \widetilde{\mathcal{K}}$. Let $(w_n)_{n \in \mathbb{N}}$ be a sequence of words in L such that x is the limit of the sequence $(\widetilde{w_n})_{n \in \mathbb{N}}$. Let l_n denote the n-th word in the ordered language L. The point $\tau^n(\mathbf{0})$ is by definition equal to

$$(\widetilde{l_n}(a_0)^\omega, \widetilde{p(l_n)}(q_0)^\omega),$$

according to notation of Proposition 7. Hence (x,y) is a limit of elements of $\{\tau^n(\mathbf{0}) \mid n \in \mathbb{N}\}$, and $\widetilde{\mathcal{K}}$, which is a closed set, is the closure of $\{\tau^n(\mathbf{0}) \mid n \in \mathbb{N}\}$.

Now if $(x,y) \in Max(\widetilde{\mathcal{K}})$, then $\tau(x,y) = \mathbf{0}$ and $\mathbf{0}$ belongs to the orbit (x,y), which implies the desired result.

Let us suppose that $x \notin Max(\widetilde{\mathcal{K}})$. Let

$$D : \widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}}) \to \mathbb{N} : x \mapsto \sup\{k \mid (x, y) \in \mathbf{max}_k\}.$$

Let us assume that D takes bounded values on the orbit of (x, y). Let C such that $D(\tau^n(x, y)) < C$, for every n. In particular, (x, y) does not have the property $\mathbf{max}_C$. By definition of the odometer, after a suitable number of iterations of τ, say n, then $\tau^n(x, y)$ is easily seen to belong to $\mathbf{max}_C$. Hence D does not take bounded values on the orbit of (x, y), which implies that there exists an increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that $\tau^{n_k}(x, y) \in \mathbf{max}_k$. By compactness of $\widetilde{\mathcal{K}}$, one can extract from $(n_k)_{k \in \mathbb{N}}$ an increasing sequence $(m_k)_{k \in \mathbb{N}}$ such that the sequence $(\tau^{m_k}(x, y))$ converges; its limit belongs to $Max(\widetilde{\mathcal{K}})$, according to Lemma 5. By continuity of τ, $(\tau^{m_k+1}(x, y))$ converges toward $\mathbf{0}$, which implies that the closure of the orbit of (x, y) contains $\mathbf{0}$ and thus equals $\widetilde{\mathcal{K}}$. □

6. Equivalence with positional systems

Let $(U_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of integers such that $U_0 = 1$. Such a sequence is called a *positional number system*. We assume furthermore that the set $L = 0^* \operatorname{rep}_U(\mathbb{N})$ of all the greedy representations of the integers is a regular language over a finite alphabet A_U (from now on $\operatorname{rep}_U(n)$ denotes the U-representation of n computed by the greedy algorithm with the most significant digit on the left). The finiteness of A_U implies that the ratio U_{n+1}/U_n is bounded. In particular, for $L = 0^* \operatorname{rep}_U(\mathbb{N})$ (or equivalently for $\operatorname{rep}_U(\mathbb{N})$) to be regular, it is shown in [36] that the sequence $(U_n)_{n \in \mathbb{N}}$ must satisfy a linear recurrence relation with constant coefficients. In [21], a sufficient condition is given in terms of the polynomials of the recurrence that $(U_n)_{n \in \mathbb{N}}$ satisfies. (The reader can also see the special case treated in [24].) As an example, the set $\operatorname{rep}_U(\mathbb{N})$ is regular whenever the sequence $(U_n)_{n \in \mathbb{N}}$ satisfies a linear recurrence relation whose characteristic polynomial is the minimal polynomial of a Pisot number [4].

In this small section, we study the link between the odometer τ_L built over the language L and the odometer τ_U presented in [19]. Notice that we allow leading zeroes in the greedy representations to obtain a language satisfying hypothesis (2).

Remark 32. Notice that, in this particular setting, as a consequence of the greedy algorithm, if uv belongs to L then v belongs also to L.

Proposition 33. *Let $(U_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of integers such that $U_0 = 1$, τ_U be the odometer associated to this sequence, and let us assume that the language $L = 0^* \operatorname{rep}_U(\mathbb{N})$ associated to the numeration system built upon the sequence $(U_n)_{n \in \mathbb{N}}$ is regular. Let $p_1 : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{L}}$ be the projection mapping (x, y) onto x. Then the following relation holds on $\widetilde{\mathcal{K}}$:*

$$p_1 \circ \tau_L = \tau_U \circ p_1.$$

Proof. Let us first observe that the set on which τ_U is defined and acts, which is the set of right infinite words satisfying the greedy property (1), is exactly $\widetilde{\mathcal{L}}$, following Remark 32.

Let $(x, y) = (x_0 x_1 \ldots, y_0 y_1 \ldots)$ be an element in $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$. Thus there exists i such that $(x, y) \notin \mathbf{max}_i$.

Notice that if u belongs to L_q, since $\mathcal{M}_L$ is accessible, then there exists v such that vu belongs to L. So thanks to Remark 32, u also belongs to L.

Therefore $x_i \cdots x_0$ belongs to both $L_{y_{i+1}}$ and L. Since $x_i \cdots x_0$ does not belong to $Max(L_{y_{i+1}})$, then it does not belong to $Max(L)$ which means that $x_i U_i + \cdots + x_0 U_0$ is strictly less than $U_{i+1} - 1$. We set $x'_i \cdots x'_0 = \operatorname{Succ}_{y_{i+1}}(x_i \cdots x_0)$, so with our

notation

$$\tau_L\left(x_0 \cdots x_i\, x_{i+1} \ldots, y\right) = \left(x_0' \cdots x_i'\, x_{i+1} \ldots, y'\right)$$

for some $y' \in Q^\omega$. We have to show that the successor of the word $x_i \cdots x_0$ in the genealogically ordered language L is $x_i' \cdots x_0'$ which means therefore that

$$x_i U_i + \cdots + x_0 U_0 + 1 = x_i' U_i + \cdots + x_0' U_0$$

and thus $\tau_U(x_0 \cdots x_i\, x_{i+1} \ldots) = (x_0' \cdots x_i'\, x_{i+1} \ldots)$. To the contrary, assume that there exists $z_i \cdots z_0 \in L$ such that $x_i \cdots x_0 < z_i \cdots z_0 < x_i' \cdots x_0'$. Let v be such that $\delta(q_0, v) = y_{i+1}$. The words $v x_i \cdots x_0$ and $v x_i' \cdots x_0'$ are accepted from q_0 and satisfy therefore the greedy condition (1). Since $z_i \cdots z_0 < x_i' \cdots x_0'$, $v z_i \cdots z_0$ satisfies the greedy condition and so it belongs to L. Since $\mathcal{M}_L$ is deterministic, $z_i \cdots z_0$ is also accepted from y_{i+1}. Therefore $x_i' \cdots x_0' \neq \mathrm{Succ}_{y_{i+1}}(x_i \cdots x_0)$ which is a contradiction.

Consequently if p_1 is the projection mapping (x, y) onto x then we have shown that on $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$, the following holds

$$(5) \qquad\qquad\qquad\qquad p_1 \circ \tau_L = \tau_U \circ p_1.$$

Observe that here, $\mathbf{0}$ is $(0^\omega, q_0^\omega)$ because ε is the representation of 0 and belongs to L. If (x, y) belongs to $Max(\widetilde{\mathcal{K}})$ then $\tau_L(x, y) = \mathbf{0}$ and it is clear that $\sum_{\ell=0}^{i} x_\ell U_\ell = U_{i+1} - 1$ for an infinite number of indices i. Therefore from [19], $\tau_U(x) = 0^\omega$ and the relation (5) holds on the whole set $\widetilde{\mathcal{K}}$. $\qquad\qquad\square$

Remark 34. A characterization of the continuity of the odometer for positional number system is given in [16], in terms of the right subsequentiality of the successor function on 0^*L. We will see in Proposition 45 that we can have τ_L continuous whereas τ_U is not continuous.

7. SUBSTITUTION NUMERATION SYSTEMS

7.1. **Definition.** Let $\Sigma = \{a_1, \cdots, a_d\}$ be an alphabet (here, Σ does not need to be totally ordered). Let $\sigma : \Sigma \to \Sigma^+$ be a *substitution*, i.e., a morphism of the free monoid Σ^* such that $\sigma(a_1) \in a_1 \Sigma^+$. To this substitution, we associate a deterministic automaton $\mathcal{M}_\sigma = (Q, a_1, A, \delta, F)$ in the classical way. The set of states is $Q = \Sigma \cup \{s\}$ where a sink state $s \notin Q$ is possibly added to Q in order to make $\mathcal{M}_\sigma$ complete when σ is not a uniform substitution (a substitution is said to be *uniform* if the images of all the letters have the same length). The alphabet of the automaton is

$$A = \{0, \ldots, \sup_{a \in \Sigma} |\sigma(a)| - 1\}.$$

There is an edge of label $i \in A$ between two states a and b, that is, $\delta(a, i) = b$ if and only if the $(i+1)$-th letter in $\sigma(a)$ is b. The initial state is a_1 and all the states are final, i.e., $F = \Sigma$.

In the literature [8, 9, 10, 11, 28, 30] the notion of *prefix automaton* (respectively *prefix-suffix automaton*) can also be found. In this latter case, the label i between a and b is replaced by the prefix of length i of $\sigma(a)$ (if $i = 0$ then the prefix is ε) (respectively, the prefix of length i of $\sigma(a)$ and the suffix of length $|\sigma(a)| - i - 1$ of $\sigma(a)$). It is well-known (see for instance [10, 11]) that each integer $n \geq 1$ has a unique decomposition of the form

$$(6) \qquad\qquad\qquad\qquad n = \sum_{i=1}^{\ell} |\sigma^{i-1}(m_i)|$$

where $m_\ell \cdots m_1$ is the label of a path read in the prefix automaton from the initial state a_1 with $m_\ell \neq \varepsilon$.

Let us recall that d denotes the cardinal of the alphabet Σ. The *incidence matrix* of the substitution σ is defined as the $d \times d$ matrix whose entry of index (a, b) counts the number of occurrences of the letter a in $\sigma(b)$. The incidence matrix of σ coincides with the transpose of the adjacency matrix of the automaton $\mathcal{M}_\sigma$.

Since the alphabet $A = \{0, 1, \ldots, \sup_{a \in \Sigma} |\sigma(a)| - 1\}$ is totally ordered by the usual ordering on $\mathbb{N}$, we can order the words of the language $L \subset A^*$ accepted by $\mathcal{M}_\sigma$ using the genealogical ordering. This leads to an abstract numeration system $S = (L, A, <)$ built upon L.

7.2. Equivalence between substitution and abstract numeration systems.
In this section, we give a new interpretation of the numeration systems built upon a substitution according to (6).

Let $\mathcal{M}'_\sigma$ be the automaton built upon $\mathcal{M}_\sigma$ but having an extra state a_0 which is the initial state of this new automaton. For $k = 2, \ldots, |\sigma(a_1)|$, we add an edge labeled by $k - 1$ from a_0 to the k-th letter of $\sigma(a_1)$. Observe that if L is the language accepted by $\mathcal{M}_\sigma$ then $L \setminus 0A^*$ is the language accepted by $\mathcal{M}'_\sigma$. This kind of construction is also classical and was for instance used in [33]. We denote from now on by L' the language accepted by $\mathcal{M}'_\sigma$.

Example 35. Consider the substitution on $\Sigma = \{a_1, a_2\}$ defined by $\sigma(a_1) = a_1 a_2 a_1$ and $\sigma(a_2) = a_1$. We have the following automata $\mathcal{M}_\sigma$ and $\mathcal{M}'_\sigma$ sketched in Figure 4. Here $A = \{0, 1, 2\}$ and the sink has not been represented.
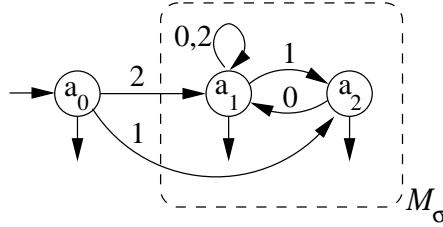

FIGURE 4. The automata $\mathcal{M}_\sigma$ and $\mathcal{M}'_\sigma$.

Naturally, we can also order the words of the language $L' \subset A^*$ accepted by $\mathcal{M}'_\sigma$ using the genealogical ordering. This leads to an abstract numeration system $S' = (L', A, <)$ built upon L'. The representation of the integer n is defined as the $(n + 1)$-th word w of L' and we write $\mathrm{val}_{S'}(w) = n$ (let us recall that the first word of L' is the empty word).

The following proposition allows us to make the connection with the substitutive numeration system as expressed in (6).

Proposition 36. *The $(n+1)$th word $w_1 \cdots w_\ell$ of the genealogically ordered language L' generates the prefix $u_0 \cdots u_{n-1}$ of length n of $\sigma^\omega(a_1)$ as follows: $u_0 \cdots u_{n-1}$ is equal to the concatenation of $\sigma^{\ell-i}[\delta(a_1, w_1 \cdots w_{i-1}0) \cdots \delta(a_1, w_1 \cdots w_{i-1}(w_i - 1))]$ in decreasing order of indices $1 \le i \le \ell$, where $w_1 \cdots w_{i-1}$ is understood as ε if $i = 1$, as well as $\delta(a_1, w_1 \cdots w_i 0) \cdots \delta(a_1, w_1 \cdots w_i(w_i - 1))$ if $w_i = 0$. In other words, $u_0 \cdots u_{n-1}$ is equal to*

$$\sigma^{\ell-1}[\delta(a_1, 0) \cdots \delta(a_1, (w_1-1))] \cdots \sigma^0[\delta(a_1, w_1 \cdots w_{\ell-1}0) \cdots \delta(a_1, w_1 \cdots w_{l-1}(w_\ell-1))]$$

and

$$n = \sum_{i=0}^{\ell-1} \left| \sigma^i[\delta(a_1, w_1 \cdots w_{i-1}0) \cdots \delta(a_1, w_1 \cdots w_{i-1}(w_i - 1))] \right|.$$

Proof. The proof is based on the fact that the prefix of $\sigma(q)$ of length $t \leq |\sigma(q)|$ read from the state $q \in \Sigma$ is equal to $\delta(q, 0) \cdots \delta(q, t-1)$.

Let us recall that for a state q, $\mathbf{u}_q(n)$ denotes the cardinal of the set of the words of length n accepted from q. If $w = w_1 \cdots w_\ell \in L'$ (this means in particular that $w_1 > 0$), then with respect to the automaton $\mathcal{M}'_\sigma$ the following formula holds (see [22, 23])

$$
\begin{aligned}
\text{(7)} \quad \text{val}_{S'}(w) = &\sum_{i=0}^{\ell-1} \mathbf{u}_{a_0}(i) + \sum_{b < w_1} \mathbf{u}_{\delta(a_0, b)}(\ell - 1) \\
&+ \sum_{b < w_2} \mathbf{u}_{\delta(a_0, w_1 b)}(\ell - 2) + \cdots + \sum_{b < w_\ell} \mathbf{u}_{\delta(a_0, w_1 \cdots w_{\ell-1} b)}(0).
\end{aligned}
$$

The interested reader can find a combinatorial interpretation of this formula in [20]. We have two immediate observations

$$
\sum_{i=0}^{\ell-1} \mathbf{u}_{a_0}(i) = \mathbf{u}_{a_1}(\ell - 1) \quad \text{and} \quad \forall q \in \Sigma, \forall n \in \mathbb{N}, \mathbf{u}_q(n) = |\sigma^n(q)|.
$$

We are now able to prove the equivalence of the two formulas (6) and (7). First notice that $\delta(a_0, 0)$ is the sink s of $\mathcal{M}'_\sigma$. Therefore $\mathbf{u}_{\delta(a_0, 0)}(n) = 0$ for all n. If $b \neq 0$ then $\delta(a_0, b) = \delta(a_1, b)$. The first two terms in (7) can be written as

$$
|\sigma^{\ell-1}(a_1)| + \sum_{0 < b < w_1} \left| \sigma^{\ell-1}[\delta(a_1, b)] \right| = \left| \sigma^{\ell-1}[\delta(a_1, 0) \cdots \delta(a_1, w_1 - 1)] \right|.
$$

Notice that for the latter equality, we have used the fact that $\delta(a_1, 0) = a_1$ and that σ is a morphism. Consequently, (7) can be written as

$$
\text{val}_{S'}(w) = \sum_{i=1}^{\ell} \left| \sigma^{\ell-i}[\delta(a_1, w_1 \cdots w_{i-1} 0) \cdots \delta(a_1, w_1 \cdots w_{i-1}(w_i - 1))] \right|.
$$

This gives another interpretation of (6).

$\square$

7.3. First properties of the odometer.

Since the infinite language L accepted by $\mathcal{M}_\sigma$ satisfies property (2), then one can consider the odometer $\widetilde{\mathcal{K}}$ built upon $(L, A, <)$. Let us observe that L' does not satisfy (2), but that that $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}'$ do coincide. The set $\widetilde{\mathcal{K}}$ is a subshift of finite type of $(A \times Q)^\omega$ since every state (except the sink) in $\mathcal{M}_\sigma$ is a final state. Let us observe

Proposition 37. *The set $\widetilde{\mathcal{K}}$ corresponds to the set of infinite paths $(x, y) \in (A \times \Sigma)^\omega$ in the automaton $\widetilde{\mathcal{M}}_\sigma$, i.e., $\forall i \geq 0$, y_{i+1} is the $(x_i + 1)$-th letter of $\sigma(y_i)$. Furthermore,*

$$
Max(\widetilde{\mathcal{K}}) = \{(x, y) \in \widetilde{\mathcal{K}} | \ \forall i \geq 0, \ x_i = |\sigma(y_i)| - 1\},
$$
$$
Min(\widetilde{\mathcal{K}}) = \{(x, y) \in \widetilde{\mathcal{K}} | \ \forall i \geq 0, \ x_i = 0\}.
$$

Proof. There is an edge in the automaton $\widetilde{\mathcal{M}}_\sigma$ of label i between two states u and v if and only if the $(i+1)$-th letter in $\sigma(v)$ is u; furthermore, all the states are initial and final. Hence $\widetilde{\mathcal{L}}$ is equal to the set of labels of infinite paths in $\widetilde{\mathcal{M}}_\sigma$, which implies the desired description of $\widetilde{\mathcal{K}}$. The characterization of $Max(\widetilde{\mathcal{K}})$ and $Min(\widetilde{\mathcal{K}})$ is immediate.

$\square$

Let us recall that the odometer τ is one-to-one from $\widetilde{\mathcal{K}} \setminus Max(\widetilde{\mathcal{K}})$ onto $\widetilde{\mathcal{K}} \setminus Min(\widetilde{\mathcal{K}})$ following Corollary 27.

Remark 38. The automaton $\mathcal{M}_\sigma$ associated to a substitution σ is not necessarily minimal. Indeed, if you consider the Thue-Morse substitution defined by $\sigma(a_1) = a_1 a_2$ and $\sigma(a_2) = a_2 a_1$ then it is easy to see that $\mathcal{M}_\sigma$ accept $\{0,1\}^*$ and is not minimal. To obtain unambiguous constructions, we have always considered the minimal automaton of a language but clearly, we could define a set $\widetilde{\mathcal{K}}$ and an odometer τ depending on the choice of a finite deterministic automaton which is not necessarily minimal. We just need a loop in the initial state a_1 labeled by the smallest letter 0 of the alphabet (it is always the case for the automaton associated to a substitution σ which satisfies $\sigma(a_1) \in a_1 A^+$).

We can say nothing in general concerning the continuity of τ as illustrated in Proposition 45.

7.4. **The Pisot case.** Nevertheless, there are some cases for which the automaton $\mathcal{M}_\sigma$ can be proved to be minimal. A substitution is said of *Pisot type* if the eigenvalues of its incidence matrix satisfy the following: there exists a dominant eigenvalue α such that for every other eigenvalue λ, one gets $0 < |\lambda| < 1 < |\alpha|$. A substitution of Pisot type is primitive and the characteristic polynomial χ_σ of its incidence matrix is irreducible over $\mathbb{Q}$ [9]. Let us recall that χ_σ is also the minimal polynomial of the adjacency matrix of $\mathcal{M}_\sigma$.

Proposition 39. *Let σ be a Pisot substitution. The automaton $\mathcal{M}_\sigma$ is minimal.*

Proof. Let σ be a substitution of Pisot type. The automaton $\mathcal{M}_\sigma$ is accessible since σ is primitive, that is, all its states can be reached from its initial state a_1. Hence the minimal polynomial χ_σ of its adjacency matrix is dividable by the minimal polynomial of the minimal automaton recognizing the language M_σ. Since χ_σ is irreducible, this implies that both polynomials do coincide, and thus that $\mathcal{M}_\sigma$, which is deterministic, is the minimal automaton recognizing the language M_σ. $\square$

In the particular case of a Pisot substitution, we are now able to give a dynamical interpretation of $(\widetilde{\mathcal{K}}, \tau)$. Let S denote the shift map on $\Sigma^{\mathbb{Z}}$: $S((w_i)_{i\in\mathbb{Z}}) = (w_{i+1})_{i\in\mathbb{Z}}$. A word $u \in \Sigma^{\mathbb{Z}}$ such that there exists a positive integer k with $S^k(u) = u$ is called a *periodic point* under the action of σ. Let us recall that the (two-sided) *symbolic dynamical system* generated by a primitive substitution σ is the pair (X_σ, S), where X_σ is the set of two-sided sequences in $\Sigma^{\mathbb{Z}}$ with the same set of factors of any periodic point u of σ; this definition does not depend on the choice of u by primitivity of σ.

We use here the notation and results of [8, 9] adapted to our framework. Following [26], every two-sided sequence v in X_σ has a unique decomposition

$$v = S^k(\sigma(w)), \text{ with } w \in X_\sigma \text{ and } 0 \le k < |\sigma(w_0)|$$

(w_0 denotes here the 0-th coordinate of w).

Let

$$\begin{cases} \theta: \ X_\sigma \to X_\sigma: \ v \mapsto w, \\ \text{where } v = S^k(\sigma(w)), \text{ with } 0 \le k < |\sigma(w_0)|. \end{cases}$$

The map θ is called the *desubstitution map*.

Let

$$\begin{cases} \gamma: \ X_\sigma \to (A \times \Sigma): \ v \mapsto (k+1, w_0), \\ \text{where } v = S^k(\sigma(w)), \text{ with } w \in X_\sigma \text{ and } 0 \le k < |\sigma(w_0)|. \end{cases}$$

In other words, if $\gamma(v) = (k, q)$, then v_0 is the $(k+1)$-th letter of $\sigma(q)$. Hence, for every $v \in X_\sigma$, the sequence $(\gamma \circ \theta^i(v))_{i\in\mathbb{N}}$ is easily seen to belong to $\widetilde{\mathcal{K}}$. Let us now define

$$\Gamma: X_\sigma \to \widetilde{\mathcal{K}}: \ v \mapsto (\gamma \circ \theta^i(v))_{i\in\mathbb{N}}.$$

The following theorem is a direct consequence of [8, 9].

Theorem 40. [8, 9] *Let σ be a Pisot substitution. The map Γ is continuous and onto $\widetilde{\mathcal{K}}$; it is one-to-one except on the orbit of periodic points of σ. Furthermore,*

$$\Gamma \circ S = \tau \circ \Gamma \ and \ \Gamma \circ \theta = S_{\widetilde{\mathcal{K}}} \circ \Gamma,$$

where $S_{\widetilde{\mathcal{K}}}$ denotes the shift map acting on elements of $(A \times \Sigma)^{\mathbb{Z}}$.

Proof. We know from [8] that (X_σ, S) is measure-theoretically isomorphic with the subshift of finite type $\mathcal{D}$ defined as the set of labels of infinite paths $\mathcal{D}$ in the prefix-suffix automaton. Let us recall that there is an edge from a to b of label (p, a, s) in the prefix-suffix automaton if $\sigma(b) = a$, and all its states (which are the letters of Σ) are both initial and final. Let us prove that $\widetilde{\mathcal{K}}$ and $\mathcal{D}$ are in one-to-one correspondence. This comes from the fact that the following map is one-to-one:

$$\widetilde{\mathcal{K}} \to \mathcal{D}, \ (x, y) \mapsto (\delta(y_{i+1}, 0) \cdots \delta(y_{i+1}, x_i - 1), y_i, s_i)_{i \in \mathbb{N}},$$

where s_i is the suffix of size $|\sigma(y_{i+1})| - x_i - 1$ of $\sigma(y_{i+1})$. Now from Remark 23, the map τ coincides with the adic transformation acting on $\mathcal{D}$. It just remains to apply the results of [8, 9]. □

Remark 41. Two dynamical systems can be built over $\widetilde{\mathcal{K}}$, i.e., $(\widetilde{\mathcal{K}}, \tau)$ and $(\widetilde{\mathcal{K}}, S_{\widetilde{\mathcal{K}}})$. Theorem 40 gives us two combinatorial interpretations for these systems: the action of the desubstitution map θ (the "inverse" of σ) on X_σ corresponds to the action of the shift $S_{\widetilde{\mathcal{K}}}$ on $\widetilde{\mathcal{K}}$, whereas the action of the shift S on X_σ corresponds to action of the odometer τ on $\widetilde{\mathcal{K}}$.

Remark 42. Furthermore, there exists a unique shift invariant measure on the dynamical system (X_σ, S) since σ is primitive ((X_σ, S) is said uniquely ergodic); for more details see for instance [29]. This measure can be naturally carried on $(\widetilde{\mathcal{K}}, \tau)$ via the map Γ (which is one-to-one except on a countable number of points). Theorem 40 means that $(\widetilde{\mathcal{K}}, \tau)$ endowed with this measure is measure-theoretically isomorphic with (X_σ, S). One interest of this approach is that it provides us some insight on a metrical study of $(\widetilde{\mathcal{K}}, \tau)$, following [6].

Remark 43. It is possible to give a combinatorial interpretation of $Min(\widetilde{\mathcal{K}})$ and $Max(\widetilde{\mathcal{K}})$ in this framework. Following [8], $Min(\widetilde{\mathcal{K}})$ and $Max(\widetilde{\mathcal{K}})$ correspond respectively to the periodic points (under the action of σ) of X_σ (we denote this set $\mathrm{Per}(X_\sigma)$) and to the preimages $S^{-1}(\mathrm{Per}(X_\sigma))$ under the shift S of those periodic points. Both sets do not have necessarily the same cardinal as illustrated for instance in Section 8, Proposition 45.

8. The case of sofic beta-numerations

This section gathers results of Section 6 and 7 within the framework of β-numeration. Let $U = (U_n)_{n \in \mathbb{N}}$ be a positional numeration system such that the ratio U_{n+1}/U_n is bounded, as defined in Section 6. Assume now that $\mathrm{rep}_U(\mathbb{N})0^*$ is included in $\mathrm{rep}_U(\mathbb{N})$; the positional number system U is said to be a *Bertrand numeration system*. Bertrand numeration systems are closely related to β-expansions as recalled below.

Let $\beta > 1$ be a positive real number. The Rényi β-expansion of a real number $x \in [0, 1]$ is defined as the sequence $(x_i)_{i \geq 1}$ with values in $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$ produced by the β-transformation $T_\beta : [0, 1] \to [0, 1] : x \mapsto \beta x \pmod 1$ as follows

$$\forall i \geq 1, \ x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \ \text{and thus} \ x = \sum_{i \geq 1} x_i \beta^{-i}.$$

Let $d_\beta(1) = (t_i)_{i\geq 1}$ denote the β-expansion of 1. Let $d_\beta^*(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and $d_\beta^*(1) = (t_1 \ldots t_{m-1}(t_m - 1))^\omega$, if $d_\beta(1) = (t_1 \ldots t_{m-1}t_m)$ is finite (with $t_m \neq 0$). The set D_β of β-expansions of numbers in $[0,1)$ is exactly the set of sequences $(c_i)_{i\geq 1}$ that satisfy:

$$\forall k \in \mathbb{Z}, \ (c_i)_{i\geq k} <_{\text{lex}} d_\beta^*(1).$$

For more details, see for instance [25]. We denote by $F(D_\beta)$ the set of finite factors of the sequences in D_β.

Numbers β such that $d_\beta(1)$ is ultimately periodic are called β-*numbers* and those such that $d_\beta(1)$ is finite are called *simple* β-*numbers*. If β is a β-number (simple or not), the minimal automaton $\mathcal{M}_\beta$ recognizing the set of factors of $F(D_\beta)$ can easily be constructed (representations of this classical automaton $\mathcal{M}_\beta$ can be found in [18] or [23]). Furthermore, let us recall that when β is assumed to be Pisot, then β is either a β-number or a simple β-number, and (X_β, S) is sofic.

Bertrand numeration systems are characterized by the following theorem:

Theorem 44. [3] *Let U be a positional number system over a finite alphabet. Then U is a Bertrand numeration system if and only if there exists a real number $\beta > 1$ such that $L = 0^* \text{rep}_U(\mathbb{N}) = F(D_\beta)$. Furthermore, L is regular if and only if β is a β-number.*

There is a natural way to associate a substitution σ_β with the β-numeration when β is a β-number (simple or not). These substitutions will be called in all what follows β-*substitutions*. The automaton $\mathcal{M}_{\sigma_\beta}$ associated with σ_β, as defined in Section 7, coincides with the minimal automaton $\mathcal{M}_\beta$ which recognizes $F(D_\beta)$. For more details, see [13, 39]. Let us note that $d_\beta(1)$ cannot be purely periodic, hence one has either $d_\beta^*(1) = (t_1 \cdots t_{n-1}(t_n - 1))^\omega$ with $t_n \neq 0$ or $d_\beta^*(1) = t_1 \cdots t_n(t_{n+1} \cdots t_{n+p})^\omega$, with $t_n \neq t_{n+p}$ and $n \geq 1$.

- Assume $d_\beta(1) = (t_1 \cdots t_{n-1}t_n)$ with $t_n \neq 0$ and thus $d_\beta^*(1) = (t_1 \cdots t_{n-1}(t_n - 1))^\omega$. Consider the substitution σ_β defined over the alphabet $\{1, 2, \ldots, n\}$ by:

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \quad \vdots \\ n-1 & \mapsto 1^{t_n - 1}n \\ n & \mapsto 1^{t_n}. \end{cases}$$

- Assume $d_\beta(1) = d_\beta^*(1) = t_1 \cdots t_n(t_{n+1} \cdots t_{n+p})^\omega$, with $t_{n+1} \cdots t_{n+p} \neq 0^p$ and $t_n \neq t_{n+p}$. Furthermore $n \geq 1$. Consider the substitution σ_β defined over the alphabet $\{1, 2, \ldots, n+p\}$ by:

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \quad \vdots \\ n+p-1 & \mapsto 1^{t_{n+p-1}}(n+p) \\ n+p & \mapsto 1^{t_{n+p}}(n+1). \end{cases}$$

¿From now on, we assume that the positional number system U is a Bertrand numeration associated with β Pisot number; thus $L = 0^* \text{rep}_U(\mathbb{N}) = F(D_\beta)$ is an infinite regular language which satisfies (2). We still denote τ_L the odometer acting on $\widetilde{\mathcal{K}}$. When β is a simple β-number, then the substitutions σ_β are of Pisot type since the characteristic polynomial of their incidence matrices coincides with the minimal polynomial of β. Hence, the results of Section 7.4 do apply.

We end now this section by proving that the odometer τ_L is continuous when β is a Pisot number, contrarily to the positional number systems case where continuity holds if and only if β is a simple β-number (see [19] and [16]).

Proposition 45. *Let β be a Pisot number and let $L = F(D_\beta)$. Then the odometer τ_L is continuous on $\widetilde{\mathcal{K}}$.*

Proof. Let $(x, y) \in Max(\widetilde{\mathcal{K}})$. Let us prove that for any sequence $(x^{(n)}, y^{(n)})_{n \in \mathbb{N}}$ with values in $\widetilde{\mathcal{K}}$ which converges toward (x, y), then $\tau_L(x^{(n)}, y^{(n)})_{n \in \mathbb{N}}$ converges toward $\mathbf{0} = (0^\omega, q_0^\omega)$. Let $(x^{(n)}, y^{(n)})_{n \in \mathbb{N}}$ be such a sequence. We assume furthermore that for n large enough, then $(x^{(n)}, y^{(n)}) \notin Max(\widetilde{\mathcal{K}})$. There exists a state q for which there exist infinitely many integers k such that $x_k x_{k-1} \cdots x_0 \in Max(L_q)$. Let N be fixed. Let $k \geq N$ such that $x_k x_{k-1} \cdots x_0 \in Max(L_q)$ with $q = y_{k+1}$. For n large enough, $(x^{(n)}, y^{(n)})$ coincides on its first N values with (x, y) and $(x^{(n)}, y^{(n)}) \notin Max(\widetilde{\mathcal{K}})$. In particular, $x_k^{(n)} x_{k-1}^{(n)} \cdots x_0^{(n)} \in Max(L_q)$, with $q = y_{k+1}^{(n)}$. Since $(x^{(n)}, y^{(n)}) \notin Max(\widetilde{\mathcal{K}})$, there exists a non-negative integer $l > k$ such that $x_l^{(n)} x_{l-1}^{(n)} \cdots x_0^{(n)} \notin Max(L_{y_{l+1}^{(n)}})$. Let l_0 denote the smallest of these integers. The successor in $L_{y_{l_0+1}^{(n)}}$ of $x_{l_0}^{(n)} x_{l_0-1}^{(n)} \cdots x_0^{(n)}$ is $(x_{l_0}^{(n)} + 1)0^{l_0}$. Furthermore, any edge labeled by 0 in $\mathcal{M}_\beta$ leads to the initial state q_0 (recall that the interested reader can find a representation of $\mathcal{M}_\beta$ in [18, 23]). Hence $\tau_L(x^{(n)}, y^{(n)})$ admits as a prefix $(0^N, y_0^N)$ for n large enough, which ends the proof.   □

Remark 46. The sets $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{L}}$ are not in one-to-one correspondence. Indeed the word 0^ω admits several representations in $\widetilde{\mathcal{K}}$, hence we cannot deduce directly continuity results from Proposition 33. Let us observe nevertheless that there is at most a countable number of antecedents to elements of $\widetilde{\mathcal{L}}$ according to the projection $p_1 : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{L}}$, $(x, y) \mapsto x$, in the particular situation described in this section.

9. REAL REPRESENTATION OF THE ODOMETER

The aim of this section is to outline the first steps of a study of a geometric representation of the dynamical system $(\widetilde{\mathcal{K}}, \tau)$. A geometric representation of the dynamical system $(\widetilde{\mathcal{K}}, \tau)$ is a continuous map φ from $\widetilde{\mathcal{K}}$ onto a geometric dynamical system (Y, T) such that $\varphi \circ \tau = T \circ \varphi$, and on which there exists a partition indexed by the alphabet $A \times Q$ such that every word (x, y) is the itinerary of a point of (Y, T) with respect to the partition.

Let L be an arbitrary regular language satisfying (2). A first representation which might be possible consists in extending the work of [22, 23, 34], where a real value is attributed to limits of finite words for abstract numeration systems built on an exponential regular language satisfying the following conditions: there exist $\beta > 1$ and $P \in \mathbb{R}[X]$ such that for all states $q \in Q$, there exists some non-negative real numbers a_q such that $\lim_{n \to \infty} \frac{\mathbf{u}_q(n)}{P(n)\beta^n} = a_q$. (We recall that $\mathbf{u}_q(n)$ represents the number of words of length n in L_q.) We assume now that $\widetilde{L}$ also satisfies the previous conditions of [22, 23]. Clearly, if L is exponential then $\widetilde{L}$ is also exponential because $\#(L \cap \Sigma^n) = \#(\widetilde{L} \cap \Sigma^n)$. The main assumptions for building a representation map rely therefore on the asymptotic behavior of the sequences $\frac{\mathbf{u}_{q'}(n)}{P(n)\beta^n}$ for all the states q' of the minimal automaton of $\widetilde{L}$. Let $\mathbf{v}(n)$ denotes the number of words of length at most n in L (or in $\widetilde{L}$), and $\mathrm{val}_{\widetilde{L}}(w)$ the numerical value of $w \in \widetilde{L}$, i.e., if $\mathrm{val}_{\widetilde{L}}(w) = n$, then w is the $(n+1)$-th word of $\widetilde{L}$. Let $(x, y) \in \widetilde{\mathcal{K}}$. Since $x \in \widetilde{\mathcal{L}}$, there exists a sequence $(w_n)_{n \in \mathbb{N}}$ of words in $\widetilde{L}$ which converges to x.

The limit

$$\lim_{n \to \infty} \frac{\mathrm{val}_{\widetilde{L}}(w_n)}{\mathbf{v}(|w_n|)}$$

does not depend on the choice of the sequence $(w_n)_{n \in \mathbb{N}}$ (see [22, Corollary 8]) and is denoted by $\mathrm{val}_\infty(x)$. We thus can define a map $r : \widetilde{\mathcal{K}} \to \mathbb{R} : (x, y) \mapsto \mathrm{val}_\infty(x)$. It remains to explore the properties of such a representation map r.

We propose now a second possible geometric representation in the framework of languages associated with substitutions. We continue here with the notation of Section 7. A substitution is said *unimodular* if the determinant of its incidence matrix equals ± 1. It is conjectured that for a Pisot unimodular substitution, the dynamical system (X_σ, S), and hence $(\widetilde{\mathcal{K}}, \tau)$ according to Remark 42, is measure-theoretically isomorphic to a rotation on the torus $\mathbb{T}^{d-1}$, where d denotes the cardinal of the alphabet Σ. For more details, see for instance Chap. 7 of [28]. There are however families of substitutions for which this result is known to hold true. One simple way to exhibit this rotation is to give a geometrical representation of $(\widetilde{\mathcal{K}}, \tau)$ as explained in the next paragraph. We follow the formalism of [8, 9].

Let L be a language associated with the substitution σ of Pisot type over the alphabet Σ of cardinal d, as explained in Section 7. Let us recall that the characteristic polynomial χ_σ of the incidence matrix of σ is also the minimal polynomial of the adjacency matrix of the minimal automaton $\mathcal{M}_L$ recognizing L. Let $\alpha_1, \ldots, \alpha_r$ denote the r real eigenvalues of χ_σ, and $\alpha_{r+1}, \ldots, \alpha_{r+s}, \overline{\alpha_{r+1}}, \ldots, \overline{\alpha_{r+s}}$, denote its $2s$ complex roots ($r + 2s = d$). Let us assume $\alpha_1 > 1$, hence, $|\alpha_i| < 1$, for $i \geq 2$.

Let $\vec{v}^{(1)}$ be a left eigenvector associated with the eigenvalue α_1 with coefficients in the field $\mathbb{Q}(\alpha_1)$. Let α_k be an eigenvalue and let ρ_k be the canonical morphism from $\mathbb{Q}(\alpha_1)$ onto $\mathbb{Q}(\alpha_k)$, extended to $\mathbb{Q}(\alpha_1)^d$. Let $\vec{v}^{(k)} = \rho_k(\vec{v}^{(1)})$. We propose as a geometric representation of the set $\widetilde{\mathcal{K}}$ in this framework the following map $\varphi : \widetilde{\mathcal{K}} \to \mathbb{R}^{r-1} \times \mathbb{C}^s$:

$$(x, y) \mapsto \left(\sum_{i \geq 0} (\vec{v}^{(2)}_{\delta(y_{i+1}, 0)} + \cdots + \vec{v}^{(2)}_{\delta(y_{i+1}, x_i - 1)}) \alpha_2^i, \cdots, (\vec{v}^{(r+s)}_{\delta(y_{i+1}, 0)} + \cdots + \vec{v}^{(r+s)}_{\delta(y_{i+1}, x_i - 1)}) \alpha_{r+s}^i \right).$$

This series is easily seen to converge. This map can be factorized as a map on the torus. Indeed, let $\mathbb{L}$ denote the lattice

$$\left\{ \sum_{k=1}^{d} n_k \vec{v}^{(k)} \mid n_k \in \mathbb{Z}, \sum_{k=1}^{d} n_k = 0 \right\}.$$

Following [8, 9], the map

$$\varphi_{\mathbb{L}} : \widetilde{\mathcal{K}} \to (\mathbb{R}^{r-1} \times \mathbb{C}^s)/\mathbb{L} \equiv \mathbb{T}^{d-1}, \ (x, y) \mapsto \varphi(x, y) \bmod \mathbb{L}$$

is well defined and continuous. Consider the toral translation

$$T : \ (\mathbb{R}^{r-1} \times \mathbb{C}^s)/\mathbb{L} \to (\mathbb{R}^{r-1} \times \mathbb{C}^s)/\mathbb{L} : z \mapsto z + \vec{v}^{(1)} \bmod \mathbb{L}.$$

One checks that $\varphi_{\mathbb{L}} \circ \tau = T \circ \varphi_{\mathbb{L}}$, and that φ is a geometrical representation of $(\widetilde{\mathcal{K}}, \tau)$, the partition being given by the sets $\varphi\{(x, y) \mid (x, y) \in \widetilde{\mathcal{K}}, y_0 = q\}$, $q \in Q$.

In particular, for some families of β-substitutions, this map is known to provide a measure-theoretical isomorphism (this is the case in particular for numbers β having the finiteness property (F) introduced in [17], which states that the set of non-negative real numbers with finite β-expansion coincides with the set of non-negative elements of $\mathbb{Z}[1/\beta]$). We deduce the following proposition from the results of [17, 1], stated in the framework of β-substitutions.

Proposition 47. *Assume that $L = F(D_\beta)$, where $\beta > 1$ is either*

- *the positive root of the polynomial $X^m - t_1 x^{m-1} - \cdots - t_m$, where $t_i \in \mathbb{Z}$, and $t_1 \geq t_2 \geq \cdots \geq t_m > 0$,*
- *the dominant root of the polynomial $X^m - t_1 x^{m-1} - \cdots - t_m$, where $t_i \in \mathbb{N}$, and $t_1 > \sum_{i=2}^{d} |t_i| > 0$, and $(t_1, t_2) \neq (2, -1)$,*
- *a cubic Pisot unit.*

Then the map $\varphi_{\mathbb{L}}$ is continuous and onto $\mathbb{T}^{m-1}$ ($m = 3$ in the third case); it is one-to-one except on a at most countable number of points; furthermore $\varphi_{\mathbb{L}}$ is a geometrical representation of $(\widetilde{\mathcal{K}}, \tau)$, the partition being given by the sets $\varphi_{\mathbb{L}}\{(x,y)|\ (x,y) \in \widetilde{\mathcal{K}},\ y_0 = q\}$, $q \in Q$.

Proof. The fact that $\varphi_{\mathbb{L}}$ is one-to-one except on a at most countable number of points comes from [17] for the first case, and from [1] for the last two points. It remains to prove that the sets $\varphi_{\mathbb{L}}\{(x,y)|\ (x,y) \in \widetilde{\mathcal{K}},\ y_0 = q\}$, $q \in Q$ are disjoint up to sets of zero Lebesgue measure. This is a direct consequence of the fact that β-substitutions satisfy the so-called strong coincidence condition, according to [2]. $\qquad\qquad\square$

10. Some special cases

In [19], the odometer is defined on a set $\mathcal{R}$ of sequences of digits. Here, we have introduced an odometer on a set $\widetilde{\mathcal{K}}$ of pairs of infinite words. In this section, we show that in some particular situations, we can restrict ourselves to unidimensional sequences. So we exhibit hypothesis where the extra information given by the sequence of states is useless, or more precisely, the projection map $p_1 : \widetilde{\mathcal{K}} \to \widetilde{\mathcal{L}}$ $(x,y) \mapsto x$ is injective, which implies, following Proposition 10, that both sets $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{L}}$ are in one-to-one correspondence. The interest relies on the fact that the odometer can be directly defined on $\widetilde{\mathcal{L}}$.

Definition 48. Let $d \geq 1$. A regular language L is said to be *d-synchronizing* if there exists a function $f : A^d \to Q$ such that for any word $w \in A^*$ of length d and any $q \in Q$, $\delta(q, w)$ is equal to $f(w)$ (let us recall that δ denotes the transition function of the minimal automaton of L). In other words, for any element $(x, y) = (x_0 x_1 \ldots, y_0 y_1 \ldots)$ in $\widetilde{\mathcal{K}}$, for all $i \geq 0$ the state y_i is completely determined by $x_i \cdots x_{i+d-1}$. A language is *synchronizing* if there exists a positive integer d such that L is d-synchronizing. Otherwise stated, this means that y can be deduced from x.

Example 49. Consider the language accepted by the automaton $\mathcal{M}_\sigma$ depicted in Figure 4 of Example 35. Here, we represent in Figure 5 the automaton $\widetilde{\mathcal{M}_\sigma}$. This
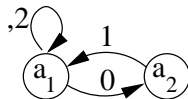

FIGURE 5. The automaton $\widetilde{\mathcal{M}_\sigma}$.

language is 1-synchronizing. Indeed, assume that (x, y) is an element in $\widetilde{\mathcal{K}}$. The factors possibly appearing in x are 00, 01, 02, 10, 12, 20 and 22. Actually, 11 and 21 cannot occur in x because no infinite path in the automaton depicted in Figure 5 contains such a factor. Clearly, if $x_i \in \{0, 2\}$ then $y_i = a_1$ and if $x_i = 1$ then $y_i = a_2$.

Example 50. Continuing Examples 35 and 49, the language accepted by $\mathcal{M}_\sigma$ depicted in Figure 4 is 1-synchronizing and we have the function

$$f : 0 \mapsto a_1,\ 1 \mapsto a_2,\ 2 \mapsto a_1.$$

For this automaton $\mathcal{M}_\sigma$, we have $\delta(a_1, 0) = \delta(a_2, 0) = f(0) = a_1$, $\delta(a_2, 1) = f(1) = a_2$, $\delta(a_2, 1)$ is the sink, $\delta(a_1, 2) = f(2) = a_1$ and $\delta(a_2, 2)$ is also the sink.

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