Dynamical Directions in Numeration

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DYNAMICAL DIRECTIONS IN NUMERATION

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ABSTRACT. We survey definitions and properties of numeration from a dynamical point of view. That is we focuse on numeration systems, their associated compactifications, and the dynamical systems that can be naturally defined on them. The exposition is unified by the notion of fibred numeration system. A lot of examples are discussed. Various numerations on natural, integral, real or complex numbers are presented with a special attention payed to beta-numeration and its generalisations, abstract numeration systems and shift radix systems. A section of applications ends the paper.

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Key words and phrases. numeration, coding, tilings, substitutions, additive functions, bounded remainder sets, odometers, subshifts, β-expansions.

1. Introduction

1.1. **Origins.** Numeration is the art of representation of numbers; primarily natural numbers, then extensions of them - fractions; negative, real, complex numbers, vectors, a.s.o. Numeration systems are algorithmic ways of coding numbers, that is, essentially, a process permitting to code elements of an infinite set with finitely many symbols.

For ancient civilisations, numeration was necessary for practical use, commerce, astronomy... Hence numeration systems have been created not only for writing down numbers, but also in order to perform arithmetical operations.

Numeration is inherently dynamical. Because it is collated with infinity as potentiality (Aristotle)[1]: if I can represent some natural number, how do I write the next one ? On that score, it is significant that motion (greek *dynamis*) and infinity are treated together in Aristotle's work (*Physics*, third book.) Furthermore, the will to deal with arbitrary large numbers requires

[1] "The infinite exhibits itself in different ways-in time, in the generations of man, and in the division of magnitudes. For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different. Again, 'being' has more than one sense, so that we must not regard the infinite as a 'this', such as a man or a horse, but must suppose it to exist in the sense in which we speak of the day or the games as existing things whose being has not come to them like that of a substance, but consists in a process of coming to be or passing away; definite if you like at each stage, yet always different." [19] translation from http://people.bu.edu/wwildman/WeirdWildWeb/courses/wphil/readings / wphil_rdg07_physics_entire.htm

some kind of invariance of the representation and a recursive algorithm which will be iterated, hence something of a dynamical kind again.

In the sequel, we briefly mention the most important historical steps of numeration. We refer to the book of Ifrah [107] for an amazing amount of information on the subject.

Numeration systems are the ultimate elaboration concerning representation of numbers. Most of the early representations are only concerned with finitely many numbers, indeed those which are of practical use. Some primitive civilisations ignore the concept of numeration and have only name for cardinals that are immediately perceptible without to perform any action of counting, that is, as anybody can experiment by himself, from 1 to 4. For example, the australian tribe Aranda say "*ninta*" for one, "*tara*" for two, "*tara-ma-ninta*" for three and "*tara-ma-tara*" for four. Larger numbers are indeterminate (many, a lot).

Many people have developped a representation of natural numbers with fingers, hands or other parts of the human body. Using phalanxes and articulations, it is then possible to represent (or show) numbers up to 10000 or more. A way of showing numbers up to 10^{10} only with both hands is attested in the XVIth century in China (*Sua fa tong zong*, 1593). Clearly, the choice of base 10 originates in those methods. Other bases are attested, like 5, 12, 20 or 60 by Babylonians. However, all representations use bases.

Bases have been developped in Egypt and Mesopotamia, about five thousand years ago. The Egyptians had a special sign for any small power of ten: a vertical stroke for 1, a kind of horseshoe for 10, a spiral for 100, a loto flower for 1000, a finger for 10000, a tadpole for 10^5 and a praying man for a million. For 45200, they drew four fingers, five loto flowers and two spirals (hieroglyphic writing). A similar princip was used by Sumerians in base 60. To avoid an indegestible representation, digits (from 1 to 59) were written in base 10. This kind of representation follows an additional logic. A more concise coding has been used by inventing a symbol for each digit from 1 to 9 in base 10. In this modified system, 431 is understood as $4 * 100 + 3 * 10 + 1 * 1$ instead of $100 + 100 + 100 + 100 + 10 + 10 + 10 + 1$. Etruscans used such a system. Hieratic and demotic handwritings in Egypt as well.

The next crucial step was the invention of positional numeration. It has been discovered independently four times, by Babylonians, in China, by the pre-Columbian Mayas and in India. However, only Indians had a distinct sign for every digit; Babylonians only had two, for 1 and 10. Positional numeration made possible the representation of arbitrary large numbers.

Nevertheless, the system was uncomplete without the most ingenuous invention: the zero. A sign for zero was necessary and it was known to those four civilisations. But to achieve the story, to be able not only to represent huge numbers but to peform arithmetic operations with any of them, one had to understand this zero not as "nothing" but as a quantity, an entity of the same type as the other numbers.

Ifrah writes: [Notre] *numération est née en Inde il y a plus de quinze siècles, de l'improbable conjonction de trois grandes idées ; à savoir :*

- l'idée de donner aux chiffres de base des signes graphiques détachés de toute intuition sensible, n'évoquant donc pas visuellement le nombre des unités représentées ;

- celle d'adopter le principe selon lequel les chiffres de base ont une valeur qui varie suivant la place qu'ils occupent dans les représentations numériques;

- et enfin celle de se donner un zéro totalement 'opératoire', c'est-à-dire permettant de remplacer le vide des unités manquantes et ayant simultanément le sens du 'nombre nul'." [107].

It was then possible to interpret the numeration system as something multiplicative: $431 = 1 + 10 * (3 + 4 * (100))$. Moreover, the representation could be obtained in a purely dynamical way and had a signification in terms of modular arithmetic. Finally, the concept of number deaply fits in with its representation. A mathematical maturation following an increasing abstraction process cumulating in the invention of the zero had been necessary to construct a satisfactory numeration system. It turned out to be the key for many further mathematical developments.

1.2. **From fractional and integral parts of real numbers and homoclinic points.**

1.2.1. *The q-adic expansion.* The representation of natural numbers with respect to a given integer base b allows to add and multiply integers easily by means of simple algorithms. We introduce the popular positional notation with a dot (a coma for french) so that, to base b, the symbol $(x_\ell \cdots x_0.x_1 \cdots x_r)_b$ $(\ell \leq 0)$ represents the b-adic rational number $\sum_{k=\ell}^{r} x_k b^{-k}$. Let us have a look on this numeration system from various approaches exhibiting several algebraic and topological objects. The first one is certainly the ring D_b of b-adic rational numbers that is to say all numbers of the form n/b^k. Addition and multiplication of integers to base b run as well in D_b with quite identical algorithms in use for natural integers, and the division (resp. multiplication) by b is performed simply by translating the dot of one position to the right (resp. to the left).

The ring D_b is also very convenient to approximate real numbers t which can be expanded as $t = \sum_{k=-m}^{\infty} t_k b^{-k}$ with uniqueness if we reject expansions which terminate with $t_k = b - 1$ if k is large enough. This approach could be as well the starting point to initiate the b-adic expansion including both integers and real numbers mod 1. To this ends the map $T_b : t \mapsto bt \pmod 1$ is introduced. It is a b-to-1 map on the torus $[0,1[\pmod 1$ which preserves the lebesgue measure. The b-adic expansion of any $t \in [0,1[$ described above is related to T_b by iterating the formula

$$t = \frac{t_1}{b} + \frac{T_b(t)}{b}\,.$$

We will go back to this point of view for generalising numeration systems.

For our convenience we introduce the set $\Omega(b)$ of the formal bi-infinite words $y = (\dots y_\ell \cdots y_1 y_0 \cdot y_1 \dots y_r \dots)$ ($y_k \in \{0,\dots,b-1\}$, for any $k \in \mathbb{Z}$) equipped with the compact weak-product topology and identify $x = (x_\ell \cdots x_0.x_1 \cdots x_r)_b$ to $({}^\omega 0 x_\ell \cdots x_0.x_1 \cdots x_r 0^\omega)$ where ${}^\omega 0$ (resp. 0^ω) denotes the left (resp. right) one-sided infinite word built from the symbol "0". Each element $x = (x_\ell \cdots x_0.x_1 \cdots x_r)_b$ in D_b has a unique decomposition $x = E(x) + F(x)$ where $E(x) = (x_\ell \cdots x_0)_b$ is the integer part of x and $(0.x_1 \cdots x_r)_b$ is its fractional part. From the left side, we get the sub-ring $\mathbb{Z} = E(D_b)$ which is naturally compactified to the ring of b-adic integers, denoted by $\mathbb{Z}_b$.

Now, E is extended to $\Omega(b)$ by taking the limit $E(y) = \lim_n \sum_{k=0}^n y_k b^k$ in $\mathbb{Z}_b$. The fractional part $F(x)$ is conveniently identified to an element of the one dimensional additive torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, so that $F(D_d)$ becomes the b-torsion subgroup $\mathbb{T}_b$ of $\mathbb{T}$. Now, we extend F to $\Omega(b)$ by continuity since

$$F(y) = \sum_{n=1}^{\infty} y_k b^{-k} = \lim_{\ell,r\to\infty} F({}^\omega 0 y_\ell \cdots y_0.y_1 \cdots y_r 0^\omega).$$

The decomposition map $\Sigma : \Omega(b) \to \mathbb{Z}_b \times \mathbb{T}$ defined by $\Sigma(y) = (E(y), F(y))$ is continuous, surjective, but cannot be one-to-one due to the fact that $\mathbb{T}$ is connected. In fact, Σ becomes injective on the subset $\Omega_0(b)$ of $\Omega(b)$ obtained by deleting all points ending to the right with $(b-1)^\omega$. The remaining points are said *regular*.

Before going back to the group D_b, we first point out that $\mathbb{Z}_b$ is the (compact) dual group of the discrete group $\mathbb{T}_b$ and $\mathbb{T}$ the dual group of $\mathbb{Z}$, and that $\Sigma(y) \in \mathbb{Z}_b \times [0,1[$.

Secondly, it is instructive to observe that the map $t \mapsto bt$ is a surjective endomorphim of $\mathbb{T}_b$, hence admits a natural extension which is precisely the map $M_b : t \mapsto bt$ viewed as an automorphism of D_b. Therefore, it is recommended to analyse the dual map M_b^* of M_b on the dual group $\Gamma_b = \widehat{D_b}$ and the effect both of the group law and of the map M_b^* with respect to the decomposition by Σ.

The dual group Γ_b is known as the b-adic solenoid (see [102]) that we represent here as the quotient group $(\mathbb{Z}_b \times \mathbb{R})/\Delta$ where $\mathbb{Z}_b \times \mathbb{R}$ is the direct product and $\Delta = \{(n,-n)\,;\; n \in \mathbb{Z}\}$. This group is usually viewed as the set

$\mathbb{Z}_b \times [0,1[$ (endowed with the compact topology which identifies $[0,1[$ and $\mathbb{T}$) and the group law $(u,t) \oplus (u',t') = (u + u' + \lfloor t + t' \rfloor, t + t' - \lfloor t + t' \rfloor)$. By construction, the map $J : \mathbb{R} \to \mathbb{Z}_b \times [0,1[$ defined by $J(t) = (\lfloor t \rfloor, t - \lfloor t \rfloor)$ is a one-to-one continous group homomorphism with a dense image, which identifies the real numbers into Γ_b. The characters of D_b are given by the paring

$$\psi(x,(u,t)) = \exp(2i\pi(x.u + xt)) \quad (x,u,t) \in D_b \times \mathbb{Z}_b \times [0,1[) \,,$$

where $x.u$ is set for $b^{-k}n(u_0 + u_1 b + \cdots + u_{k-1}b^{k-1})$ if $x = n/b^k$. The map M_b^* is then given by

$$M_b^*(u,t) = (bu + \lfloor bt \rfloor, bt - \lfloor bt \rfloor),$$

that the reader can check easil, as well the fact that if S denote the shift to the left on $\Omega(b)$ $(Sy = (\cdots y_\ell \cdots y_{-1}y_0y_1 \cdot y_2y_3 \cdots y_r \cdots))$, then

$$\Sigma(Sy) = M_b^*(\Sigma(y))\,.$$

To finish this glance at the b-adic expansion, we point out that M_b^* is an automorphism of the compact group Γ_b and the homoclinic group$^{(2)}$ of M_b^* is exactly $J(D_b)$ $(= \mathbb{Z} \times \mathbb{T}_b)$.

1.2.2. *The Zeckendorf expansion.* We may replace the integer base b by any real number $\beta > 1$ and asking the question to built a numeration system based on β. That is the so-called β-expansion developed in Example 2 and that goes through the whole paper. We may observe that, due to computing considerations, the cases of algebraic numbers β furnish the most interesting systems to investigate.

Let us consider the concrete and fascinating case of the golden ratio $\theta = \frac{1+\sqrt{5}}{2}$. We look for the smallest sub-ring D_θ of $\mathbb{Q}$ containing $\mathbb{Z}$, θ and θ^{-1}. Element in D_θ is said θ-rational. From the equation $\theta^2 = \theta + 1$ we readily obtain $D_\theta = \mathbb{Z} + \theta\mathbb{Z}$. The dual group $\widehat{D_\beta}$ is the two dimensional torus $\mathbb{T}^2$ and characters are furnished with the paring $\psi : D_\theta \times \mathbb{T}^2 \to \mathbb{U}$ defined by

$$\psi(x + y\theta),(u,v)) = \exp(2i\pi(xu + yv))\,.$$

The dual map $M_\theta^* : \mathbb{T}^2 \to \mathbb{T}^2$ of $M_\theta : z \mapsto \theta z$ $(z \in \mathbb{Z} + \theta\mathbb{Z})$ is given by

$$M_\theta^* \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u + v \end{pmatrix}.$$

The homoclinic group of M_θ^* is well known (see [197] for example), namely

$$\frac{1}{\sqrt{5}}(\mathbb{Z} + \theta\mathbb{Z}) \begin{pmatrix} 1/\theta \\ 1 \end{pmatrix}.$$

[2] Recall that for any automorphism α of a compact abelian group A a point $a \in A$ is homoclinic if $\lim_{|n|\to\infty} \alpha^n(x) = 0_A$. They form a subgroup of A.

This approach does not yet lead the map $T_\theta : t \mapsto \theta t - \lfloor \theta t \rfloor$ that we introduce now to produce the so-called θ-expansion

$$t = \sum_{n=1}^{\infty} \frac{t_k}{\theta^k} \tag{1.1}$$

analysed in details later on. The digits in (1.1) verify the constraint $t_k t_{k+1} \neq 11$ for all indexes k and this expansion is unique if we delete those which end with the sequence of digits that alternates 0 and 1. A θ-expansion ending with $t_k = 0$ for all k is said to be finite. Moreover, successive powers of M_θ^* lead the Fibonacci sequence $(F_n)_n$ that we define by $F_0 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. This sequence can be use to expand natural numbers N as $N = e_0 F_0 + e_1 F_1 + \cdots + e_m F_m$ with constraints $e_k e_{k+1} \neq 11$ ensuring the uniqueness of this expansion.

It is therefore natural to introduce the set $\Omega(\theta)$ of formal bi-infinite words $e = (\dots e_\ell \cdots e_{-1} e_0 \cdot e_1 \dots e_r \dots)$ with $e_k \in \{0,1\}$ and $e_k e_{k+1} \neq 11$, for any $k \in \mathbb{Z}$, equipped with the compact weak-product topology and to consider the compact space $\Omega^-(\theta)$ (resp. $\Omega^+(\theta)$ of left (resp. right) infinite words $(\dots e_\ell \cdots e_1 e_0)$ (resp. $(e_1, e_2, \dots))$ with no two consecutive digits equal to 1. The addition of 1 performed on natural numbers given by their Fibonacci expansion (or Zeckendorf expansion, see Example 6 *supra*) can be formally extented on $\Omega^-(\theta)$ to define the Fibonacci odometer $\tau : \Omega^-(\theta) \to \Omega^-(\theta)$. To visualise the odometer as a familiar transformation, let us introduce the Mona map $\Phi^- : \Omega^-(\theta) \to [-1/\theta, 1 - 1/\theta[$ mod 1 by

$$\Phi^-(\cdots e_{-2} e_{-1} e_0) = \sum_{k=0}^{\infty} e_{-k}(F_k \theta - F_{k+1}) \text{ mod } 1.$$

Then, Φ^- is continuous and conjugates the odometer with the translation $R_\theta : x \mapsto x + \theta$ on $\mathbb{T}$, *i.e.*

$$\Phi^- \circ \tau = R_\theta \circ \Phi\,.$$

We are ready to send $\Omega(b)$ into $\mathbb{T}^2$ as follows:

$$\Phi : (\cdots e_{-2} e_{-1} e_0 \cdot e_1 e_2 \dots) \mapsto \big(\Phi^-(\cdots e_{-2} e_{-1} e_0), \Phi^+(e_1 e_2 \dots)\big),$$

with $\Phi^+ : \Omega^+(\theta) \to [0,1[$ defined by $\Phi^+(e_1 e_2 \dots) = \sum_{k=1}^{\infty} e_k \theta^{-k}$. It can be shown by hand that first $\Phi(\Omega(\theta)) = [0,1] \times [0, \theta^{-1}] \cup [0, \theta^{-1}] \times [\theta^{-1}, 1]$ and if S denote the shift to the left on $\Omega(\theta)$ then $\Phi \circ S = \widetilde{T}_\beta \circ \Phi$ where $\widetilde{T}_\beta$ is defined on $\Phi(\Omega(\theta))$ by

$$\widetilde{T}_\beta(s,t) = (\theta^{-1}(s + \lfloor \theta t \rfloor), \theta t - \lfloor \theta t \rfloor)\,. \tag{1.2}$$

Due to the fact that $\theta^k + \left(\frac{-1}{\theta}\right)^k$ is an integer the double series $\sum_{k=-\infty}^{\infty} e_k \theta^{-k}$ converge mod 1. Now, following [197] a bijective map $\varphi : \Omega(\theta) \to \mathbb{T}^2$ can be defined using homoclinic points, for example

$$\varphi(e) = \frac{1}{\sqrt{5}} \sum_{k=-\infty}^{\infty} e_k \theta^{-k} \cdot \begin{pmatrix} 1/\theta \\ 1 \end{pmatrix}.$$

With these constructions the commutation relation $\varphi \circ S = M_\theta^* \circ \varphi$ holds.

1.2.3. *The Gaussian integers.* Another direction for numeration systems is to consider the complex numbers. To the scope of this introduction we replace $\mathbb{Z}$ by the ring of Gauß integers $\mathbb{Z}[i]$ and select the base $\alpha = -1 + i$ which is said canonical in the literature (see section 4 for details and references) due to the fact that any Gauß integers z admit a unique expansion of the form

$$z = c_{-\ell}\alpha^\ell + c_{\ell-1} + \cdots + c_{-1}\alpha + c_0\,, \quad c_{-m} \neq 0 \tag{1.3}$$

where the digits c_{-k} $(k \in \{0, 1, \ldots, m\}$ are 0 or 1. Notice that there are no digital constraints in the expansion (1.3) so that the map $z \mapsto {}^\omega 0 a_{-\ell} \cdots a_{-1} a_0$ is one-to-one between $\mathbb{Z}[i]$ and the set of left infinite binary words that end with ${}^\omega 0$.

We proceed as for the ordinary base two. Let D_α the sub-ring of α-adic rational complex numbers with are those of the form z/α^n, $z \in \mathbb{Z}[i]$, leading to the alpha-expansion $(z_{-\ell} \cdots z_{-1} z_0 \cdot z_1 \ldots z_r)_\alpha$ and the unique decomposition $E(z) + F(z)$ where $E(z) = \sum_{k=0}^{\ell} z_{-k}\alpha^k$ is called the integer part of z and $F(z)$ its fractional part.

Let $\mathcal{K}(\alpha)$ be the α-adic ring of integers viewed as well as the α-compactification of $\mathbb{Z}[i]$ or the projective limit $\mathcal{K}(\alpha) = \operatorname{proj\,lim}_n \mathbb{Z}[i]/(\alpha)^n$. By natural identification, $\mathbb{Z}[i]$ is viewed as a sub-ring of $\mathcal{K}(\alpha)$ and the map E is extended continuously to $\Omega(2)$ by the convergent series in $\mathcal{K}(\alpha)$: $E(x) = \sum_{\ell=0}^{\infty} x_{-\ell}\alpha^\ell$. We do the same with F and built a continuous map $F : \Omega(2) \to \mathbb{C}/\mathbb{Z}[i]$, putting $F(x) = \sum_{r=1}^{\infty} x_r \alpha^{-r} \pmod{\mathbb{Z}[i]}$.

The decomposition map $\Sigma : \Omega(2) \to \mathcal{K}(\alpha) \times (\mathbb{C}/\mathbb{Z}[i])$ defined by $\Sigma(y) = (E(y), F(y))$ is continuous, surjective, and one-to-one on $\Omega_0(2)$. The analogy with the usual base 2 is not on surface, but in deep. In fact, set $\Omega(2)^- = \{x \in \Omega(2)\,;\ \cdots x_1 x_0 \cdot 0^\omega\}$ and $\Omega(2)^+ = \{x \in \Omega(2)\,;\ {}^\omega 0 \cdot x_1 x_2 \cdots\}$. The restriction of E on $\Omega(2)^-$ is an homeomorphism. Moreover, any complex number ζ has a unique expansion of the form $\sum_{k=-\ell}^{\infty} \zeta_k \alpha^{-k}$ if we assume that $(\zeta_{-\ell} \cdots \zeta_0 \cdot \zeta_1 \zeta_2 \cdots)$ belongs to $\Omega_0(2)$. This allows to define the integer part $E(\zeta) := \sum_{k=0}^{\ell} \zeta_k \alpha^{-k}$ of ζ. The image $F(\Omega_0(2))$ (in $\mathbb{C}$) is a fundamental domain for the discrete group $\mathbb{Z}[i]$ in $\mathbb{C}$ and its closure $F(\Omega_0(2))$ coincide with the popular twin-dragon $\mathcal{T}_0$ depicted Figure 4.2. Subsection 4.6 investigated tiling properties of the twin-dragon.

We are ready to describe the dual group Γ_α of D_α like a $\mathbb{C}$-solenoidal group wich contains all the dynamics associated to the complex numeration to base α. Introduce the direct product $\mathcal{K} \times \mathbb{C}$ and the discrete subgroup

$$\Delta = \{(z, -z)\,;\; z \in \mathbb{Z}[i]\}.$$

The quotient group $(\mathcal{K}(\alpha) \times \mathbb{C})/\Delta$ is compact and can be identified to the product $\mathcal{K}(\alpha) \times (\mathbb{C}/\mathbb{Z}[i])$ endowed with the group law $(u, \zeta) \oplus (u', \zeta') = (u + u' + E(\zeta + \zeta'), \zeta + \zeta' - E(\zeta + \zeta'))$. By construction the map $J : \mathbb{C} \to \mathcal{K}(\alpha) \times (\mathbb{C}/\mathbb{Z}[i])$ defined by $J(\zeta) = (E(\zeta), \zeta - E(\zeta))$ is a one-to-one continuous group homomorphism with a dense image. The characters of D_α are now given by the paring

$$\psi(x, (u, \zeta)) = \exp(2i\pi\Re(x.u + x\zeta)) \quad (x, u, \zeta) \in D_\alpha \times (\mathcal{K}(\alpha) \times (\mathbb{C}/\mathbb{Z}[i]))\,,$$

where $x.u$ is set for $\alpha^{-k} z(u_0 + u_1\alpha + \cdots + u_{k-1}\alpha^{k-1})$ if $x = z/\alpha^k$.

Finally, the dual homomorphism M_α^* of the group automorphism $M_\alpha : x \mapsto \alpha x$ of D_α is then given by

$$M_\alpha^*(u, \zeta) = (\alpha u + E(\alpha\zeta), \alpha\zeta - E(\alpha\zeta)).$$

The analogy with the numeration system to base two is complete.

1.3. **What this survey is (not) about.** The subject of representing natural integers, real numbers or any suitable extension, generalisation or analogon of them (complex numbers, integers of a number field, elements of a quotient ring, vectors of a finite dimensional vector space, points of a function field, aso) is too vast to be embraced in a single paper. Hence we made choices.

Choices among the possible arts to think about numbers and their representations. Our point of view is essentially dynamic: we are more interested in the transformations yielding representations than in the representations themselves. We focuse as well our attention on dynamical systems built on these representations, since we think they give some insight in their understanding: as we explained through our historical considerations, numeration is in itself a dynamical concept. The key concept will be that of fibred numeration system, inspired from Schweiger, that we present in Section 3.1.

We do not treat subjects that already have been developped in previous surveys or books, even if the dynamical dimension is thereby not absent. Amoung others, we do not study in details arithmetic operations on numeration systems, how to perform them, the questions of normalization, and whether they are computable by a finite transducer or not. In that direction, we refer for example to [88]. We do neither developp too much the questions of recognizability of the underlying languages. The analysis of the β-expansions from the point of view of Vershik has been already done by Sidorov in the very clear paper [195]. It would then not make sense to insist on that subject.

Usually, papers on numeration deal with numeration on some special set of numbers: $\mathbb{N}$, $\mathbb{Z}$, $[0,1]$, $\mathbb{Z}[i]$... Our purpose is to introduce a general setting in which those examples can take place. In fact, a suitable concept already exists and it turns out that the notion of fibred system, due to Schweiger, is a powerful object to subsume most of the various numerations under a unified concept. Hence, the concepts we define in Section 3 originates directly from his book [192] [3] or have been naturally built up from it. A second source of inspiration has been the survey of Kátai [140].

These notions - essentially those of fibred system and of numeration system - are very general and helpful in order to think what quite different types of numerations may have in common. Simultaneously, they are flexible in that sense that they can be enriched with different structures. Addording to our needs, that is describing the classical examples of numeration, we will equip them progressively with a topology, a sigma-algebra or an algebraic structure, giving rise to new questions. In other words, our purpose is not to study properties of fibred numeration systems, but to use them as a frame of mind in thinking numeration.

After having given the principal definitions, we present in details some questions we will handle along the paper and discuss a series of significative examples. Each of the three next sections is devoted to a specific direction of generalisation of usual numeration and the last one deals with applications in number theory and theoretical physics. The next section is concerned with generalities on dynamical systems that we use in the sequel. Although Section 3 is essential to understand the vocabulary used along the paper, some of the notions above are not of constant use and it is possible for the reader to go back to them in case of need.

2. Tools from dynamical systems

2.1. **Ergodic notions.** We first recall some basic notions on topological dynamics and measure-theoretic dynamical systems. For more details about all of the notions defined in this section, the reader is refered to [63, 99, 220].

A *topological dynamical* system is defined as a compact metric space X together with a continuous map T defined onto the set X. A Borel measure

[3] "The notion of a fibred system arose form successive attempts to extend the so-called metrical number theory of decimal expansions and continued fractions to more general types of algorithms. [...]

Another source for this theory is ergodic theory, especially the interest in providing examples for one-sided subshifts, topological Markov chains, sofic systems and the like." [192], pages 1-2. For other applications of fibred systems and relevant references, see Preface an Chapter 1 of the book and the later book of the same author [193].

μ defined over X is said T-invariant if $\mu(T^{-1}(B)) = \mu(B)$, for every measurable set B. Let us note that a topological dynamical system always has an invariant probability measure. Indeed, given a point $x \in X$, any cluster point for the weak-star topology of the sequence of probability measures $\frac{1}{N}\sum_{n<N}\delta_{T^n x}$ is a T-invariant probability measure, where δ_y denotes the Dirac measure at point y. The topological dynamical system (X,T) is said *uniquely ergodic* if there exists a unique T-invariant probability measure on X.

A *measure-theoretic dynamical system*, or simply a *dynamical system* is defined as a system $(X, T, \mu, \mathcal{B})$, where μ is a probability measure defined on the σ-algebra $\mathcal{B}$ of subsets of X, and $T : X \to X$ is a measurable map which preserves the measure μ. A measure-theoretic dynamical system $(X, T, \mu, \mathcal{B})$ is *ergodic* if every measurable set B of X such that $T^{-1}(B) = B$ has zero measure or full measure.

A topological dynamical system (X,T) or a measure-theoretic dynamical system $(X, T, \mu, \mathcal{B})$ is *minimal* if every non-empty closed shift-invariant subset equals the whole set.

Two dynamical systems (X,S) and (Y,T) are said to be *topologically conjugate* (or *topologically isomorphic*) if there exists an homeomorphism f from X onto Y which conjugates S and T, that is: $f \circ S = T \circ f$. A topological dynamical system (Y,T) is a *topological factor* of (X,S) if there exists a continuous map from X onto Y which conjugates the maps S and T. The system (X,T) is then called an *extension* of (Y,S).

There also exists an equivalent of the notion of topological conjugacy for measure-theoretic dynamical systems. The idea consists in removing sets of measure zero in order to conjugate the spaces via an invertible measurable transformation. Two measure-theoretic dynamical systems $(X_1, T_1, \mu_1, \mathcal{B}_1)$ and $(X_2, T_2, \mu_2, \mathcal{B}_2)$ are said to be *measure-theoretically isomorphic* if there exist two sets of full measure $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$, a measurable map $f : B_1 \to B_2$ called *conjugacy map* such that

- the map f is one-to-one,
- the reciprocal map of f is measurable,
- f conjugates T_1 and T_2 over B_1 and B_2,
- μ_2 is the image $\mu_1 \circ f_{-1}$ of the measure μ_1 with respect to f, that is, $\forall B \in \mathcal{B}_2,\ \mu_1(f^{-1}(B)) = \mu_2(B)$.

If the map f is only onto, then $(X_2, T_2, \mu_2, \mathcal{B}_2)$ is said to be a *measure-theoretic factor* of $(X_1, T_1, \mu_1, \mathcal{B}_1)$, and $(X_2, T_2, \mu_2, \mathcal{B}_2)$ is said to be an *extension* of $(X_1, T_1, \mu_1, \mathcal{B}_1)$.

For a nice exposition of connected notions of isomorphism, see [220].

2.2. **Natural extension.** We recall below a classical construction [180], called the *natural extension* that, given a non-invertible transformation T :

$(X, \mu, \mathcal{B}) \to (X, \mu, \mathcal{B})$, associates to it an invertible transformation $\widetilde{T} : (\widetilde{X}, \widetilde{\mu}, \widetilde{\mathcal{B}}) \to (\widetilde{X}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ with similar ergodic properties, defined as an inductive limit. For more details, see for instance [63, 171, 180].

Theorem 2.1. [180] *Let $(X, T, \mu, \mathcal{B})$ be a measure-theoretic dynamical system. Up to measure-theoretic isomorphism, there exists a unique invertible measure-theoretic dynamical system $(\widetilde{X}, \widetilde{T}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ that satisfies:*

(1) *$(\widetilde{X}, \widetilde{T}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ is an extension of $(X, T, \mu, \mathcal{B})$; we denote by $\widetilde{\pi} \colon \widetilde{X} \to X$ the corresponding factor map;*
(2) *any invertible extension $(\hat{X}, \widehat{T}, \widehat{\mu}, \widehat{\mathcal{B}})$ of $(X, T, \mu, \mathcal{B})$ with factor map $\hat{\pi} \colon \hat{X} \to X$ is an extension of $(\widehat{X}, \widehat{T}, \widehat{\mu}, \widehat{\mathcal{B}})$ with factor map $g \colon \widehat{X} \to \tilde{X}$ satisfying $\widehat{\pi} = \widetilde{\pi} \circ g$.*

Ergodic properties such that ergodicity or mixing are preserved by the natural extension [180, 63].

We now detail the construction of a particular invertible measure-theoretic dynamical system $(\widetilde{X}, \widetilde{T}, \widetilde{\mu}, \widetilde{\mathcal{B}})$ that satisfies conditions (1) and (2) of Theorem 2.1. This particular construction corresponds to the so-called *natural extension.* One sets

$$\widetilde{X} := \{(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} \; ; \; T(x_n) = x_{n-1}, \; n \geq 1\}.$$

We transform $\widetilde{X}$ into a measure space by considering the σ-algebra generated by the following sets B_k, for $k \in \mathbb{N}$, and $B \in \mathcal{B}$

$$B_k := \{(x_n)_{n \in \mathbb{N}} \in \widetilde{X} \; ; \; x_k \in B\} \text{ for } B \in \mathcal{B}.$$

For any set B_k, put $\widetilde{\mu}(B_k) = \mu(B)$. This defines, according to Kolmogorov's consistency theorem, a probability measure $\widetilde{\mu}$ on $\widetilde{\mathcal{B}}$. The transformation $\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is then defined as

$$\widetilde{T}(x_0, x_1, x_2, \cdots) = (Tx_0, x_0, x_1, \cdots).$$

One checks that $\widetilde{T}$ is measurable and invertible, and that the measure $\widetilde{\mu}$ is $\widetilde{T}$-invariant. Let $\pi : \widetilde{X} \to X$ be the projection map defined by $\pi((x_n)_{n \in \mathbb{N}}) = x_0$. One easily checks that $\pi \circ \widetilde{T} = T \circ \pi$. Any measure-theoretic dynamical system satisfying conditions (1) and (2) of Theorem 2.1, and that is thus measure-theoretically isomorphic to $(\widetilde{X}, \widetilde{T}, \widetilde{\mu}, \widetilde{\mathcal{B}})$, is called a *realisation of the natural extension.*

The point is now to define suitable realisations of the natural extension in order to deduce for instance information on the invariant measure. In the case of fibred systems, the existence of dual systems provides an efficient way of exhibiting a realisation of the natural extension such as described in [192].

2.3. **Spectral notions.** An efficient way to obtain explicit factors is to study the spectral properties of the system.

2.3.1. *Eigenvalues.* Let $(X, T, \mu, \mathcal{B})$ be a measure-theoretic dynamical system. To simplify we assume that T is *invertible* (that is, T^{-1} is also measurable and measure-preserving). One can associate to it in a natural way an operator U acting on the Hilbert space $\mathcal{L}^2(X, \mu)$ defined as the following map:

$$\begin{array}{rccc} U: & \mathcal{L}^2(X,\mu) & \to & \mathcal{L}^2(X,\mu) \\ & f & \mapsto & f \circ T. \end{array}$$

Since T preserves the measure, the operator U is easily seen to be a *unitary operator*. Note that the surjectivity of the operator U comes from the invertibility of the map T.

The *eigenvalues* of $(X, T, \mu, \mathcal{B})$ are defined as being those of the map U. By abuse of language, we will call *spectrum* the set of eigenvalues of the operator U. It is a subgroup of the unit circle. The *eigenfunctions* of (X, T, μ) are defined to be the eigenvectors of U. Let us note that the map U always has 1 as an eigenvalue and any non-zero constant function is a corresponding eigenfunction. (X, T, μ) is ergodic if and only if the eigenvalue 1 has multiplicity. The spectrum is said to be *irrational* (respectively *rational*) if it is included in $\exp(2i\pi\mathbb{R} \setminus \mathbb{Q})$ (respectively in $\exp(2i\pi\mathbb{Q})$). The spectrum is said to be *discrete* if $\mathcal{L}^2(X, \mu)$ admits an Hilbert basis of eigenfunctions, that is, if the eigenfunctions span $\mathcal{L}^2(X, \mu)$. Hence, if $\mathcal{L}^2(X, \mu)$ is separable (this is the case for instance if X is a compact metric separable space), then there are at most a countable number of eigenvalues.

If the spectrum contains only the eigenvalue 1, with multiplicity 1, the system is said to be *weakly mixing* or to have a *continuous spectrum.* This implies in particular that T is ergodic.

One interest of this notion relies in the close connections between the spectrum and the determination of factors. Indeed, on the one hand, the spectrum of a measure-theoretic dynamical system contains the spectrum of any of its measure-theoretic factors. On the other hand, the knowledge on the existence of some eigenvalues of the dynamical system allows the determination of some rotation factors. In particular, if β is an eigenvalue of $(X, T, \mu, \mathcal{B})$, then according to the fact that the argument of β is rational or not, we deduce that the system (X, T, μ) admits as a factor a rotation on a finite group or on the torus $\mathbb{T}$. Furthermore, Von Neumann proved that, restricting to invertible ergodic system with discrete spectrum, the spectrum is a complete measure-theoretic invariant, that is, two invertible and ergodic transformations with identical discrete spectrum are measure-theoretically isomorphic; furthermore, any invertible and ergodic system with discrete spectrum is measure-theoretically isomorphic to a rotation on a compact abelian group, equiped with the Haar measure.

One can similary define a notion of topological spectrum. Let (X, T) be a topological dynamical system, where T is an homeomorphism. A nonzero complex-valued continuous function f in $\mathcal{C}(X)$ is an *eigenfunction* for T if there exists $\lambda \in \mathbb{C}$ such that $\forall x \in X,\ f(Tx) = \lambda f(x)$. The set of the eigenvalues corresponding to those eigenfunctions is called the *topological spectrum* of the operator U. If two systems are topologically conjugate they have the same group of eigenvalues. The operator U is said to have *topological discrete spectrum* if the eigenfunctions span $\mathcal{C}(X)$. Two minimal topological dynamical systems (X_1, T_1) and (X_2, T_2) with discrete spectrum, where T_1 and T_2 are homeomorphisms, are topologically conjugate if and only if they have the same eigenvalues. Von Neumann's Theorem becomes in this framework: two minimal topological dynamical systems (X_1, T_1) and (X_2, T_2) with discrete spectrum, where T_1 and T_2 are homeomorphisms, are topologically conjugate if and only if they have the same eigenvalues. any invertible and minimal topological dynamical system with topological discrete spectrum is topologically conjugate to a minimal rotation on a compact abelian metric group.

2.3.2. *Spectral measures.* To exhibit harmonic and spectral properties of dynamical systems related to numeration systems, we first recall some classical results on unitary representation of an abelian discrete countable group Γ. In current applications Γ is $\mathbb{Z}^k$ $(k \geq 1)$ but there is no particular inconvenient to work in the general situation.

Let $V : \gamma \mapsto V^\gamma$ be a unitary representation of Γ on an Hilbert space H. According to the theorem of Bochner-Herglotz, for any element h in H there exists a measure ρ_h on the dual group $\widehat{\Gamma}$ with $\rho_h(\widehat{\Gamma}) = ||h||^2$ which is defined by its Fourier transform

$$s \mapsto \hat{\rho}_h(s) = \langle V^s h, h \rangle \quad (s \in \Gamma)\,.$$

Notice that $\rho_h(\widehat{\Gamma}) = ||h||^2$. There are several maner to built ρ_n. To do so in our general setting, we use the notion of convergence along a Følner sequence which is by definition an increasing sequence of non-empty finite subsets A_n of Γ such that, for all $\gamma \in \Gamma$, the limit

$$\lim_n \frac{\#((A_n + \gamma)\Delta A_n))}{\# A_n} = 0$$

holds.

Theorem 2.2. *The sequence of following measures on* $\hat{\Gamma}$

$$\nu_n(d\chi) := \frac{1}{\# A_n} || \sum_{z \in A_n} \chi(-z) V^z h||^2 h_{\hat{\Gamma}}(d\chi)$$

weakly converge to ρ_h.

The proof is done by straightforward computations. The discrete part of ρ_h can be captured by the following formula, valuable for any given measure ν on $\hat{\Gamma}$ and due to Wiener and Schoenberg in the classical case:

$$\sum_{\chi \in \hat{\Gamma}} |\nu(\chi)|^2 = \lim_n \frac{1}{\# A_n} \sum_{z \in A_n} |\hat{\nu}(z)|^2.$$

In short, the proof is derived from the basic equality

$$\int_{\hat\Gamma\times\hat\Gamma} \sum_{z\in A_n} \chi(z)\psi(-z)\, d\nu(\chi)\, d\nu(\psi) = \sum_{z\in A_n} |\hat\nu(z)|^2,$$

the limit

$$\lim_n \frac{1}{\# A_n} \sum_{z\in A_n} \chi(z) = \begin{cases} 1 & \text{if } \chi = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases}$$

for character χ of Γ, and the Lebesgue dominated convergence theorem.

2.3.3. *Rigidity and spectral disjointness.* Let ζ be a complex number of modulus < 1. A subset S of Γ is called a ζ-rigid time for V, if $(V^s)_{s\in S}$ weakly converges to ζI according to the filter of cofinite parts. By polarisation, S is a ζ-rigid time for V means that

$$\forall\, h \in H \ , \ \lim_{s\in S} \langle V^s h|h\rangle = \zeta ||h||^2.$$

Theorem 2.3. *A subset S of Γ is a ζ-rigid time for V, if and only if, for any $h \in H$ and all $\varphi \in L^1(\hat\Gamma, \rho_h)$,*

$$\lim_{s\in S} \int_{\hat\Gamma} \varphi(u)u(s)\rho_h(du) = \zeta \int_{\hat\Gamma} \varphi(u)\rho_h(du) \tag{2.1}$$

Proof. (in short) Denote by $< \cdot|\cdot >$ the duality between Γ and $\widehat\Gamma$. From definitions,

$$\langle V^{s+\gamma}h|h\rangle = \int_{\widehat\Gamma} < s+\gamma|u > \rho_h(du) = \int_{\widehat\Gamma} u(s)u(\gamma)\rho_h(du),$$

and moreover

$$\lim_{s\in S} \langle V^{s+\gamma}h|h\rangle = \zeta\langle V^\gamma h|h\rangle = \zeta \int_{\widehat\Gamma} < \gamma|u > \rho_h(du).$$

Hence (2.1) holds for characters $u \mapsto < \gamma|u >$ of $\widehat\Gamma$ and then by standard argument, holds for all φ in $L^1(\hat\Gamma, \rho_h)$. To prove the converse, just set $\varphi = \mathbf{1}$ in (2.1). □

Theorem 2.3 furnishes a simple test to get spectral disjointness of two unitary representations of Γ:

Theorem 2.4. *Let V, V' two unitary representations of Γ (on H and H' respectively) and let $S \subset \Gamma$ be a ζ-rigid time for V and a ζ'-rigid time for V'. If $\zeta \neq \zeta'$, then V and V' are spectrally disjoint.*

2.3.4. *Skew products.* As we shall see, many arithmetical functions from numeration systems can be produced from orbits under suitable dynamical systems. One basic construction of interest is the skew product that we explain now. To begin, let K be an abelian compact metrisable group with law noted additively and let λ be the Haar measure on K. We consider an aperiodic ergodic action $\gamma \mapsto \tau^\gamma$ of Γ on K by means of translations. The aperiodicity says that $\lambda(\{x \in X\,;\ \exists \gamma \in \Gamma \setminus \{0_\gamma\}, ; \tau^\gamma(x) = x\}) = 0$ and allows to indentify any γ in Γ with $\theta(\gamma) = \tau^\gamma(0_K)$.

Definition 2.1. *A measurable map $\varphi : \Gamma \times K \to \mathbf{U}$ is called a (multiplicative) τ-**cocycle** if*

$$\varphi(\gamma + \gamma', x) = \varphi(\gamma, \tau^{\gamma'} x)\varphi(\gamma', x) \quad \lambda\text{–a.e.}$$

and

$$\varphi(0_\Gamma, \cdot) = 0 \qquad \lambda\text{–}a.e..$$

This definition can be readily extended to the case where $\mathbb{U}$ is replaced by a locally compact metrisable abelian group.

Definition 2.2. *The skew product above τ with cocycle φ is the action τ_φ on $K \times \mathbb{U}$ defined by*

$$(\tau_\varphi)^\gamma(x, z) = (x + \theta(\gamma), z\varphi(\gamma, x)).$$

The dynamical system τ_φ induced a unitary representation of Γ on $L^2(K \times \mathbf{U})$ that admits the orthogonal decomposition

$$L^2(K \times \mathbf{U}) = \bigoplus_{n \in \mathbf{Z}} L^2(K) \otimes \chi_n \qquad (\chi_n : z \to z^n).$$

which is invariant under τ_φ. Precisely, the action of τ_φ restricted $L^2(K) \otimes \chi_n$, but identified to $L^2(K)$, is $f \mapsto \varphi(\gamma, x)h(x + \theta(\gamma))$. We introduce the general weighted operator U_φ^γ by

$$U_\varphi^\gamma(h)(x) = \varphi(\gamma, x)h(x + \theta(\gamma)).$$

On one side, going back to the rigidity, one has

Theorem 2.5. *Soit S un temps 1-rigide pour τ. Alors S est un temps ζ-rigide de U_φ si et seulement si pour tout $\chi \in \widehat{K}$,*

$$\lim_{s \in S} \int_K \varphi(s, x)\chi(x)\mu(dx) = \begin{cases} \zeta & \text{if } \chi \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

On the over side, the representation U_φ verifies the Weyl commutation property, namely, for any character χ_n of $\mathbf{U}$,

$$U_{\varphi^n}^\gamma(\chi.h) = \chi(\theta(\gamma)\chi.U_{\varphi^n}^\gamma(h)$$

Which has the following main consequence:

Theorem 2.6. *The spectrum of $U_{\varphi^n}^\gamma$ is pur.*

The proof is rather analoguous to the one given by Helson in [101].

2.3.5. *The method of K. Schmidt.* [190]

Let $\gamma \to T^\gamma$ be an aperiodic ergodic group action ergodique of Γ on a Lebesgue space $(X, \mathcal{L}, \mu)$ and let $\varphi : \Gamma \times K \to A$ be a cocycle and T_φ the skew product on $(X \times A, \mu \otimes h_A)$ define by

$$T_\varphi^\gamma(x, \alpha) = (T^\gamma x, \alpha + \varphi(\gamma, x).$$

We note $\overline{A} = A \cup \{\infty\}$ the one point compactification of A. An element α in $\overline{A}$ is called an essential value of φ if for all neighborhood $N(\alpha)$ of α, all measurable subset B of K with $\mu(B) > 0$, one has

$$\mu\Big(\bigcup_{\gamma \in \Gamma} (B \cap T^{-\gamma}(B) \cap \{x\,;\ \varphi(\gamma, x) \in N(\alpha)\})\Big) > 0.$$

Element of the set $E(\varphi) \cup A$ are called essential values at finite distance. K. Schmidt shown that $E(\varphi)$ is a closed subgroup of A, that if $\psi(x, \gamma) = f(T^\gamma x) - f(x)$ for a measurable map $f : X \to A$, then $E(\varphi + \psi) = E(\varphi)$. Two main properties of essential values explain the interest of this notion. One is that the equality $\overline{E}(\varphi) = \{0_A\}$ is equivalent to the existence of a measurable function $g : X \to A$ such that $\varphi(x, \gamma) = f \circ T^\gamma - f$. The other one characterise the ergodicity:

Theorem 2.7 (Fondamental Theorem (K. Schmidt))**.**

$$T_\varphi \text{ is ergodic } \Leftrightarrow\ E(a) = A.$$

2.4. **Symbolic dynamics. Words.** From a dynamical point of view, numeration systems are closely related with symbolic dynamics. Hence we recall some basic definitions on shifts. Let X be a topological space. Endow $X^{\mathbb{N}}$ with the product topology. We note its elements indifferently $(x_1, x_2, \ldots, x_n, \ldots)$ or as an infinite word $x_1 x_2 \cdots x_n \cdots$. Similarly, the elements of the product space $X^{\mathbb{Z}}$ are written $(\ldots, x_{-2}, x_{-1}, x_0; x_1, x_2, \ldots)$ or $x_{-2}x_{-1}x_0.x_1x_2\cdots$ The so called right-sided (resp. double sided) shift operator S acts on the product spaces $X^{\mathbb{N}}$ (resp. $X^{\mathbb{Z}}$) by

$$S(x_1 x_2 \cdots) = x_2 x_3 x_4 \cdots$$
$$(\text{resp. } S(\cdots x_{-2}x_{-1}x_0.x_1x_2\cdots) = \cdots x_{-2}x_{-1}x_0x_1.x_2x_3\cdots).$$

The dynamical systems $(X^{\mathbb{N}}, S)$ (resp. $(X^{\mathbb{Z}}, S)$) are the right-sided full shift (resp. two-sided full shift). Taking any subset of $X^{\mathbb{N}}$, we consider its orbit under the action on the shift operator and the (topological) closure of this orbit Σ. It is easy to check that Σ is stable under the action of the shift, hence an other dynamical system (Σ, S), called *subshift.*

If there exists subsets A_n of X^2 such that $\Sigma = \big\{x\ ;\ \forall n, (x_n, x_{n+1}) \in A_n\big\}$, one says that the subshift is markovian. It is usual to require additionaly homogenity, that is A_n independant of n. One says that the subshift is of finite type if there exists $k \geq 2$ and $A \subset X^k$ such that Σ can be described as the subset of the whole product space no elements of which has a factor belonging to A. If X is finite, the subshift is sofic if the set of its word form a regular language, that is is recognizable by a finite automaton.

We will also need to work with finite words. Our conventions are usual. If $u = u_1u_2\cdots u_r$ and $v = v_1v_2\cdots v_s$, uv denotes the concatenation of u and v, that is the word of length $r+s$: $uv = u_1u_2\cdots u_rv_1v_2\cdots v_s$. Powers are understood with respect to the concatenation operation, u^n is the word of length rn obtained by concatenation of n copies of u. Furthermore, u^ω is the right-sided infinite word obtained by concatenation of infinitely many copies of u; for instance, $(10)^\omega = 101010101010\ldots$. A *factor* of a finite or infinite word $u_0u_1\ldots$ is a subword occuring in u formed by consecutive letters, that is of the type $u_ku_{k+1}\cdots u_{k+s}$. A *prefix* of u is the beginning of u, that is $u_0u_1\cdots u_\ell$.

3. Numeration systems

3.1. **Fibred systems.** We introduce in the sequel the terminology we will follow along the paper concerning numeration systems. We begin with a definition from [192].

Definition 3.1. *A* fibred system *is a set X and a transformation $T\colon X\to X$ for which there exists a finite or countable set I and a partition $X = \biguplus_{i\in I} X_i$ of X such that the restriction T_i of T on X_i is injective for any $i\in I$.*

This yields a well defined map $\varepsilon\colon X\to I$ that associates to $x\in X$ the index i such that $x\in X_i$.

Suppose (X,T) is a fibred system with the associated objects above and set $\Omega = I^{\mathbb{N}}$. Let $\varphi\colon X\to\Omega$ defined by $\varphi(x) = (\varepsilon(T^nx))_{n\geq 0}$. We will write $\varepsilon_n = \varepsilon\circ T^{n-1}$ for short. Recall that S denotes the (right sided) shift operator on Ω. These definitions yield a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \varphi\downarrow & & \downarrow\varphi \\ \Omega & \xrightarrow[S]{} & \Omega \end{array} \tag{3.1}$$

Definition 3.2. *Let (X,T) be a fibred system and $\varphi\colon X\to\Omega$ a map as above. If the function φ is injective, we call the quadruple $\mathcal{N} = (X,T,I,\varphi)$ a* fibred numeration system *(FNS for short). Then I is the digits set of the numeration system; the map φ is the* representation map *and $\varphi(x)$ the* $\mathcal{N}$-representation *of x. Those elements of Ω which are contained in $\varphi(X)$ are called* admissible*.*

In general, the representation map is not surjective. The set of prefixes of the $\mathcal{N}$-representations is the underlying language $\mathcal{L} = \mathcal{L}(\mathcal{N})$ *of the (fibred) numeration system and its elements are said to be admissible. The images $\varphi(x)$ for $x\in X$ are the* admissible sequences*.*

Note that we could have taken the quadruple (X, T, I, ε) instead of the quadruple (X, T, I, φ) in the defintion. A FNS is substantially a *coding* of the elements of a set with a sequence of digits. In almost all examples, the digits set is finite. It may happen that it is countable.

By definition of a partition, $X_i \neq \emptyset$ for each $i \in I$; hence all digits are admissible. Moreover, *set of prefixes* and *set of factors* are here synonymous:

$$\begin{aligned} \mathcal{L} &= \big\{(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x))\,;\, n \in \mathbb{N}, x \in X\big\} \\ &= \big\{(\varepsilon_{m+1}(x), \varepsilon_{m+2}(x), \ldots, \varepsilon_{m+n}(x))\,;\, (m,n) \in \mathbb{N}^2, x \in X\big\}. \end{aligned} \tag{3.2}$$

The representation map transports cylinders from the product space Ω to X and one may define for $(i_0, i_1, \ldots, i_{n-1}) \in I^n$ the cylinder

$$X \supset C(i_0, i_1, \ldots, i_{n-1}) := \bigcap_{0 \leq j < n} T^{-j}(X_{i_j}) = \varphi^{-1}[i_0, i_1, \ldots, i_{n-1}]. \tag{3.3}$$

In particular, the earlier assumption that the restriction of T to X_i is injective ensures that the application $x \mapsto (\varepsilon(x), T(x))$ is itself injective and $\mathcal{N}$ is a fibred numeration system if and only if

$$\forall x \in X \ : \ \bigcap_{n \geq 0} C(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_n(x)) = \{x\}. \tag{3.4}$$

If X is a metric space, a sufficient condition for (3.4) to hold is that, for any admissible sequence $(i_1, i_2, \ldots, i_n, \ldots)$, the diameter of the cylinders $C(i_1, i_2, \ldots, i_n)$ tends to zero when n tends to infinity. In that case, if F is a closed subset of X, then

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1,\ldots,i_n) \in \mathcal{L} \\ C(i_1,\ldots,i_n) \cap F \neq \emptyset}} C(i_1, \ldots, i_n),$$

which proves that the σ-algebra $\mathcal{B}$ generated by the cylinders is the Borel algebra. In general, T is $\mathcal{B}$-measurable.

The representations introduced in Definition 3.2 are by nature infinite ones. Now it is suitable to have finite expansions at one's disposal.

Definition 3.3. *A fibred numeration system $\mathcal{N}$ is* finite *(FFNS), if and only if both conditions below are satisfied.*

1. *The transformation T possesses exactly one fixed point, x_0, with $\varepsilon(x_0) = i_0$, say.*
2. *For every element $x \in X$, there exists a natural integer n_0 such that $\varepsilon_n(x) = i_0$ for all $n \geq n_0$.*

A fibred numeration system $\mathcal{N}$ is quasi-finite*, if and only if it is not finite and every $\mathcal{N}$-representation is ultimately periodic.*

The attractor *of the system is the set $\mathcal{A} = \{x \in X \,;\, \exists k \geq 1 : T^k(x) = x\}$.*

In a FFNS, the representation $\varphi(x) = (\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n_0-1}(x), i_0, i_0, \ldots)$ of an element can be identified with the finite one $(\varepsilon_1(x), \varepsilon_2(x), \ldots, \varepsilon_{n_0-1}(x))$.

With this convention, a FFNS is a FNS where every element has a finite representation. Then one can see the representation map as a map $\varphi\colon X \to I^{(\mathbb{N})}$, where $I^{(\mathbb{N})}$ is the set of finite sequences on I.

By injectivity of φ the attractor is also the set of those elements having a purely periodic representation. A FNS is finite if and only if $\mathcal{A}$ is a singleton. A FNS is quasi-finite if every orbit falls in the attractor: $\forall x \in X, \exists k$ such that $T^k x \in \mathcal{A}$.

Note that if we want (what seems to be quite natural) that the admissibility of $(i_1, i_2, \ldots, i_n, \ldots)$ implies the admissibility of $(i_1, i_2, \ldots, i_n, i_0, i_0, \ldots)$, it is useful to add a further condition in Definition 3.3: for all $i \in I$, $T_i(X)$ contains X_{i_0}.

If the $\mathcal{N}$-representation of $x \in X$ is ultimately periodic, with preperiod $i_1 i_2 \cdots i_r$ and period $i_{r+1} i_{r+2} \cdots i_\ell$, we might code it as $(i_1 i_2 \cdots i_r, T^r x)$ and it would be the shortest coding of the form

$$(\varepsilon_1(x)\varepsilon_2(Tx)\cdots\varepsilon_s(T^{s-1}x), T^s x),$$

where $T^s x \in \mathcal{A}$. If there exist several fixed points and if every element's representation ends with the representation of some of them, all representations are finitely codable. However, it would be confusing to speak of finite expansion, as shows classical Example 1-2. By the way, the set of digits is often a subset of a set endowed with some algebraic structure and contains a zero element such that $\varphi^{-1}(0, 0, \ldots)$ is T-invariant. In that case, it is natural to speak of finite representation for those elements x such that $\varepsilon_n(x) = 0$ for n sufficiently large, even if there are others fixed points.

We introduce now a general notion of numeration system.

Definition 3.4. *A* numeration system *- NS - (resp. a* finite numeration system *- FNS) is a triple (X, I, φ), where X is a set, I a finite or countable set and an injective map $\varphi\colon X \hookrightarrow I^{\mathbb{N}}$ (resp. $\varphi\colon X \hookrightarrow I^{(\mathbb{N})}$). The vocabulary of Definition 3.2 extends to numeration systems.*

In other words, a numeration system is a coding of the elements of a set X with finite (FNS) or infinite (NS) sequences of digits, I still being the set of them. The difference with definition 3.2 is that we do not request this coding to arise from the iteration of a transformation on X. Section 6 is devoted to numeration systems that are not fibred; see also Example 6. Equation (3.2) is not valid for NS anymore. Hence the *language* is defined as the set of prefixes.

Numeration systems deal with *representations*, taken in the meaning of *codings*. In most of the cases, those representations are interpreted in a purely algebraic or an algebraic and topological way, giving rise to *expansions*.

Definition 3.5. *For a numeration system (X, I, φ), the representation $\varphi(x) = (\varepsilon_n(x))_{n\geq 1}$ of $x \in X$ gives rise to an* expansion *if X is equipped with a monoid structure $(X, *)$ (in the case of a FNS) or a topological monoid structure for a NS, and if there exist functions $\psi_j \colon I \to X$ such that for every $x \in X$, $x = *_j \psi_j(\varepsilon_j(x))$.*

In practice, X is often a part of a A-module and I a subset of the ring A. Furthermore, with the notation of Definition 3.5, the functions ψ_j take the form $\psi_j(i) = \nu(i)\xi_j$. In such a case, according to the usual terminology, the sequence $(\xi_j)_j$ is called *base* or *scale*.

Endowing I with a suitable topology, one may see Ω as a topological space equipped with the product topology. This yields the following definition.

Definition 3.6. *For a numeration system $\mathcal{N} = (X, I, \varphi)$, with a Hausdorff topological space I as digits set, the associated $\mathcal{N}$-compactification $X_{\mathcal{N}}$ is the closure of $\varphi(X)$ in the product space Ω.*

Hence one can build the $\mathcal{N}$-shift $(\Sigma_{\mathcal{N}}, S)$, where $\Sigma_{\mathcal{N}}$ is as usual the topological closure in $I^{\mathbb{N}}$ of the shift action on $\varphi(X)$, on which the shift operator still acts. By (3.2), if the numeration system is a FNS, then $\Sigma_{\mathcal{N}}$ and $X_{\mathcal{N}}$ coincide. Hence we will speak on the subshift $(X_{\mathcal{N}}, S)$.

Assume $\mathcal{N}$ is a FNS. If, as almost always, I is endowed with the discrete topology, then

$$X_{\mathcal{N}} = \{\omega = (\omega_0, \omega_1, \ldots)\ ;\ \forall n \geq 0\ :\ C(\omega_0, \ldots, \omega_{n-1}) \neq \emptyset\}.$$

3.2. **Questions.** Some natural questions arise when a fibred system (X, T) and a representation map φ are introduced.

Question 1. First of all, is φ injective ? In other words: do we have a FNS ? In some cases, X and I are totally ordered sets and injectivity of the representation map is a consequence of its increasingness with respect to the order on X and to the lexicographical order on Ω.
If we have a FNS, do we have a FFNS, a quasi-FFNS ? Are there interesting characterisations of the attractor ? The set of elements $x \in X$ whose $\mathcal{N}$-representation is stationary equal to i_0 is stable under the action of T. So does the set of elements with ultimately periodic $\mathcal{N}$-representation. In case we have a FNS, but not a FFNS, this observation interprets the problem of finding the elements having finite or ultimately periodic representations as finding induced FFNS and induced quasi-FFNS.

Question 2. The determination of the language is trivial when the representation map is surjective. Otherwise, the language can be described with some simple rules... or not. Hence the question: given a FNS, describe the underlying language. The structure of the language reflects that of the numeration system and it even often happens that the combinatorics of the language has a translation in terms of arithmetic properties of the numeration.

It is usual and meaningful to distinguish between different level of complexity. *Independence* of the digits if $\mathcal{L} = I^{\mathbb{N}}$, *markovian structure* if $\mathcal{L}$ can be described with the rule $(\varepsilon_n(x), \varepsilon_{n+1}(x)) \in A$, *of finite type* if there is a positive integer k and a finite set $A \subset I^k$ such that $\omega \in \mathcal{L}$ if and only if, for all n, $(\omega_n, \dots, \omega_{n+k-1}) \in A$, *sofic* if the language is recognizable by an automaton (*i.e.* rational). We refer to Section 5 for relevant results and examples.

Question 3. The list of properties of the language above corresponds to properties of the subshift $(X_{\mathcal{N}}, S)$. The dynamical structure of this subshift is an interesting question as well. It is not independant on the previous one: suppose $X_{\mathcal{N}}$ is endowed with some S-invariant measure. Then the digits can be seen as random variables $E_n(\omega) = \omega_n$ (the n-th projection). Their distribution can be investigated and reflect the properties of the digits. For instance, a markovian structure of the digits *vs* the sequence of coordinates as a Markov chain.

Question 4. This question addresses to not necessarily numeration systems. The transfer of some operations on X. In particular, if X is indeed a group or a semi-group $(X, *)$, is it possible to define an inner law on $X_{\mathcal{N}}$ by $x{*}y = \lim(\varphi(x_n * y_n))$, where $\lim \varphi(x_n) = x$ and $\lim \varphi(y_n) = y$? Or if T' is a further transformation on X, does it yield a transformation $\mathcal{T}$ on $X_{\mathcal{N}}$ by

$$\mathcal{T}(x) = \lim_{x_n \to x} \varphi(T'(x_n))?$$

According to these transformations on $X_{\mathcal{N}}$, one may define on $X_{\mathcal{N}}$ some probability measures. Then coordinates might be seen as random variables the distribution of which also reflects the dependance questions asked above.

Question 5. The dynamical system (X, T) is itself of course of interest. The precise study of the commutative diagram issued from (3.1) by replacing Ω by $X_{\mathcal{N}}$

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ X_{\mathcal{N}} & \xrightarrow[S]{} & X_{\mathcal{N}} \end{array} \tag{3.5}$$

can make (X, T) a factor or even a conjugated dynamical system of $(X_{\mathcal{N}}, S)$. As mentionned above, also other transformations (like the addition of 1) or algebraic operations on X can be considered and transfered to the $\mathcal{N}$-compactification, giving commutative diagrams similar to (3.5):

$$\begin{array}{ccc} X & \xrightarrow{T'} & X \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ X_{\mathcal{N}} & \xrightarrow[\mathcal{T}]{} & X_{\mathcal{N}} \end{array} \tag{3.6}$$

Results on X can be sometimes proved in this way (cf. Section 7).

The shift on the symbolic dynamical system $(X_{\mathcal{N}}, S)$ is usually not a one-to-one map. It is natural to try to look for a two-sided subshift that would project onto $((X_{\mathcal{N}}, S)$. Classical applications are for instance the determination of the invariant measure [166], or the characterisation of elements of X having a purely periodic $\mathcal{N}$-representation, see for instance in the β-numeration case [115, 181, 113, 42]. Nevertheless, the compactification $X_{\mathcal{N}}$ of X opens a large source of dynamical questions in connection with the numeration.

Question 6. Recognise the rotations (discrete spectrum) among the encountered dynamical systems.

More precisely, let $(X, T, \mu, \mathcal{B})$ be a dynamical system. We first notice that if T has a discrete spectrum, then T has a rigid time, that is to say, there exists an increasing sequence $(n_k)_{k\geq 0}$ of integers such that the sequence $k \mapsto T^{n_k}$ weakly converges to the identity. In other words, for any f and g in $L^2(X, \mu)$, one has

$$\lim_k (T^{n_k} f|g) = (f|g).$$

Such a rigid time can be selected in order to characterise T up to an isomorphism. In fact it is proved in [48] that for any countable subgroup G of $\mathbb{U}$, there exists a sequence $(a_n)_n$ of integers such that for any complex number ξ the sequence $n \mapsto \xi^{a_n}$ converge to 1 if and only if $\xi \in G$. Such a sequence, called *characteristic* for G, is a rigid time for any Dynamical system $(X, T, \mu, \mathcal{B})$ of discrete spectrum such that G is the group of eigenvalues. In case G is cyclic generated by $\zeta = e^{2i\pi\alpha}$ a characteristic sequence is built explicitly from the continued fraction expansion of α (see [48], Theorem 1*). Clearly if $(n_k)_{k\geq 0}$ is a rigid time for T with the group of complex numbers z such that $\lim_n z^{a_n} = 1$ reduced to $\{1\}$, then T is weak mixing. The following proposition is extracted from [203]:

Proposition 3.1. *Let $\mathcal{T} = (X, T, \mu, \mathcal{B})$ be a dynamical system and assume there exists an increasing sequence $(q_n)_n$ of integers such that the group of complex numbers z verifying $\lim_n z^{a_n} = 1$ is countable and there exists a dense subset D of $L^2(X\mu)$ such that for all $f \in D$ the series*

$$\sum_{n\geq 0} ||f \circ T^{q_n} - f||_2^2$$

converge, then T has discrete spectrum.

Question 7. A FNS produces the following situation:

$$X \xrightarrow[\varphi]{\sim} \varphi(X) \overset{i}{\hookrightarrow} X_{\mathcal{N}}.$$

Assume further that X is a Hausdorff topological space. Suppose that $\varphi^{-1} \colon \varphi(X) \to X$ admits a continuous extension $\psi \colon X_{\mathcal{N}} \to X$. We note $\overline{\varphi} = i \circ \varphi$. We have $\psi \circ \overline{\varphi} = \mathrm{id_X}$. Elements y of $X_{\mathcal{N}}$ different from $\varphi(x)$ such that $\psi(y) = x$ (if any) are called *improper representations of*

x. Then natural questions are to characterise the $x \in X$ having improper ($\mathcal{N}$-)representations, to count the number of improper representations, to find them, and so on. In other words, study the equivalence relation $\mathcal{R}$ on $X_{\mathcal{N}}$ defined by $u\mathcal{R}v \Leftrightarrow \psi(u) = \psi(v)$. In many cases, X is connected, $X_{\mathcal{N}}$ is completely disconnected, φ is not continuous, but ψ (by definition) is and X is homeomorphic to the quotient space $X_{\mathcal{N}}/\mathcal{R}$. The improper representations are naturally understood as expansions.

There are further questions, which only make sense in determined types of numeration systems and require further special and accurate definitions. They will be stated in the introduction of the corresponding section.

3.3. **Examples.**

Example 1. q-adic representations

(1) Let $X = \mathbb{N}$, $I = \{0, 1, \ldots, q-1\}$, $X_i = i+\mathbb{N}$, and $T(n) = (n-\varepsilon(n))/q$. Then 0 is the only fixed point of T. We have a FFNS, with language, set of representations and compactification

$$\mathcal{L}_q = \bigcup_{n \geq 0} \{0, 1, \ldots, q-1\}^n$$

$$\varphi(X) = I^{(\mathbb{N})} = \big\{(i_0, \ldots, i_{n-1}, 0, 0, 0, \ldots)\,; n \in \mathbb{N}, i_j \in \{0, 1, \ldots, q-1\}\big\}$$

$$X_{\mathcal{N}} = \{0, 1, \ldots, q-1\}^{\mathbb{N}}$$

The addition can be extended to $X_{\mathcal{N}}$, and gives the additive (profinite) group $\mathbb{Z}_q = \varprojlim \mathbb{Z}/q^n\mathbb{Z}$. On the commutative ring $\mathbb{Z}_q$, the shift operator is the multiplication by q. The coordinates are independant and identically uniformly distributed on I w.r.t. Haar measure μ_q, which fulfills $\mu_q[i_0, \ldots, i_{k-1}] = q^{-k}$.

(2) Let $X = \mathbb{Z}$ and everything else as in the first example. This is again a FNS, and a actually a quasi-FFNS, since

$$\varphi(X) = \{(i_0, \ldots, i_{n-1}, a, a, a, \ldots); n \in \mathbb{N},\ a = 0 \text{ or } a = q-1\}.$$

In other words, there are two T-invariant points, which are 0 and -1. The other sets are like in the first case: $\mathcal{L} = \mathcal{L}_q$ and $X_{\mathcal{N}} = \mathbb{Z}_q$.

(3) Same example, with $X = \mathbb{Z}$ and $T(n) = (n - \varepsilon(n))/(-q)$. Curiously, this is again a FFNS, with the same language, set of representations and compactification as in the first example (see Theorem 4.1).

(4) Generalise the second example by modifying the set of digits and taking any complete set of representants modulo q, and $q \in \mathbb{Z}$, $|q| \geq 2$. Then one always get a quasi-FFNS. This is due to the observation that for

$$L = \max\{|i|\,;\, i \in I\}/(|q| - 1),$$

the interval $[-L, L]$ is stable by T and $|T(n)| < |n|$ whenever $|n| > L$ (in fact, $|T(n)| \simeq |n|/q$ if n is large). The compactification is $I^{\mathbb{N}}$, the language and the set of representations hardly depend on the digits set (see [140] for a detailed study with many examples, Lemma 1 therein for the fact that it is a quasi-FFNS).

(5) $X = [0, 1)$, $I = \{0, 1, \ldots, q-1\}$, $X_i = [i/q, (i+1)/q)$, and $T(x) = qx - \lfloor qx \rfloor$. This defines a FNS, which becomes a quasi-FFNS if one restricts the space to $[0, 1) \cap \mathbb{Q}$. Lebesgue measure is T-invariant.

The language is $\mathcal{L}_q$ and the compactification $\mathbb{Z}_q$ in both cases. The set of representations is the whole product space without the sequences ultimately equal to $q-1$ in the first case, the subset of ultimately periodic sequences in the second case. The attractor is the set $\{a/b\ ;\ a < b \text{ and } \gcd(b, q) = 1\}$. If $x = a/b$, with a and b coprime integers, write $b = b_1 b_2$, with b_1 the highest divisor of b whose prime factors divide q. Then the length of the preperiod is $\min\{m\ ;\ b_1 | q^m\}$ and the length of the period is the order of q in $(\mathbb{Z}/b_2\mathbb{Z})^*$.

The continous extension ψ of φ^{-1} is defined on $\mathbb{Z}_q$ by $\psi(y) = \sum_{n\geq 1} y_n q^{-n}$. Elements of X possessing improper representations are the so-called "q-rationals", that is the numbers of the form a/q^m with $a \in \mathbb{N}$, $m \geq 0$ and $a/q^m < 1$. If the (proper) expansion is $(i_1, i_2, \ldots, i_s, 0^\omega)$, then the improper one is

$$(i_1, i_2, \ldots, i_{s-1}, i_s - 1, (q-1)^\omega).$$

Example 2. beta-representations

(1) It is possible in the latter example to replace q by any real number $\beta > 1$. Namely, $X = [0, 1)$, $I_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$, and $T(x) = T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ $\varepsilon(x) = \lfloor \beta x \rfloor$. This way of producing β-representations (which is actually a β-expansion $\sum_{n\geq 1} \varepsilon_n(x)\beta^{-n}$) is traditionnally called "greedy", since the digit choosed at step n is always the greetest possible, that is

$$\max \left\{ \epsilon \in I; \sum_{j=1}^{n-1} \varepsilon_j(x)\beta^{-j} + \epsilon\beta^{-n} < x \right\}.$$

It is due to Rényi (see Example 4 for a discussion on this seminal paper).

According to Parry [167], the set of admissible sequences $\varphi(X)$ is simply characterised in terms of one particular β-expansion. For that purpose, let us first extend the definition of T to $[0, 1]$ and consider $d_\beta(1) = (t_n)_{n\geq 1}$ defined as the Renyi β-expansion of 1. We then set $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and

$$d^*_\beta(1) = (t_1 \ldots t_{m-1}(t_m - 1))^\omega,$$

if $d_\beta(1) = t_1 \dots t_{m-1} t_m 0^\omega$ is finite ($t_m \neq 0$). The set $\varphi(X)$ of β-expansions of real numbers in $[0,1)$ is exactly the set of sequences $(u_n)_{n\geq 1}$ with values in I_β such that

$$\forall k \geq 1,\ (u_n)_{n\geq k} <_{\text{lex}} d^*_\beta(1). \tag{3.7}$$

The set $X_\mathcal{N} = \overline{\varphi([0,1))}$ is called the (right) *one-sided β-shift*. It is equal to the set of sequences $(u_i)_{i\geq 1}$ which satisfy

$$\forall k \geq 1,\ (u_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1). \tag{3.8}$$

Numbers β such that $d_\beta(1)$ is ultimately periodic are called *Parry numbers* and those such that $d_\beta(1)$ is finite are called *simple Parry numbers*. Both are clearly algebraic integers: Parry showed that they are Perron numbers (all conjugates have modulus at most 2) [167]. For example, the golden mean ρ is a simple Parry number with $d_\rho(1) = 110^\omega$. Applying the scheme of Question 7, simple Parry numbers are those, that produce improper expansions. To any sequence $(u_n)_{n\geq 1} \in X_\mathcal{N}$, we can associate the number $\psi(x) = \sum_{n\geq 1} x_n \beta^{-n}$. Then $\psi(x) \in [0,1]$ and numbers with two expansions are exactly those with finite expansion:

$$\psi(x_1 \dots x_{s-1} x_s 0^\omega) = \psi((x_1 \dots x_{s-1}(x_s - 1) d^*_\beta(1)).$$

If β is assumed to be a Pisot number, then every element of $\mathbb{Q}(\beta) \cap [0,1]$ admits a ultimately periodic expansion [188, 45], hence β is either a Parry number or a simple Parry number [45]. One deduces from the characterisation (3.8) that the shift $X_\mathcal{N}$ is of finite type if and only if β is a simple Parry number, and it is sofic if and only if $X_\mathcal{N}$ is a Parry number [116, 45].

Rényi proved that $([0,1), T_\beta)$ possesses a unique absolutely continuous invariant probability measure $h_\beta(x)d\lambda$, and computed it explicitly when β is the golden mean. Parry [167] extended this computation to the general case and proved that the Radon-Nikodym derivative of the measure is a step function, with a finite number of steps if and only if β is a Parry number. For more details on the β-numeration, see for instance [46, 50, 152, 88, 195].

(2) Note that $\sum_{n\geq 1}(\lceil\beta\rceil - 1)\beta^{-n} = (\lceil\beta\rceil - 1)/(\beta - 1) > 1$ if β is not an integer. This leaves some freedom in the choice of the digit. The "lazy" choice corresponds to the smallest possible digit, that is

$$\min\left\{\epsilon \in I; x - \left(\sum_{j=0}^{n-1} \varepsilon_j(x)\beta^{-j-1} + \epsilon\beta^{-n-1}\right) < (\lceil\beta\rceil - 1)/\beta^n(\beta - 1)\right\}.$$

This corresponds to $\varepsilon(x) = \left\lceil \beta x - \dfrac{\lceil\beta\rceil - 1}{\beta - 1} \right\rceil$ and $T(x) = \beta x - \varepsilon(x)$. These transformations are conjugated: wite φ_g and φ_ℓ for the greedy and lazy reprensentations respectively. Then

$$\varphi_\ell\left(\frac{\lceil\beta\rceil - 1}{\beta - 1} - x\right) = (\lceil\beta\rceil - 1, \lceil\beta\rceil - 1, \ldots) - \varphi_g(x).$$

(3) It is also possible to make a choice at any step: lazy or greedy. If this choice is made in alternance, we still have a FNS (with transformation T^2 and couple of digits). More complicated choices (for instance random) are also of interest (but not FNS). See [65] and the papers of Sidorov [195] and the references therein.

(4) For β the dominating root of some polynomial of the type

$$X^d - a_0X^{d-1} - a_1X^{d-2} - \cdots - a_{d-1}$$

with integral coefficients $a_0 \geq a_1 \geq \cdots \geq a_{d-1} \geq 1$, the restriction of the first tranformation ($T(x) = \beta x - \lfloor \beta x \rfloor$) on $\mathbb{Z}[\beta^{-1}]_+ = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$ yields a FFNS. Such number β are said to satisfy *Property* (F) (introduced in [89]). They will take a consequent place in this survey. An extensively studied question is to find characterisation of those β (see Section 5).

Example 3. continued fractions

Continued fractions have been an important source of inspiration in founding fibred systems. The classical continued fractions use $X = [0, 1)$, the so-called Gauß transformation $T(x) = 1/x - \lfloor 1/x \rfloor$ and $\varepsilon(x) = \lfloor 1/x \rfloor$. The set of digits is $\mathbb{N}^*$ and the representation function is surjective. The restriction to rational numbers yields an FFNS and the restriction to rational and quadratic numbers a quasi-FFNS (Lagrange). The expansion of special numbers (nothing is known about the continued fraction expansion of $\sqrt[3]{2}$), as well as the distribution properties of the digits (partial quotients) have been extensively studied since Gauß and provide still today many publications. The regularity of T allows to use Perron-Frobenius operators, which yield interesting asymptotic results as Gauß-Kuzmin-Wirsing's result, that we cite as example:

$$\lambda\{x \,;\, T^n(x) < t\} = \frac{\log(1+t)}{\log 2} + \mathcal{O}(q^n), \text{ with } q = 0.303663...$$

(Here λ is the Lebesgue measure. The limit is due to Gauß, the first error term and the first published proof to Kuzmin, the best possible to Wirsing, in 1974). For an example of a spectacular and very recent result, we refer to [3]. There is a huge number of variants (with even, odd, or negative digits for example), generalisations to higher dimensions. For that and further references, we refer to [192] and [193]. Due to the huge litterature, including books, it is not our purpose to say much about the theory of continued fractions.

Example 4. f-expansions

It is often referred to the paper of Rényi as the first occurrence of β-expansions. It is rarely mentionned that β-expansions only occupy the fourth section of this famous paper and are seen as an example of the today less popular f-expansions. [4]

The idea is to represent the real numbers $x \in [0,1]$ as

$$x = f(a_1 + f(a_2 + f(\cdots + f(a_n))\cdots), \text{ with } a_i \in \mathbb{N}. \tag{3.9}$$

It originates in noting that both continued fractions and q-adic expansions are special type of an f-expansion, with $f(x) = 1/x$ for the continued fractions and $f(x) = x/q$ for the q-adic expansions. Furthermore, the coefficients are given in both cases by $a_1 = \lfloor f^{-1}(x) \rfloor$ and it is clear that existence of an algorithm and convergence in (3.9) occur under suitable assumptions of general type on f (injectivity and regularity).

More precisely, let $f\colon J \to [0,1]$ an injective function, where $J \subset \mathbb{R}_+$ is an interval. Let $\varepsilon(x) = \lfloor f^{-1}(x) \rfloor$ for $0 \le x \le 1$ and $T\colon [0,1] \to [0,1]$ defined by $T(x) = f^{-1}(x) - \varepsilon(x)$. For $1 \le k \le n$, let us introduce

$$u_{k,n}(x) = f(\varepsilon_k(x) + f(\varepsilon_{k+1}(x) + \cdots + f(\varepsilon_n(x) + T^n(x))\cdots)$$
$$v_{k,n}(x) = f(\varepsilon_k(x) + f(\varepsilon_{k+1}(x) + \cdots + f(\varepsilon_n(x))\cdots).$$

Then $u_{1,n}(x) = x$, $u_{k,n}(x) = u_{1,n-k+1}(T^{k-1}(x))$, $v_{k,n}(x) = v_{1,n-k+1}(T^{k-1}(x))$ and we are interested in the convergence of $(v_{1,n}(x))_n$ to x. Indeed,

$$x - v_{1,n}(x) = T^n(x) \prod_{k=1}^{n} \frac{f(v_{k,n}) - f(u_{k,n})}{v_{k,n} - u_{k,n}} \tag{3.10}$$

Convergence to 0 in (3.10) yields a fibred numeration system and f-expansions according to Definition 3.5. This question seems to have been raised for the first time by Kakeya in 1924 [118]. Independently, Bissinger treated the case of a decreasing function f [49] and Everett the case of an increasing function f two years later [117] before the already cited synthesis of Rényi [178]. Since one needs the function f to be injective and continuous, there are two cases, whenever f is increasing or decreasing.

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[4]The term β-expansion does not occur in the AMS review of Thron, who just evoques "more general f-expansions" [than the q-adic one]. In Zentralblatt, the one full page long review of Hartman shortly says: "*Der schwierige Fall: $T < \infty$, T nicht ganz, wird nicht allgemein untersucht, jedoch kann Verf. für den Sonderfall $f(x) = x/\beta$ (bei $0 \le x \le \beta$) oder 1 (bei $\beta < x$), β nicht ganz, d.h. für die systematischen Entwickelungen nach einer gebrochenen Basis, den Hauptsatz noch beweisen.*" (The difficult case: $T < \infty$, T not integral, is not investigated in general. However, the author is able to prove the principal theorem for the special case $f(x) = x/\beta$ (for $0 \le x \le \beta$) or 1 (for $\beta < x$), β not an integer, that is for systematic expansions w.r.t. a fractional base.) [This "principal theorem" is concerned with the absolutely continuous invariant measure (see Example 2) - the case T finite non-integer is not treated in general.]

The usual assumptions are either $f\colon [1,g] \to [0,1]$, decreasing, with $2 < g \le +\infty$, $f(1) = 1$ and $f(g) = 0$, or $f\colon [0,g] \to [0,1]$, increasing, with $1 < g \le +\infty$, $f(0) = 0$, and $f(g) = 1$. In both cases, the set of digits is $I = \{1, \ldots, \lceil g \rceil - 1\}$. In case $T = +\infty$, the set of digits is infinite and there is a formal problem at the extremities of the interval. Let us consider the decreasing case. Then T is not well defined at 0. It is possible to consider the transformation T on $[0,1] \setminus \cup_{j\ge 0} T^{-j}\{0\}$. It is also valid to set $T(0) = 0$ and $\varepsilon(0) = \infty$, say. Then, we say that the f-representation of x is finite if the digit ∞ occurs. In terms of expansions, for $(\varepsilon_n(x))_n = (i_1, \ldots, i_n, \infty, \infty, \ldots)$, we have a finite expansion $x = f(i_1 + f(i_2 + \cdots + f(i_n))\cdots)$. For the special case of continued fractions, the first choice considers the Gauß's transformation on $[0,1] \setminus \mathbb{Q}$ and the second one obtains the so called regular continued fraction expansion of rational numbers. The case f increasing is similar.

The convergence in Equation (3.10) is clearly ensured under the hypothesis that f is contracting (s-lipschitz with $s < 1$). There are several results in that direction, which are variants of this hypothesis and depend on the different cases (d decreasing or increasing, g finite or not). We refer to the references cited above and to the paper of Parry [168] for more details.

The rest of Rényi's paper is devoted to the ergodic study of the dynamical system $([0,1], T)$. Considering the case of independant digits ($g \in \mathbb{N}$ or $g = \infty$), and assuming that there exists a constant C such that for all x, one has $\sup_t |H_n(x,t)| \le C \inf_t |H_n(x,t)|$, where

$$H(x,t) = \frac{d}{dt} f(\varepsilon_1(x) + f(\varepsilon_2 + \cdots + f(\varepsilon_n(x) + t))\cdots)$$

he proves that there exists a unique T-invariant absolutely continuous measure $\mu = h d\lambda$ such that $C^{-1} \le h(x) \le C$. Note that the terminology "independant" is troublesome, since as random variables defined on $([0,1], \mu)$, the digits ε_n are not necessarily independant. They are in the q-adic case, but they are not in the continued fractions case. We refer to [192] for further developpements.

Example 5. Signed numeration systems

Such representations have been introduced in order to facilitate the performing of arithmetical operations. To our knowledge, the first apparition of negative digits is due to Cauchy, whose title "*Sur les moyens d'éviter les erreurs dans les calculs numériques*" is significant. Cauchy proposes explicit examples of additions and multiplications of natural numbers using digits i with $-5 \le i \le 5$. He also explains verbally how one proceeds to perform the conversion between both representations, using what was not yet called a transducer at that time.

The interest is double. Reduce considerably the size of the multiplication tables; decrease dramatically the carries propagation."*Les nombres étant exprimés, comme on vient de le dire, par des chiffres dont la valeur numérique* *ne surpasse pas 5, les additions, soustractions, mlultiplications, divisions, les conversions de fractions ordinaires en fractions décimales, et les autres opérations de l'arithmétique, se trouveront notablement simplifiées. Ainsi, en particulier, la table de multiplication pourra être réduite au quart de son étendue, et l'on n'aura plus à effectuer de multiplications partielles que par les seuls chiffres 2, 3,* $4 = 2 \times 2$, *et* $5 = 10/2$. *Ainsi, pour être en état de multiplier l'un par l'utre deux nombres quelconques, il suffira de savoir doubler ou tripler un nombre, ou en prendre la moitié.* $[\cdots]$ *Observons en outre que, dans les additions, multiplications, élévations aux puissances, etc., les reports faits d'une colonne à l'autre seront généralement très faibles, et souvent nuls, attendu que les chiffres positifs et négatifs se détruiront mutuellement en grande partie, dans une colonne verticale composée de plusieurs chiffres.*"

Nowadays, signed representations have two interests. The first one is still algorithmic - as for Cauchy, the title of the book where Knuth mentions them is significant (see [128]). The second interest lies in the associated dynamical systems.

The representation considered by Cauchy is redundant - for example, $5 = 1\bar{5}$, where $\bar{n} = -n$. In the sequel, we restric ourselves to base 2 with digits $\{\bar{1}, 0, 1\}$. Reitwiesner proved that any integer $n \in \mathbb{Z}$ can be uniquely written as a finite sum $\sum_{0 \le i \le \ell} a_i 2^i$ with $a_i \in \{-1, 0, 1\}$ and $a_i \times a_{i+1} = 0$. This yields the compactification

$$X_{\mathcal{N}} = \{x_0 x_1 x_2 \ldots \in \{-1, 0, 1\} \; ; \; \forall i \in \mathbb{N} : x_i x_{i+1} = 0\}.$$

This numeration system is a finite fibred numeration system. The elements of this FNS are $X = \mathbb{Z}$, with partition $X_0 = 2\mathbb{Z}$, $X_{-1} = -1 + 4\mathbb{Z}$ and $X_1 = 1 + 4\mathbb{Z}$, and transformation $T(n) = (n - \varepsilon(n))/2$. Two natural transformations act on $X_{\mathcal{N}}$, the shift S and the addition of 1 imported from $\mathbb{Z}$ by (3.6), that we note τ. Then $(X_{\mathcal{N}}, S)$ is a topological mixing Markov chain, whose Parry measure is the Markov probability measure with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}.$$

and initial distribution $(1/6, 2/3, 1/6)$. Furthermore, $(X_{\mathcal{N}}, S)$ is conjugated to the dynamical system $([-2/3, 2/3], u)$ by

$$\Psi(x_0 x_1 x_2 \ldots) = \sum_{k=0}^{\infty} x_k 2^{-k-1},$$

where $u(x) = 2x \mod 1$. A realisation of the natural extension is given by $(\overline{X}, \overline{S})$ with

$$\overline{X} = \big([-2/3, -1/3) \times [-1/3, 1/3]\big) \cup \\ \cup \big([-1/3, 1/3) \times [-2/3, 2/3]\big) \cup \big([1/3, 2/3) \times [-1/3, 1/3]\big)$$

and $\overline{S}(x, y) = \big(2x - a(x), (a(x) + y)/2\big)$, where $a(x) = -1$ if $-2/3 \le x < -1/3$, $a(x) = 0$ if $-1/3 \le x < 1/3$ and $a(x) = 0$ if $1/3 \le x \le 2/3$.

The odometer $(X_{\mathcal{N}}, \tau)$ is topologically conjugate to the usual dyadic odometer $(\mathbb{Z}_2, x \mapsto x + 1)$. The study of this FNS and related arithmetical functions is due to Dajani, Kraaikamp and Liardet [64].

Example 6. Zeckendorf and Ostrowski representation

Let $(F_n)_n$ be the (shifted) Fibonacci sequence $F_0 = 1$, $F_1 = 2$ and $F_{n+2} = F_{n+1} + F_n$. Then any natural integer can be represented as a sum $n = \sum_j \varepsilon_j(n) F_j$. That representation is unique if one assumes that $\varepsilon_j(n) \in \{0, 1\}$ and $\varepsilon_j(n)\varepsilon_{j+1}(n) = 0$. It is then called *Zeckendorf expansion*. If $\rho = (1 + \sqrt{5})/2$ is the golden mean, the map f given by $f(n) = \sum_{j \ge 0} \varepsilon_j(n) \rho^{-j-1}$ embedds $\mathbb{N}$ into $[0, 1]$, the righthand side of the latter equation being the greedy β-expansion of its sum (for $\beta = \rho$, see Example 2). The representation of the real number $f(n)$ is then given by a FNS. But not the natural number n itself. Indeed, one needs the Zeckendorf representation of n to be able to compute the real number $f(n)$. One obtains it by the greedy algorithm. The compactification is the set of $0 - 1$ sequences without consecutive 1's. The addition cannot be extended to this compact space as $x + y = \lim(x_n + y_n)$ for integer sequences $(x_n)_n$ and $(y_n)_n$ tending to x and y respectively, but the addition of 1 can (see Section 6).

The Ostrowski representation of the integers is a generalisation of the latter. Suppose $0 < \alpha < 1/2$, $\alpha \notin \mathbb{Q}$. Let $\alpha = [a_1, a_2, \ldots, a_n, \ldots]$ its continued fraction expansion with convergents $p_n/q_n = [a_1, a_2, \ldots, a_n]$. Then every non-negative integer n has a representation $n = \sum_{j \ge 0} \varepsilon_j(n) q_n$, which becomes unique under the condition

$$\begin{cases} e_0(m) \le a_1 - 1; \\ \forall\, j \ge 1,\ e_j(m) \le a_{j+1}; \\ \forall\, j \ge 1,\ (e_j(m) = a_{j+1} \Rightarrow e_{j-1}(m) = 0\,). \end{cases} \tag{3.11}$$

The set above describes exactly the representations. Although this numeration system is not fibred, Definition 3.6 gives here

$$\begin{aligned} X_{\mathcal{N}} &= \{(x_n)_{n \ge 0} \in \mathbb{N}^{\mathbb{N}}\ ;\ \forall j \ge 0 : x_0 q_0 + \cdots + x_j q_j < q_{j+1}\} \\ &= \{(x_n)_{n \ge 0} \in \mathbb{N}^{\mathbb{N}}; x_0 \le a_1 - 1 \text{ and} \\ &\qquad\qquad \forall j \ge 1 :\ x_j \le a_{j+1}\ \&\ [x_j = a_{j+1} \Rightarrow x_{j-1} = 0]\}. \end{aligned}$$

On $X_{\mathcal{N}}$, the addition of the unity $\tau\colon x \mapsto x+1$ can be realised continuously by extending the addition of 1 for the integers. The map

$$f(n) := \sum_{j=0}^{\infty} \varepsilon_j(n)(q_j\alpha - p_j). \tag{3.12}$$

associates a real number $f(n) \in [-\alpha, 1-\alpha[$ to n. If $\alpha = [0; 2, 1, 1, 1, \ldots] = \rho^{-2} = (3-\sqrt{5})/2$, then the sequence of denominators $(q_n)_n$ of the convergents is exactly the Fibonacci sequence and the map of (3.12) coincides with the map given above in the discussion on Zeckendorf expansion up to a multiplicative constant.

In general, this map extends by continuity to $X_{\mathcal{N}}$ and realises an almost topological isomorphism in the sense of Denker & Keane [68] between the odometer $(X_{\mathcal{N}}, \tau)$ and $([1-\alpha, \alpha], R_\alpha)$, where R_α denotes the rotation with angle α. Explicitly, we have a commutative diagram

$$\begin{array}{ccc} X_{\mathcal{N}} & \xrightarrow{\tau} & X_{\mathcal{N}} \\ \rho\big\downarrow & & \big\downarrow\rho \\ [-\alpha, 1-\alpha] & \xrightarrow[R_\alpha]{} & [-\alpha, 1-\alpha], \end{array} \tag{3.13}$$

where ρ induces an homeomorphism between $X_{\mathcal{N}} \setminus \mathcal{O}_{\mathbb{Z}}(0^\omega)$ and $[-\alpha, 1-\alpha] \setminus \alpha\mathbb{Z} \pmod 1$, that is the spaces without the (countable) two-sided orbit of 0. In particular, the odometer $(X_{\mathcal{N}}, \tau)$ is strictly ergodic (uniquely ergodic and minimal).

4. Canonical numeration systems, β-expansions and shift radix systems

4.1. Canonical numeration systems in number fields. This subsection is mainly devoted to numeration systems located in a residue class ring

$$X := A[x]/p(x)A[x]$$

where $p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1x + p_0 \in A[x]$ is a polynomial over the commutative ring A. By reduction modulo p we see that each element $q \in X$ has a representative of the shape

$$q(x) = q_0 + q_1x + \cdots + q_{d-1}x^{d-1} \qquad (q_j \in A)$$

where d is the degree of the polynomial $p(x)$. In order to define a fibred numeration system on X, we consider the mapping

$$\begin{array}{rccc} T: & X & \to & X, \\ & q & \mapsto & \frac{q-\varepsilon(q)}{x} \end{array}$$

where the *digit* $\varepsilon(q) \in X$ is defined in a way that

$$T(q) \in X. \tag{4.1}$$

Note that this requirement generally leaves some freedom for the definition of ε. In the cases considered in this subsection, the image I of ε will always be finite. Moreover, the representation map $\varphi = (\varepsilon(T^n x))_{n\geq 0}$ defined in Subsection 3.1 will be surjective, i.e. all elements of $I^{\mathbb{N}}$ are admissible.

If we iterate T for ℓ times starting with an element $q \in X$ we obtain the representative

$$q(x) = \varepsilon(q) + \varepsilon(Tq)x + \cdots + \varepsilon(T^{\ell-1}q)x^{\ell-1} + T^{\ell}(q)x^{\ell}. \tag{4.2}$$

According to Definition 3.2 the triple $\mathcal{N} := (X, T, \varphi)$ is an FNS. Moreover, following Definition 3.3 we call $\mathcal{N}$ an FFNS if for each $q \in X$ there exists an $\ell \in \mathbb{N}$ such that $T^k(q) = 0$ for each $k \geq \ell$.

Once we have fixed the ring A, the definition of $\mathcal{N}$ only depends on p and ε. Moreover, in what follows, the image I of ε will always be chosen to be a complete set of coset representatives of A/p_0A. With this choice the requirement (4.1) determines the value of $\varepsilon(q)$ uniquely for each $q \in X$. In other words, in this case $\mathcal{N}$ is determined by the pair (p, I). Motivated by the shape of the representation (4.2) we will call p the *base* of the numeration system (p, I) and I its *set of digits*.

The pair (p, I) defined in this way still provides a fairly general notion of numeration system. By further specialization we will obtain from it the notion of *canonical numeration systems* as well as a notion of digit systems over finite fields that will be discussed in Subsection 4.5.

Historically, the term *canonical numeration system* is due to Kovács [133]. He used it for numeration systems defined in the ring of integers of an algebraic number field.[5] Meanwhile, Pethő [170] generalized this notion to numeration systems in certain polynomial rings and it is this notion of numeration system to which we will attach the name *canonical numeration system* in the present survey.

Before we precisely define Kovács' as well as Pethő's notion of numeration system and link it to the general numeration systems in residue classes of polynomial rings, we discuss some earlier papers on the subject.

In fact, instances of numeration systems in rings of integers have been studied long before Kovács' paper. The first paper on these objects seems to be Grünwald's treatise [97] dating back to 1885 which is devoted to numeration systems with negative bases. In particular Grünwald showed the following result.

Theorem 4.1. *Each $n \in \mathbb{Z}$ admits a unique finite representation w.r.t. the base number $-q$, i.e.,*

$$n = c_0 + c_1(-q) + \cdots + c_\ell(-q)^\ell$$

where $0 \leq c_i < q$ for $i \in \{0, \ldots, \ell\}$ and $c_\ell \neq 0$ for $\ell \neq 0$ if and only if $q \geq 2$.

We can say that Theorem 4.1 describes the bases of number systems in the ring of integers $\mathbb{Z}$ of the number field $\mathbb{Q}$. It is natural to ask whether

[5] With the word "canonical" Kovács wanted to emphasize the fact that the digits he attached to these numeration systems were chosen in a very simple "canonical" way.

this concept can be generalized to other number fields. Knuth [126] and Penney [169] observed that $b = -1 + \sqrt{-1}$ serves as a base for a numeration system with digits $\{0, 1\}$ in the ring of integers $\mathbb{Z}[\sqrt{-1}]$ of the field of Gaussian numbers $\mathbb{Q}(\sqrt{-1})$, *i.e.*, each $z \in \mathbb{Z}[\sqrt{-1}]$ admits a unique representation of the shape

$$z = c_0 + c_1 b + \cdots + c_\ell b^\ell$$

with digits $c_i \in \{0, 1\}$ and $c_\ell \neq 0$ for $\ell \neq 0$. Knuth [128] also observed that this numeration system is strongly related to the famous twin-dragon fractal which will be discussed in Subsection 4.6. It is not hard to see that Grünwald's as well as Knuth's examples are special cases of FFNS.

We carry out the details of this correspondence for a more general definition of numeration systems in the ring of integers Z_K of a number field K. In particular, we claim that the pair $(b, \mathcal{D})$ with $b \in Z_K$ and $\mathcal{D} = \{0, 1, \ldots, |N(b)| - 1\}$ defines an FFNS in Z_K if each $z \in Z_K$ admits a unique representation of the shape

$$z = c_0 + c_1 b + \cdots + c_\ell b^\ell \qquad (c_i \in \mathbb{N}) \tag{4.3}$$

if $c_\ell \neq 0$ for $\ell \neq 0$ (note that this requirement just ensures that there occur no leading zeros in the representations). To see this set $X = Z_K$ and define $T : Z_K \to Z_K$ by

$$T(z) = \frac{z - \varepsilon(z)}{b}$$

where $\varepsilon(z)$ is the unique element of $\mathcal{D}$ with $T(z) \in Z_K$. Note that $\mathcal{D}$ is uniquely determined by b. The first systematic study of FFNS in rings of integers of number fields is due to Kátai and Szabó [124]. They proved that the only bases in $\mathbb{Z}[i]$ are the numbers $b = -n + \sqrt{-1}$ with $n \geq 1$. Later Kátai and Kóvacs [122, 123] (see also Gilbert [91]) characterised all (bases of) canonical numeration systems in quadratic number fields. A. Kovács, B. Kovács, Pethő and Scheicher [133, 134, 136, 183, 131] studied numeration systems in rings of integers of algebraic number fields of higher degree and proved some partial characterisation results (some further generalised concepts of numeration systems can be found in [137, 135]). In [138] an estimate for the length ℓ of the CNS representation (4.3) of z w.r.t. base b in terms of the modulus of the conjugates of z as well as b is given.

Pethő [170] observed that the notion of numeration systems in number fields can easily be extended using residue class rings of polynomials. In particular, he gave the following definition.

Definition 4.1. *Let*

$$p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0 \in \mathbb{Z}[x], \quad \mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$$

and $X := \mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ *and denote the image of* x *under the canonical epimorphism from* $\mathbb{Z}[x]$ *to* X *again by* x. *If every non-zero element* $q(x) \in X$

can be written uniquely in the form

$$q(x) = c_0 + c_1 x + \cdots + c_\ell x^\ell \tag{4.4}$$

with $c_0, \ldots, c_\ell \in \mathcal{D}$*, and* $c_\ell \neq 0$ *if* $\ell \neq 0$*, we call* $(p, \mathcal{D})$ *a canonical number system (CNS for short).*

Let p be irreducible and assume that b is a root of p. Let $K := \mathbb{Q}(b)$ and assume further that $Z_K = \mathbb{Z}[b]$, *i.e.*, Z_K is monogenic. Then $\mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ is isomorphic to Z_K and this definition is easily seen to agree with the above definition of numeration systems in rings of integers of number fields.

On the other hand, canonical numeration systems turn out to be a special case of the more general definition given at the beginning of this section. To see this we choose the commutative ring A occurring there to be $\mathbb{Z}$. The value of $\varepsilon(q)$ is defined to be the least non-negative integer satisfying the requirement that

$$T(q) = \frac{q - \varepsilon(q)}{x} \in X.$$

Note that this definition implies that $\varepsilon(X) = \mathcal{D}$, as required. If

$$q(x) = q_0 + q_1 x + \cdots + q_{d-1} x^{d-1} \qquad (q_j \in \mathbb{Z})$$

is a representative of q then T takes the form

$$T(q) = \sum_{i=0}^{d-1} (q_{i+1} - c p_{i+1}) X^i, \tag{4.5}$$

where $q_d = 0$ and $c = \lfloor q_0/p_0 \rfloor$. Then

$$q(x) = (q_0 - c p_0) + x T(q), \text{ where } q_0 - c p_0 \in \mathcal{D}.$$

Thus the iteration of T yields exactly the representation (4.4) given above. The iteration process of T can become divergent (e.g. $q(x) = -1$ for $p(x) = x^2 + 4x + 2$), ultimately periodic (e.g. $q(x) = -1$ for $p(x) = x^2 - 2x + 2$) or can terminate at 0 (e.g. $q(x) = -1$ for $p(x) = x^2 + 2x + 2$). For the reader's convenience we will give the details for the last constellation.

Example 7. Let $p(x) = x^2 + 2x + 2$ be a polynomial. We want to calculate the representation of $q(x) = -1 \in \mathbb{Z}[x]/p(x)\mathbb{Z}(x)$. To this matter we need to iterate the mapping T defined in (4.5). Setting $d_j := \varepsilon(T^j(q))$ this yields

$$\begin{array}{rclrlll}
q & = & -1, & & & c=-1, & \\
T(q) & = & (0-(-1)\cdot 2)+(0-(-1)\cdot 1)x & = & 2+x, & c=1, & d_0=1, \\
T^2(q) & = & (1-1\cdot 2)+(0-1\cdot 1)x & = & -1-x, & c=-1, & d_1=0, \\
T^3(q) & = & (-1-(-1)\cdot 2)+(0-(-1)\cdot 1)x & = & 1+x, & c=0, & d_2=1, \\
T^4(q) & = & (1-0)+(0-1\cdot 1)x & = & -1-x, & c=-1, & d_3=1, \\
T^5(q) & = & 0, & & & c=0, & d_4=1. \\
T^k(q) & = & 0 \quad \text{for } k \geq 6. & & & &
\end{array}$$

Thus

$$-1 = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 = 1 + x^2 + x^3 + x^4$$

is the unique finite representation (4.4) of -1 with respect to the base $p(x)$.

Note that $(p, \mathcal{D})$ is a canonical numeration system if and only if the attractor of T is $\{0\}$.

The fundamental problem which we want to addres is to exhibit all polynomials p that give rise to a CNS. There exist many partial results on this problem. Generalizing the above-mentioned results for quadratic number fields Brunotte [55] characterised all quadratic CNS polynomials. In particular, he obtained the following result.

Theorem 4.2. *The pair $(p(x), \mathcal{D})$ with $p(x) = x^2 + p_1x + p_0$ and set of digits $\mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$ is a CNS if and only if*

$$p_0 \geq 2 \qquad \textit{and} \qquad -1 \leq p_1 \leq p_0. \tag{4.6}$$

For CNS polynomials of general degree, Kovács [133] (see also the more general treatment in [11]) proved the following theorem.

Theorem 4.3. *The polynomial*

$$p(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_1x + p_0$$

gives rise to a CNS if its coefficients satisfy the "monotonicity condition"

$$p_0 \geq 2 \quad \textit{and} \quad p_0 \geq p_1 \geq \cdots \geq p_{d-1} > 0. \tag{4.7}$$

More recently, Akiyama and Pethő [12], Scheicher and Thuswaldner [186] as well as Akiyama and Rao [13] showed characterisation results under the condition

$$p_0 > |p_1| + \cdots + |p_{d-1}|.$$

Moreover, Brunotte [55, 56] has results on trinomials that give rise to CNS.

It is natural to ask whether there exists a complete description of all CNS polynomials. For the case $d = 3$ of cubic polynomials this characterisation problem was studied extensively. Some special results on cubic CNS are contained in Körmendi [129]. Brunotte [57] characterised cubic CNS polynomials with three real roots. Akiyama *et al.* [9] studied the problem of describing all cubic CNS systematically. Their results indicate that the structure of cubic CNS polynomials is very irregular.

Recently, Akiyama *et al.* [8] invented a new notion of numeration system, so called shift radix systems. All recent developments on the characterisation problem of CNS have been done in this new framework. Shift radix systems will be discussed in Subsection 4.4.

4.2. **Generalisations.** There are some quite immediate generalisations of canonical number systems. First we mention that there is no definitive reason to study only the set of digits $\mathcal{D} = \{0, 1, \ldots, |p_0| - 1\}$. More generally each set $\mathcal{D}$ containing one of each cosets of $\mathbb{Z}/p_0\mathbb{Z}$ can serve as digit sets. numeration systems of this more general kind can be studied in rings of integers of number fields as well as residue class rings of polynomials. For quadratic numeration systems Farkas, Kátai and Steidl [83, 119, 206] showed that for all but finitely many quadratic integers there exists a digit set such that each element of the corresponding number field has a finite representation. In particular, Steidl [206] proves the following result for numeration systems in Gaussian integers.

Theorem 4.4. *If $K = \mathbb{Q}(i)$ and b is an integer of Z_K satisfying $|\alpha| > 1$ with $\alpha \neq 2, 1 \pm i$ then one can effectively construct a residue system $\mathcal{D}$ (mod b) such that each $z \in Z_K$ admits a finite representation*

$$z = c_0 + c_1 b + \ldots + c_\ell b^\ell$$

with $c_0 \ldots, c_\ell \in \mathcal{D}$.

Another way of generalising canonical numeration systems runs *via* an embedding into an integer lattice. Let $(p(x), \mathcal{D})$ be a canonical numeration system. As mentioned above, each $q \in X := \mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ admits a unique representation of the shape

$$q_0 + q_1 x + \ldots + q_{d-1} x^{d-1}$$

with $q_0, \ldots, q_{d-1} \in \mathbb{Z}$ and $d = \deg(p)$. Thus the bijective group homomorphism

$$\begin{array}{rccc} \Phi: & X & \rightarrow & \mathbb{Z}^d \\ & q & \mapsto & (q_0, \ldots, q_{d-1}) \end{array}$$

is well defined. Besides being a homomorphism of the additive group in X, Φ satisfies

$$\Phi(xq) = B\Phi(x)$$

with

$$B := \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & & \vdots & -p_1 \\ 0 & \ddots & \ddots & & \vdots & -p_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -p_{d-1} \end{pmatrix}. \tag{4.8}$$

Exploiting the properties of Φ we easily see the following equivalence. Each $q \in X$ admits a CNS representation of the shape

$$q = c_0 + c_1 x + c_2 x^2 + \ldots + c_\ell x^\ell \qquad (c_0, \ldots, c_\ell \in \mathcal{D})$$

if and only if each $z \in \mathbb{Z}^d$ admits a representation of the form

$$z = d_0 + Bd_1 + B^2 d_2 + \cdots + B^\ell d_\ell \qquad (d_0, \ldots, d_\ell \in \Phi(\mathcal{D})).$$

Thus $(B, \Phi(\mathcal{D}))$ is a special case of the following notion of numeration system.

Definition 4.2. *Let $B \in \mathbb{Z}^{d\times d}$ be an expanding matrix (i.e., all eigenvalues of B are greater than 1 in modulus). Let $\mathcal{D} \subset \mathbb{Z}^d$ be a complete set of cosets in $\mathbb{Z}^d/B\mathbb{Z}^d$ such that $0 \in \mathcal{D}$. Then the pair $(B, \mathcal{D})$ is called a matrix numeration system if each $z \in \mathbb{Z}^d$ admits a unique representation of the shape*

$$z = d_0 + Bd_1 + \cdots + B^\ell d_\ell$$

with $d_0, \ldots, d_\ell \in \mathcal{D}$ and $d_\ell \neq 0$ for $\ell \neq 0$.

Matrix numeration systems have been studied for instance By Kátai, Kovács and Thuswaldner in [120, 130, 132, 212]. Apart from some special classes it is quite hard to obtain characterisation results because the number of parameters to be taken into account (namely the entries of B and the elements of the set $\mathcal{D}$) is very large. However, matrix number systems will be our starting point for the definition of lattice tilings in Subsection 4.6.

4.3. **On the finiteness property of β-expansions.** β-expansions have already been defined in the introduction. They admit the representation of elements of $[0, \infty)$ with respect to a real base number β and with a finite set of positive integer digits. It is natural to ask when these representations are finite. Let $\mathrm{Fin}(\beta)$ be the set of all $x \in [0, \infty)$ having a finite β-expansion. Since finite sums of the shape

$$\sum_{j=m}^{n} c_j \beta^{-j} \qquad (c_j \in \mathbb{N})$$

are always contained in $\mathbb{Z}[\beta^{-1}] \cap [0, \infty)$ we always have

$$\mathrm{Fin}(\beta) \subseteq \mathbb{Z}[\beta^{-1}] \cap [0, \infty). \tag{4.9}$$

Following Frougny and Solomyak [89] we say that a number β satisfies *property (F)* if equality holds in (4.9). Using the terminology of the introduction property (F) is equivalent to the fact that (X, T) with

$$X = \mathbb{Z}[\beta^{-1}] \cap [0, \infty) \quad \text{and} \quad T(x) = \beta x - \lfloor \beta x \rfloor$$

is an FFNS (see Definition 3.3).

In [89, Lemma 1] it was shown that (F) can hold only if β is a Pisot number. However, there exist Pisot numbers that do not fulfill (F). This raises the problem of exhibiting all Pisot numbers having this property. Up to now no complete characterisation of all Pisot numbers satisfying (F) is in existence. In what follows we would like to present some partial results that have been achieved. In [89, Proposition 1] it is proved that each quadratic Pisot number has property (F). Akiyama [6] could characterise (F) for all cubic Pisot units. In particular, he obtained the following result.

Theorem 4.5. *Let $x^3 - a_1x^2 - a_2x - 1$ be the minimal polynomial of a cubic Pisot unit β. Then β satisfies (F) if and only if*

$$a_1 \geq 0 \qquad \textit{and} \qquad -1 \leq a_2 \leq a_1 + 1. \tag{4.10}$$

If β is an arbitrary Pisot number the complete characterisation result is still unknown. Recent results using the notion of shift radix system suggest that even the characterisation of the cubic case is very involved (cf. [8, 11]). For details on this approach we refer to the next subsection. Here we just want to give some partial characterisation results for Pisot numbers of arbitrary degree. The following result is contained in [89, Theorem 2].

Theorem 4.6. *Let*

$$x^d - a_1 x^{d-1} - \cdots - a_{d-1}x - a_d \tag{4.11}$$

be the minimal polynomial of a Pisot number β. If the coefficients of (4.11) *satisfy the "monotonicity condition"*

$$a_1 \geq \ldots \geq a_d \geq 1 \tag{4.12}$$

then β fulfills property (F).

Moreover, Hollander [103] proved the following result on property (F) under a condition on the representation $d(1,\beta)$ of 1.

Theorem 4.7 ([103, Theorem 3.4.2])**.** *A Pisot number β has property (F) if $d(1,\beta) = .d_1 \ldots d_l$ with $d_1 > d_2 + \cdots + d_l$.*

In the next subsection the most important concepts introduced in this section, namely CNS and β-expansions, will be unified.

4.4. **Shift radix systems.** Looking at the definition, canonical numeration systems in polynomial rings over $\mathbb{Z}$ and β-expansions are quite different objects. However, despite they differ concerning the space of representations as well in the independency resp. dependency of their digits the characterisation results for their respective finiteness properties resemble each other. As example we mention (4.6) and (4.7) on the one hand and (4.10) and (4.12) on the other.

The notion of shift radix system which is discussed in the present subsection will shed some light on this resemblance. Indeed, it turns out that canonical numeration systems in polynomial rings over $\mathbb{Z}$ as well as β-expansions are special instances of a class of very simple dynamical systems. The most recent studies of canonical numeration systems as well as β-expansions make use of this more general concept which allows to gain results on canonical numeration systems as well as β-expansions at once. We start with a definition of shift radix systems (cf. Akiyama *et al.* [8, 11]).

Definition 4.3. *Let $d \geq 1$ be an integer, $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$ and define the mapping $\tau_{\mathbf{r}}$ by*

$$\begin{array}{rrcl} \tau_{\mathbf{r}}: & \mathbb{Z}^d & \to & \mathbb{Z}^d \\ & \mathbf{a} = (a_1, \ldots, a_d) & \mapsto & (a_2, \ldots, a_d, -\lfloor \mathbf{r}\mathbf{a} \rfloor), \end{array}$$

where $\mathbf{r}\mathbf{a} = r_1 a_1 + \cdots + r_d a_d$, i.e., the inner product of the vectors $\mathbf{r}$ and $\mathbf{a}$. Let $\mathbf{r}$ be fixed. If

$$\text{for all } \mathbf{a} \in \mathbb{Z}^d \text{ there exists } k > 0 \text{ with } \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \tag{4.13}$$

we will call $\tau_{\mathbf{r}}$ *a shift radix system (SRS for short). For simplicity, we write* $0 = (0, \ldots, 0)$.

Let

$$\mathcal{D}_d^0 := \left\{ \mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\}$$

be the set of all SRS parameters in dimension d and set

$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\}.$$

It is easy to see that $\mathcal{D}_d^0 \subseteq \mathcal{D}_d$.

In [8] (cf. also Hollander [103]) it was observed that SRS correspond to CNS and β-expansions in the following way.

Theorem 4.8. *The following correspondences hold between CNS as well as* β*-expansions and SRS.*

- *Let* $p(x) := x^d + p_{d-1}x^{d-1} + \cdots + p_1 x + p_0 \in \mathbb{Z}[x]$. *Then* $p(x)$ *gives rise to a CNS if and only if*

$$\mathbf{r} := \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0} \right) \in \mathcal{D}_d^0. \tag{4.14}$$

- *Let* $\beta > 1$ *be an algebraic integer with minimal polynomial* $X^d - a_1 X^{d-1} - \cdots - a_{d-1}X - a_d$. *Define* $r_1, \ldots, r_d$ *by*

$$\begin{array}{rcl} r_1 & := & 1, \\ r_j & := & a_j \beta^{-1} + a_{j+1}\beta^{-2} + \cdots + a_d \beta^{j-d-1} \quad (2 \leq j \leq d). \end{array} \tag{4.15}$$

Then β *has property (F) if and only if* $(r_d, \ldots, r_2) \in \mathcal{D}_{d-1}^0$.

In particular $\tau_{\mathbf{r}}$ *is conjugate to the mapping* T *defined in* (4.5) *if* $\mathbf{r}$ *is chosen as in* (4.14) *and conjugate to the* β*-transformation* $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ *for* $\mathbf{r}$ *as in* (4.15).

This theorem draws our attention to the problem of describing the set $\mathcal{D}_d^0$. Describing this set would solve the problem of understanding the finiteness properties of CNS as well as β-expansions. We start with some considerations on the set $\mathcal{D}_d$. It is not hard to see (cf. [8, Section 4]) that

$$\mathcal{E}_d \subseteq \mathcal{D}_d \subseteq \overline{\mathcal{E}_d} \tag{4.16}$$

where

$$\begin{array}{rcl} \mathcal{E}_d & := & \big\{ (r_1, \ldots, r_d) \in \mathbb{R}^d \mid x^d + r_d x^{d-1} + \cdots + r_1 \\ & & \text{has only roots } y \in \mathbb{C} \text{ with } |y| < 1 \big\} \end{array}$$

denotes the Schur-Cohn region (see Schur [191]). The only problem in describing $\mathcal{D}_d$ consists in characterising its boundary. This problem turns out to be very hard and contains as a special case the following conjecture of Schmidt [188, p. 274].

Conjecture 4.1. Let β be a Salem number and $x \in \mathbb{Q}(\beta) \cap [0, 1)$. Then the orbit $(T_\beta^k(x))_{k \geq 0}$ of x under the β-transformation T_β is eventually periodic.

Up to now Boyd [51, 52, 53] could verify some special instances of this conjecture (see also [10] where the problem of characterising $\partial\mathcal{D}_d$ is addressed). As quoted in [50], there exist Parry numbers which are neither Pisot nor even Salem; consider e.g., $\beta^4 = 3\beta^3 + 2\beta^2 + 3$ with $d_\beta(1) = 3203$; a *Salem number* is a Perron number, all conjugates of which have absolute value less than or equal to 1, and at least one has modulus 1. It is conjectured in [188] that every Salem number is a Parry number. This conjecture is sustained by the fact that if each rational in $[0, 1)$ has a ultimately periodic β-expansion, then β is either a Pisot or a Salem number. In particular, it is proved in [51] that if β is a Salem number of degree 4, then β is a Parry number; see [52] for the case of Salem numbers of degree 6. Note that the algebraic conjugates of a Parry number $\beta > 1$ are smaller than $\frac{1+\sqrt{5}}{2}$ in modulus, this upper bound being sharp [85, 204].

We would like to characterise $\mathcal{D}_d^0$ by starting from $\mathcal{D}_d$. This could be achieved by removing all parameters $\mathbf{r}$ from $\mathcal{D}_d$ for which the mapping $\tau_{\mathbf{r}}$ admits nontrivial periods. We would like to do this "period wise". Let

$$\mathbf{a}_j := (a_{1+j}, \dots, a_{d+j}) \qquad (0 \le j \le L-1) \tag{4.17}$$

with $a_{L+1} = a_1, \dots, a_{L+d} = a_d$ be L vectors of $\mathbb{Z}^d$. We want to describe the set of all parameters $\mathbf{r} = (r_1, \dots, r_d)$ that admit the period $\pi = (\mathbf{a}_0, \dots, \mathbf{a}_{L-1})$, *i.e.,* the set of all $\mathbf{r} \in \mathcal{D}_d$ with

$$\tau_{\mathbf{r}}(\mathbf{a}_0) = \mathbf{a}_1,\ \tau_{\mathbf{r}}(\mathbf{a}_1) = \mathbf{a}_2,\ \dots, \tau_{\mathbf{r}}(\mathbf{a}_{L-2}) = \mathbf{a}_{L-1},\ \tau_{\mathbf{r}}(\mathbf{a}_{L-1}) = \mathbf{a}_0.$$

According to the definition of $\tau_{\mathbf{r}}$ this is the set given by

$$0 \le r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \qquad (0 \le j \le L-1).$$

We call this set $\mathcal{P}(\pi)$. Since $\mathcal{P}(\pi)$ is a (possibly degenerate or even empty) convex polyhedron we call it the *cutout polyhedron* of π. Because 0 is the only permitted period for element of $\mathcal{D}_d^0$ we obtain $\mathcal{D}_d^0$ from $\mathcal{D}_d$ by cutting out all polyhedra $\mathcal{P}(\pi)$ corresponding to non-zero periods, *i.e.*,

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \ne 0} \mathcal{P}(\pi). \tag{4.18}$$

Thus describing $\mathcal{D}_d^0$ is tantamount to describing the cutout polyhedra coming from non-zero periods. It can be easily seen from the definition that

$$\tau_{\mathbf{r}}(\mathbf{x}) = R(\mathbf{r})\mathbf{x} + \mathbf{v}.$$

Here $R(\mathbf{r})$ is a $d \times d$ matrix whose characteristic polynomial is $x^d + r_d x^{d-1} + \cdots + r_1$. The vector $\mathbf{v}$ is an "error term" coming from the floor function occurring in the definition of $\tau_{\mathbf{r}}$ and always fulfills $|\mathbf{v}| < 1$. The further away from the boundary of $\mathcal{D}_d$ the parameter $\mathbf{r}$ is chosen, the smaller are the eigenvalues of $R(\mathbf{r})$. Since for each $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ the mapping $\tau_{\mathbf{r}}$ is contracting apart from the error term $\mathbf{v}$ one can easily prove that the norms of the elements $\mathbf{a}_0, \dots, \mathbf{a}_{L-1}$ forming a period $\pi = (\mathbf{a}_0, \dots, \mathbf{a}_{L-1})$ of $\tau_{\mathbf{r}}$ can become large only if the parameter $\mathbf{r}$ is chosen near the boundary. Therefore the

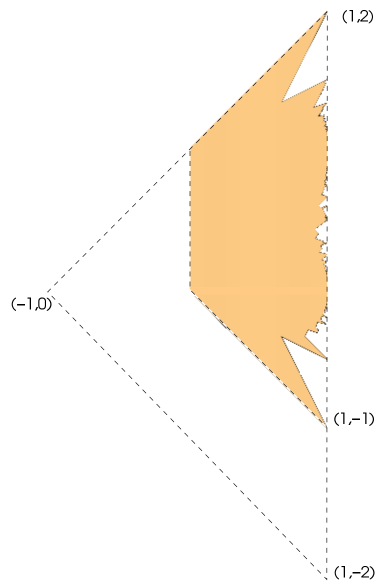

FIGURE 4.1. An approximation of $\mathcal{D}_2^0$

number of periods corresponding to a given $\tau_{\mathbf{r}}$ with $\mathbf{r} \in \operatorname{int}(\mathcal{D}_d)$ is bounded. The bound depends on the largest eigenvalue of $R(\mathbf{r})$.

This fact was used in order to derive the following algorithm which allows to describe $\mathcal{D}_d^0$ in whole regions provided that they are at some distance away from $\partial\mathcal{D}_d$. In particular, in [8] the following result was proved.

Theorem 4.9. *Let $\mathbf{r}_1, \ldots, \mathbf{r}_k \in \mathcal{D}_d$ and denote by H the convex hull of $\mathbf{r}_1, \ldots, \mathbf{r}_k$. We assume that $H \subset \operatorname{int}(\mathcal{D}_d)$ and that H is sufficiently small in diameter. For $\mathbf{z} \in \mathbb{Z}^d$ take $M(\mathbf{z}) = \max_{1\le i\le k}\{-\lfloor \mathbf{r}_i\mathbf{z}\rfloor\}$. Then there exists an algorithm to create a finite directed graph (V, E) with vertices $V \subset \mathbb{Z}^d$ and edges $E \in V \times V$ which satisfy*

(1) *each d-dimensional standard unit vector $(0, \ldots, 0, \pm 1, 0, \ldots, 0) \in V$,*
(2) *for each $\mathbf{z} = (z_1, \ldots, z_d) \in V$ and*
$$j \in [-M(-\mathbf{z}), M(\mathbf{z})] \cap \mathbb{Z}$$
we have $(z_2, \ldots, z_d, j) \in V$ and a directed edge $(z_1, \ldots, z_d) \to (z_2, \ldots, z_d, j)$ in E.
(3) *$H \cap \mathcal{D}_d^0 = H \setminus \bigcup_\pi P(\pi)$, where the union is taken over all nonzero primitive cycles of (V, E).*

This result has been heavily used in [11] in order to describe large parts of $\mathcal{D}_2^0$. Since it is fairly easy to show that $\mathcal{D}_2^0 \cap \partial\mathcal{D}_d = \emptyset$ the difficulties related to the boundary of $\mathcal{D}_2$ do not cause problems. However, it turned out that $\mathcal{D}_2^0$ has a very complicated structure. To get an impression of this structure we refer the reader to Figure 4.1.

The big isoceles triangle is $\mathcal{E}_2$ and thus by (4.16) apart from its boundary equal to $\mathcal{D}_2$. The gray figure is an approximation of $\mathcal{D}_2^0$ which has been constructed by using (4.18) and Theorem 4.9. It is easy to see that the periods $(1,1)$ and $(1,0),(0,1)$ correspond to cutout polygons cutting away from $\mathcal{D}_2^0$ the area to the left and below the approximation. Since Theorem 4.9 can be used to treat regions far enough away from $\partial\mathcal{D}_2$ it remains to describe $\mathcal{D}_2^0$ near to the upper and right boundary of $\partial\mathcal{D}_2$.

Large parts of the region near to the upper boundary could be treated in [11, Section 4] showing that this region indeed belongs to $\mathcal{D}_2^0$. Near the right boundary of $\mathcal{D}_2$, however, the structure of $\mathcal{D}_2^0$ is much more complicated.

For instance, in [8] it has been proved that infinitely many different cutouts are needed in order to describe $\mathcal{D}_2^0$. Moreover, the period lengths of $\tau_{\mathbf{r}}$ are not uniformly bounded. The shape of some infinite families of cutouts as well as some new results on $\mathcal{D}_2^0$ can be found in Surer [209]. In view of Theorem 4.8 this difficult structure of $\mathcal{D}_2^0$ implies that in cubic β-expansions of elements of $\mathbf{Z}[\beta^{-1}] \cap [0,\infty)$ there may occur periods of arbitrarily large length.

SRS exist for parameters varying in a continuum. In [11, Section 4] this fact was used in order to exploit a certain structural stability occurring in the orbits of $\tau_{\mathbf{r}}$ when varying $\mathbf{r}$ continuously near to the point $(1,-1)$. This lead to a description of $\mathcal{D}_2^0$ in a big area.

Theorem 4.10. *We have*

$$\{(r_1,r_2) \mid r_1 > 0, -r_1 \le r_2 < 1 - 2r_1\} \subset \mathcal{D}_2^0.$$

In view of Theorem 4.8 this yields a large class of Pisot numbers β satisfying property (F).

The description of $\mathcal{D}_2^0$ itself is not so interesting for the characterisation of CNS since quadratic CNS are already well understood (see Theorem 4.2). The set $\mathcal{D}_3^0$ is not yet well studied. However, Scheicher and Thuswaldner [186] made some computer experiments to exhibit a counterexample to the following conjecture which (in a slightly different form) appears in [136]. It says that

$$p(x) \text{ CNS polynomial } \Longrightarrow p(x)+1 \text{ CNS polynomial.}$$

In particular, they found that this is not true for

$$p(x) = x^3 + 173x^2 + 257x + 198.$$

This counterexample was found by studying $\mathcal{D}_3^0$ near a degenerate cutout polyhedron that cuts out the parameter corresponding to $p(x)+1$ in view of Theorem 4.8, but not the parameter corresponding to $p(x)$. Since Theorem 4.9 can be used to prove that no other cutout polygon cuts out regions near this parameter the counterexample can be confirmed.

The characterisation of cubic CNS polynomials $p(x) = x^3+p_2x^2+p_1x+p_0$ with fixed, large p_0 is related to certain cuts of $\mathcal{D}_3^0$ which resemble $\mathcal{D}_2^0$ very strongly. In view of Theorem 4.8 this indicates that the characterisation of cubic CNS polynomials is also very difficult. In particular, according to the *Lifting theorem* ([8, Theorem 6.2]), each of the periods occurring for two dimensional SRS also occurs for cubic CNS polynomials. Thus CNS representations of elements of $\mathbb{Z}[x]/p(x)\mathbb{Z}[x]$ with respect to a cubic polynomial $p(x)$ can have infinitely many periods. Moreover, there is no bound for the period length (see [8, Section 7]). For the family of dynamical systems T in (4.5) this means that their attractors can be arbitrarily large if p varies over the cubic polynomials.

Recently, Akiyama and Scheicher [15, 14] study a variant of $\tau_{\mathbf{r}}$. In particular, they consider the family

$$\begin{array}{cccc} \tilde{\tau}_{\mathbf{r}}: & \mathbb{Z}^d & \to & \mathbb{Z}^d \\ & (a_1,\dots,a_d) & \mapsto & (a_2,\dots,a_d,-\lfloor \mathbf{r}\mathbf{a}+\frac{1}{2}\rfloor), \end{array}$$

of dynamical systems. In the same way as above they attach the sets $\tilde{\mathcal{D}}_d$ and $\tilde{\mathcal{D}}_d^0$ to it. However, interestingly, it turns out that the set $\tilde{\mathcal{D}}_2^0$ can be described completely in this modified setting. In particular, it can be shown that $\tilde{\mathcal{D}}_2^0$ is an open triangle together with some parts of its boundary. Judging from computer experiments it seems that even $\tilde{\mathcal{D}}_3^0$ is of a shape that gives some hope to achieve a complete characterisation results. As in the case of ordinary SRS this variant has relations to numeration systems. Namely, some modifications of CNS and β-expansions fit into this framework (see [15]).

4.5. **numeration systems defined over finite fields.** In this subsection we would like to present other notions of number system. The first one is defined in residue classes of polynomial rings as follows. Polynomial rings $\mathbb{F}[x]$ over finite fields share many properties with the ring $\mathbb{Z}$. Thus it is natural to ask for analogues of canonical numeration systems in finite fields. Kovács and Pethő [136] studied special cases of the following more general concept introduced by Scheicher and Thuswaldner [185].

Let $\mathbb{F}$ be a finite field and and $p(x,y)=\sum b_j(x)y^j \in \mathbb{F}[x,y]$ be a polynomial in two variables and let $\mathcal{D}=\{p\in\mathbb{F}[x]\,:\,\deg p(x)<\deg b_0(x)\}$. We call $(p(x,y),\mathcal{D})$ a *digit system* with base $p(x,y)$ if each element q of the quotient ring $X=\mathbb{F}[x,y]/p(x,y)\mathbb{F}[x,y]$ admits a representation of the shape

$$q=c_0(x)+c_1(x)y+\ldots+c_\ell(x)y^\ell$$

with $c_j(x)\in\mathcal{D}$ $(0\le j\le\ell)$.

Obviously, these numeration systems fit into the framework defined at the beginning of this subsection by setting $A=F[y]$ and defining $\varepsilon(q)$ to be the polynomial of least degree satisfying the requirement that $T(q)\in X$.

It turns out that the characterisation of the bases of these digit systems is quite easy. Indeed, in [185] the following result is proved.

Theorem 4.11. *$(p(x,y),\mathcal{D})$ is a digit systems if and only if* $\max_{i=1}^n \deg b_i < \deg b_0$.

β-expansions have also been extended to the case of finite fields independently by Scheicher [182] as well as Hbaib and Mkaouar [100]. Let $\mathbb{F}((x^{-1}))$ be the field of formal Laurent series over $\mathbb{F}$ and denote by $|\cdot|$ some absolute value. Choose $\beta \in \mathbb{F}((x^{-1}))$ with $|\beta| > 1$. Let $z \in \mathbb{F}((x^{-1}))$ with $|z| < 1$. A *β-representation* of z is an infinite sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{F}[x]$ with

$$z = \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

The most important β-representation (called β-expansion), is determined by the "greedy algorithm"

$$\begin{aligned} r_0 &:= z, \\ d_j &:= \lfloor \beta r_{j-1} \rfloor, \\ r_j &:= \beta r_{j-1} - d_j. \end{aligned}$$

Here $\lfloor \cdot \rfloor$ cuts off the negative powers of a formal Laurent series.

In [182] several problems related to β-expansions are studied. An analogue of property (F) of Frougny and Solomyak [89] is defined. Contrary to the classical case, all β satisfying this condition can be characterised. In [182, Section 5] it is shown that (F) is true if and only if β is a Pisot element of $\mathbb{F}((x^{-1}))$, i.e., if β is an algebraic integer over $\mathbb{F}[x]$ with $|\beta| > 1$ all whose Galois conjugates β_j satisfy $|\beta_j| < 1$ (see [43]).

Furthermore the analogue of the conjecture of Schmidt [188] concerning periodic β representations of Pisot and Salem numbers discussed above (see Conjecture 4.1) could be settled in the finite fields setting. In [100] the "representation of 1", which is defined in terms of an analogue of the β-transformation, is studied.

4.6. **Lattice tilings.** Consider Knuth's numeration system $(-1+\sqrt{-1}, \{0,1\})$ discussed in Subsection 4.1. We are interested in the set of all complex numbers admitting a representation w.r.t. this numeration system having zero "integer part", *i.e.*, in all numbers

$$z = \sum_{j\geq 1} c_j(-1+\sqrt{-1})^{-j} \qquad (c_j \in \{0,1\}).$$

Define the set (cf. [128])

$$\mathcal{T} := \left\{ z \in \mathbb{C} \,\middle|\, z = \sum_{j\geq 1} c_j(-1+\sqrt{-1})^{-j} \qquad (c_j \in \{0,1\}) \right\}.$$

From this definition we easily see that $\mathcal{T}$ satisfies the functional equation $(b = -1+\sqrt{-1})$

$$\mathcal{T} = b^{-1}\mathcal{T} \cup b^{-1}(\mathcal{T}+1). \tag{4.19}$$

Since $f_0(x) = b^{-1}x$ and $f_1(x) = b^{-1}(x+1)$ are contractive similarities in $\mathbb{C}$ w.r.t. the Euclidean metric, (4.19) asserts that $\mathcal{T}$ is the union of contracted copies of itself. Because the contractions are similarities in our case, $\mathcal{T}$ is a

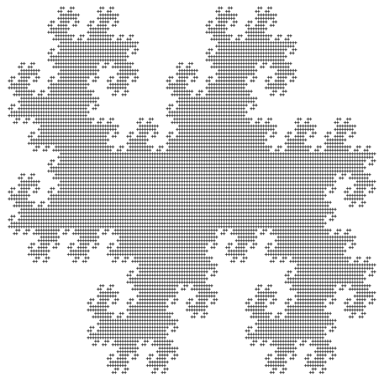

FIGURE 4.2. Knuth's twin dragon

self-similar set. From the general theory of self-similar sets (see for instance Hutchinson [106]) we are able to draw several conclusions on $\mathcal{T}$. Indeed, according to a simple fixed point argument $\mathcal{T}$ is uniquely defined by the set equation (4.19). Furthermore, $\mathcal{T}$ is a non-empty compact subset of $\mathbb{C}$. The set $\mathcal{T}$ is depicted in Figure 4.2. It is the well-known twin-dragon.

We want to mention some interesting properties of $\mathcal{T}$. It is the closure of its interior ([16]) and its boundary is a fractal set whose Hausdorff dimension is given by

$$\dim_H \partial\mathcal{T} = 1.5236\ldots$$

([92, 109]). Furthermore, it induces a tiling of $\mathbb{C}$ in the sense that

$$\bigcup_{z\in\mathbb{Z}[i]} (\mathcal{T} + z) = \mathbb{C} \tag{4.20}$$

where $(\mathcal{T}+z_1)\cap(\mathcal{T}+z_2)$ has zero Lebesgue measure if z_1 and z_2 are disjoint elements of $\mathbb{Z}[i]$ ([121]). Note that this implies that the Lebesgue measure of $\mathcal{T}$ is equal to 1. We also mention that $\mathcal{T}$ is homeomorphic to the closed unit disk ([17]).

These properties make $\mathcal{T}$ a so-called self-similar lattice tile. Tiles can be associated to numeration systems in a more general way. After Definition 4.2 we already mentioned that matrix numeration systems admit the definition of tiles. Let $(B, \mathcal{D})$ be a matrix numeration system. Since all eigenvalues of B are larger than one in modulus, each of the mappings

$$f_d(x) := B^{-1}(x+d) \qquad (d \in \mathcal{D})$$

is a contraction w.r.t. a suitable norm. This justifies the following definition.

Definition 4.4. *Let $(B, \mathcal{D})$ be a matrix numeration system in $\mathbb{Z}^d$. Then the non-empty compact set T which is uniquely defined by the set equation*

$$BT = \bigcup_{d\in\mathcal{D}} (T + d) \tag{4.21}$$

is called the self-affine tile associated to $(B, \mathcal{D})$.

Because $\mathcal{D} \subset \mathbb{Z}^d$ is a complete set of cosets in $\mathbb{Z}^d/B\mathbb{Z}^d$ these self-affine tiles are often called *self-affine tiles with standard digit set* (see for instance [141]). The literature on these objects is vast. It is not our intention here to survey this literature. We just want to link numeration systems and self-affine lattice tiles and give some of their most important properties. (For surveys on lattice tiles we refer for instance to [219, 221].)

In [30] it is shown that each self-affine tile with standard digit set has positive d-dimensional Lebesgue measure. Together with [142] this implies the following result.

Theorem 4.12. *Let T be a self-affine tile associated to a matrix numeration system $(B,\mathcal{D})$ in $\mathbb{Z}^d$. Then T is the closure of its interior. Its boundary ∂T has d-dimensional Lebesgue measure zero.*

As mentioned above, the twin-dragon induces a tiling of $\mathbb{C}$ in the sense mentioned in (4.20). It is natural to ask whether all self-affine tiles associated to matrix numeration systems share this property. In particular, let $(B,\mathcal{D})$ be a matrix numeration system. We say that the self-affine tile T associated to $(B,\mathcal{D})$ tiles $\mathbb{R}^n$ with respect to the lattice $\mathbb{Z}^d$ if

$$T + \mathbb{Z}^d = \mathbb{R}^d$$

such that $(T+z_1)\cap(T+z_2)$ has zero Lebesgue measure if $z_1, z_2 \in \mathbb{Z}^d$ are disjoint.

It turns out that it is a difficult question to describe all tiles having this property. Lagarias and Wang [141] and independently Kátai [140] found the following criterion.

Proposition 4.1. *Let $(B,\mathcal{D})$ be a matrix numeration system in $\mathbb{Z}^d$ and set*

$$\Delta(B,\mathcal{D}) := \bigcup_{k\geq 1}\left\{ \sum_{j=1}^{k} B^j(d_j - d_j') \;\middle|\; d_j, d_j' \in \mathcal{D}\right\}.$$

The self-affine tile T associated to $(B,\mathcal{D})$ tiles $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$ if only if

$$\Delta(B,\mathcal{D}) = \mathbb{Z}^d.$$

In [143] methods from Fourier analysis have been used in order to derive the tiling property for a very large class of tilings. We do not state the theorem in full generality here (see [143, Theorem 6.1]). We just want to give a special case. To state it we need some notation. Let A_1 and A_2 be two $d \times d$ integer matrices. We write $A_1 \equiv A_2$ to mean A_1 is integrally similar to A_2, *i.e.*, there exists $Q \in \mathrm{GL}(d,\mathbb{Z})$ such that $A_2 = QA_1Q^{-1}$. We say that A is *(integrally) reducible* if

$$A \equiv \begin{pmatrix} A_1 & 0 \\ C & A_2 \end{pmatrix}$$

holds with A_1, A_2 non-empty. We call A *irreducible* if it is not reducible. Note that a sufficient condition for the irreducibility of an integer matrix A is the irreducibility of its minimal polynomial over $\mathbb{Q}$.

From [143, Corollary 6.2] the following result follows.

Theorem 4.13. *Let $(B, \mathcal{D})$ be a matrix numeration system in $\mathbb{Z}^d$ with associated self-affine tile T. Then T tiles $\mathbb{R}^d$ with respect to the lattice $\mathbb{Z}^d$.*

A special case of this result can also be found in [96]. Theorem 4.13 ensures for instance that each canonical numeration system with irreducible base polynomial $p(x)$ yields a tiling of $\mathbb{R}^d$ with $\mathbb{Z}^d$-translates. Indeed, just observe the matrix B in (4.8) has minimal polynomial $p(x)$.

Many more properties of self-affine tiles associated to number systems have been investigated so far. The boundary of these tiles can be represented as a graph directed iterated function system (see [82, Chapter 3] for a definition). Indeed, let $(B, \mathcal{D})$ be a matrix numeration system and let T be the associated self-affine tile. Suppose that T tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$-translates. The set of neighbors of the tile T is defined by

$$S := \{s \in \mathbb{Z}^d \mid T \cap (T + s) \neq \emptyset\}.$$

Since T and its translates form a tiling of $\mathbb{R}^d$ we may infer that

$$\partial T = \bigcup_{s \in S \setminus \{0\}} T \cap (T + s).$$

Thus in order to describe the boundary of T it suffices to describe the sets $B_s := T \cap (T + s)$ for $s \in S \setminus \{0\}$. Using the set equation (4.21) for T we easily derive that (cf. [187, Section 2])

$$B_s = A^{-1} \bigcup_{d, d' \in \mathcal{D}} B_{As+d'-d} + d.$$

Here $B_{As+d'-d}$ is empty only if the index is an element of S. Now label the elements of S as $S = \{s_1, \ldots, s_J\}$ and define the graph $G(S) = (V, E)$ with set of states $V := S$ in the following way. Let $E_{i,j}$ be the set of edges leading from s_i to s_j. Then

$$E_{i,j} := \left\{ s_i \xrightarrow{d|d'} s_j \mid As_i + d' = s_j + d \text{ for some } d' \in \mathcal{D} \right\}.$$

In an edge $s_i \xrightarrow{d|d'} s_j$ we call d the *input digit* and d' the *output digit*. This yields the following result.

Proposition 4.2. *$\partial\mathcal{T}$ is a graph-directed iterated function system directed by the graph $G(S)$. In particular,*

$$\partial\mathcal{T} = \bigcup_{s \in S} B_s$$

where

$$B_s = \bigcup_{\substack{d\in\mathcal{D},\, s'\in S \\ s\xrightarrow{d}s'}} A^{-1}(B_{s'} + d).$$

The union is extended over all d, s' such that $s \xrightarrow{d} s'$ is an edge in the graph $G(S)$.

This description of ∂T is useful in several regards. In particular, the graph $G(S)$ contains a lot of information on the underlying numeration system and its associated tile. Before we give some of its application we mention that there exist simple algorithms for constructing $G(S)$ (see for instance [208, 187]).

In [221, 208] the graph $G(S)$ was used to derive a formula for the Hausdorff dimension of ∂T. The result reads as follows.

Theorem 4.14. *Let $(B, \mathcal{N})$ be a matrix numeration system in $\mathbb{Z}^d$ and T the associated self-affine tile. Let ρ be the spectral radius of the accompanying matrix of $G(S)$. If B is a similarity then*

$$\dim_B(\partial T) = \dim_H(\partial T) = \frac{d \log \rho}{\log |\det A|}.$$

Similar results are contained in [80, 108, 219, 213, 184]. There they are derived using a certain subgraph of $G(S)$. In [213, 184] also dimension calculations for the case where B is not a similarity are contained.

In [96] a subgraph of $G(S)$ is used in order to set up an algorithmic tiling criterion. In [143] this criterion was used as a basis for a proof of Theorem 4.13.

More recently, the importance of $G(S)$ for the topological structure of the tile T was discovered. We mention a result of Bandt and Wang [29] that yields a criterion for a tile to be homeomorphic to a disk. Roughly it says that a self-affine tile is homeomorphic to a disk if it has 6 or 8 neighbors and satisfies some additional easy-to-check conditions. Very recently Luo and Thuswaldner [154] established criteria for the triviality of the fundamental group of a self-affine tile. Also in these criteria the graph $G(S)$ plays an important role.

At the end we would like to show the relation of $G(S)$ to the matrix numeration system $(B, \mathcal{D})$ itself. If we change the direction of all edges in $G(S)$ and add the state 0 we obtain the transposed graph $G^T(S \cup \{0\})$. Suppose we have a representation of an element $z \in \mathbb{Z}^d$ of the shape

$$z = d_0 + Bd_1 + \cdots + B^\ell d_\ell \qquad (d_j \in \mathcal{D}).$$

To this representation we associate the digit string $(\ldots 00d_\ell \ldots d_0)$. Select a state s of the graph $G^T(S \cup 0)$. It can be shown that a walk in $G(S)$ is uniquely defined by its starting state and a sequence of input digits. Now we run through the graph $G(S)$ starting at s along a path of edges whose input digits agree with the digit string $(\ldots 00d_\ell \ldots d_0)$ starting with d_0. This yields an output string $(\dots 00d'_{\ell'} \dots d'_0)$. From the definition of $G(S)$ one easily checks that this output string is the B-ary representation of $z+s$, *i.e.*,

$$z + s = d'_0 + Bd'_1 + \cdots + B^{\ell'} d'_{\ell'} \qquad (d'_j \in \mathcal{D}).$$

Thus $G(S)$ is an adding automaton that allows to perform additions of the B-ary representations (see for instance [93, 184]). In [212] the graph $G(S)$ was used to get characterisation results for matrix numeration systems.

5. Some sofic Fibered Numeration Systems

This section is devoted to a particular class of FNS for which the subshift $X_{\mathcal{N}}$ is sofic. This class includes in particular β-numeration for β assumed to be a Parry number (see Example 2), the Dumont-Thomas numeration associated to a primitive substitution (see Section 5.1), as well as some exponential abstract numeration systems (see Section 5.2). Analogical with Section 4.6 we focuse on the construction of central tiles and Rauzy fractals in Section 5.3. We strongly use in the present section the algebraicity of the associated parameters of the FNS (e.g., β for the β-numeration), and hence the self-similarity properties of $X_{\mathcal{N}}$. We focus in particular on the Pisot case and end this section by discussing the Pisot conjecture in Section 5.4.

5.1. **Substitutions and Dumont-Thomas numeration.** We now introduce a second class of examples of a sofic FNS: the Dumont-Thomas numeration. For that purpose, we first recall some basic facts on the notion of substitution and substitutive dynamical systems. For more results on substitutions, the reader is refered to [173, 86, 18].

A *substitution* σ is an endomorphism of the free monoid $\mathcal{A}^*$. A substitution naturally extends to the set of two-sided sequence $\mathcal{A}^{\mathbb{Z}}$. A one-sided *σ-periodic point* of σ is a sequence $u = (u_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ that satisfies $\sigma^n(u) = u$ for some $n > 0$. A two-sided *σ-periodic point* of σ is a two-sided sequence $u = (u_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ that satisfies $\sigma^n(u) = u$ for some $n > 0$, and such that $u_{-1}u_0$ belongs to the image of some letter by some iterate σ^n of σ. This notion of σ-periodicity should not be confused with the usal notion of periodicity for sequences.

A substitution over $\mathcal{A}$ is said of *constant length* if the images of all letters of $\mathcal{A}$ have the same length. The *incidence matrix* $\mathbf{M}_\sigma = (m_{i,j})_{1\le i,j\le n}$ of the substitution σ has for entries $m_{i,j} = |\sigma(j)|_i$, where the notation $|w|_i$ stands for the number of occurrences of the letter i in the word w. A substitution σ is called *primitive* if there exists an integer n such that $\sigma^n(a)$ contains at least one occurrence of the letter b for every pair $(a,b) \in \mathcal{A}^2$. This is equivalent to the fact that its incidence matrix is primitive, that is, there exists a nonnegative integer n such that $\mathbf{M}_\sigma^n$ has only positive entries. If σ is primitive, then the *Perron-Frobenius theorem* ensures that the incidence matrix $\mathbf{M}_\sigma$ has a simple real positive dominant eigenvalue β. A substitution σ is called *unimodular* if $\det \mathbf{M}_\sigma = \pm 1$. A substitution σ is said to be *Pisot* if its incidence matrix $\mathbf{M}_\sigma$ has a dominant eigenvalue β such that for every other eigenvalue λ, one gets: $0 < |\lambda| < 1 < \beta$. The characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$. We deduce [86] that the dominant eigenvalue β is a Pisot number, and that Pisot substitutions are primitive.

Every primitive substitution has at least one periodic point [173]. In that case, if u is a periodic point for σ, then the closure in $\mathcal{A}^{\mathbb{Z}}$ of the shift orbit of u does not depend on u. We thus denote it by X_σ. The *symbolic dynamical system generated by* σ is defined as (X_σ, S). The system (X_σ, S) is *minimal* and *uniquely ergodic* [173]; it is made of all the two-sided sequences, whose set of factors coincides with the set of factors u (which does not depend on the choice of u by primitivity).

Example 8. Let $\beta > 1$ be a Parry number as defined in Example 2. As introduced for instance in [210] and in [81], one can associate in a natural way to (X_β, S) a substitution σ_β called *β-substitution* defined as follows according to the two cases, β simple and β non-simple:

- Assume $d_\beta(1) = t_1 \dots t_{m-1} t_m$ is finite, with $t_m \neq 0$. Thus $d^*_\beta(1) = (t_1 \dots t_{m-1}(t_m - 1))^\infty$. One defines σ_β over the alphabet $\{1, 2, \dots, m\}$ as
$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1} 2 \\ 2 & \mapsto 1^{t_2} 3 \\ \vdots & \vdots \\ m-1 & \mapsto 1^{t_{m-1}} m \\ m & \mapsto 1^{t_m}. \end{cases}$$
- Assume $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 [152]). Hence $d_\beta(1) = d^*_\beta(1) = t_1 \dots t_m (t_{m+1} \dots t_{m+p})^\infty$, with $m \geq 1$, $t_m \neq t_{m+p}$ and $t_{m+1} \dots t_{m+p} \neq 0^p$. One defines σ_β over the alphabet $\{1, 2, \dots, m+p\}$ as
$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1} 2 \\ 2 & \mapsto 1^{t_2} 3 \\ \vdots & \vdots \\ m+p-1 & \mapsto 1^{t_{m+p-1}} (m+p) \\ m+p & \mapsto 1^{t_{m+p}} (m+1). \end{cases}$$

One checks that in both cases the substitutions σ_β are primitive.

In the particular case that $\beta > 1$ is the root of $X^3 - X^2 - X - 1$, called *Tribonacci number*, then $d(1, \beta) = 111$, and
$$\sigma_\beta : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1.$$
The Tribonacci substitution has been introduced and studied in detail in [174], where it is proved that the symbolic dynamical system generated by the Tribonacci substitution is measure-theoretically isomorphic to a translation of the torus $\mathbb{T}^2$, the isomorphism being a continuous onto map. This latter translation first occured in [20], where the Tribonacci substitution

was used to model an interval exchange map of 6 intervals and to build explicitly a continuous and surjective conjugacy between this interval exchange map and the Rauzy translation (see also [26]); these results have led to the introduction of the family of Arnoux-Rauzy words in [25], to which the Tribonacci word belongs, as a generalisation of the family of Sturmian words. For more results and references on the Tribonacci substitution, see [111, 159, 160, 86, 153]. Let us also quote [21] and [160, 161] for an extension of the Tribonacci recurrence relation to the Fibonacci multiplication introduced in [127].

The connections between substitutions and numeration systems are numerous (see for instance [78, 77, 81]) and natural. We describe now a numeration system associated to a primitive substitution σ, known as the Dumont-Thomas numeration [74, 75, 177]. This numeration allows one to expand prefixes of the fixed point of the substitution, as well as real numbers in an noninteger base associated to the substitution; in this latter case, one gets a FNS providing expansions of real numbers with digits in a finite subset of the number field $\mathbb{Q}(\beta)$, β being the Perron-Frobenius eigenvalue of the substitution σ.

Let σ be a primitive substitution. We denote by β its dominant eigenvalue. Let $\delta_\sigma : \mathcal{A}^* \to \mathbb{Q}(\beta)$ be the morphism defined by

$$\forall a \in \mathcal{A},\ \delta_\sigma(a) = \lim_{n\to\infty} |\sigma^n(a)|\beta^{-n}.$$

Let us note that the convergence is ensured by Perron Frobenius' theorem. In other words, one checks that up to a multiplication constant, the map δ_σ sends the letter a to the corresponding coordinate of a left eigenvector $\mathbf{v}_\beta$ of the incidence matrix $\mathbf{M}_\sigma$.

Let $b \in \mathcal{A}$. Let $x \in [0, \delta_\sigma(a))$. One has $\beta x \in [0, \delta_\sigma(\sigma(a)))$. There exist a unique letter c in $\mathcal{A}$, and a unique word $p \in \mathcal{A}^*$ such that $\delta_\sigma(p) \le x < \delta_\sigma(pc)$. One checks that $\beta x - \delta_\sigma(p) \in [0, \delta_\sigma(c))$.

We thus define the following map T:

$$\begin{array}{rcl} T : \bigcup_{a\in\mathcal{A}}[0, \delta_\sigma(a)) \times \{a\} & \to & \bigcup_{a\in\mathcal{A}}[0, \delta_\sigma(a)) \times \{a\} \\ (x, b) & \mapsto & (\beta x - \delta_\sigma(p), c) \text{ with } \left\{ \begin{array}{l} \sigma(b) = pcs \\ \beta x - \delta_\sigma(p) \in [0, \delta_\sigma(c)). \end{array} \right. \end{array}$$

Furthermore, one checks that (X, T) is a fibred system, by setting $X = \bigcup_{a\in\mathcal{A}}[0, \delta_\sigma(a)) \times \{a\}$,

$$I = \{(p, c, s) \in \mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*;\ \exists\, b \in \mathcal{A},\ \sigma(b) = pcs\}$$

and $\varepsilon(x, b) = (p, c, s)$ where (p, c, s) is uniquely determined by $\sigma(b) = pcs$ and $\beta x - \delta_\sigma(p) \in [0, \delta_\sigma(c))$. According to [74], one checks that $\varphi = (\varepsilon(T^n x))_{n\ge 0}$ is injective, hence we get a FNS $\mathcal{N}$. Let us note that at first sight, a more natural choice in the framework of numeration could have been to define ε as $(x, b) \mapsto \delta_\sigma(p)$; nevertheless, we would have lost injectivity for the map φ by using such a definition.

In order to describe the subshift $X_{\mathcal{N}} = \overline{\varphi(X)}$, we need to introduce the notion of prefix-suffix automaton. The *prefix-suffix automaton* $\mathcal{M}_\sigma$ of the substitution σ is defined in [60, 61] as the oriented graph that has as set of vertices the alphabet $\mathcal{A}$ and whose edges satisfy the following: there exists an edge labeled by $(p, c, s) \in I$ from b toward c if $\sigma(b) = pcs$. Prefix automata have also been considered in the literature by just labelling edges with the prefix p [74, 177], but we need here the full information (p, c, s), in particular for Theorem 5.2 below: the main difference between the prefix automaton and the prefix-suffix automaton is that the subshift generated by the second one is of finite type, while the one generated by the first automaton is only sofic. For more details, see the discussion in Chap. 7 of [86].

Theorem 5.1 ([74])**.** *Let σ be a primitive substitution on the alphabet $\mathcal{A}$. Let us fix $a \in \mathcal{A}$. Every real number $x \in [0, \delta_\sigma(a))$ can be uniquely expanded as $x = \sum_{n\geq 1} \delta_\sigma(p_n)\beta^{-n}$, where the sequence of digits $(p_n)_{n\geq 1}$ is the projection on the first component of an infinite path $(p_n, a_n, s_n)_{n\geq 1}$ in the prefix-suffix automaton $\mathcal{M}_\sigma$ with p_1a_1 prefix of $\sigma(a)$, and with the extra condition that there exist infinitely integers n such that p_na_n is a proper suffix of $\sigma(a_{n-1})$, that is, s_n is not equal to the empty word.*

In other words, $\varphi(X)$ is equal to the set of labels of infinite paths $(p_n, a_n, s_n)_{n\geq 1}$ in the prefix-suffix automaton such that there exist infinitely integers n such that p_na_n is a proper suffix of $\sigma(a_{n-1})$, whereas $X_{\mathcal{N}} = \overline{\varphi(X)}$ is equal to the set of labels of infinite paths in the prefix-suffix automaton.

Let us note that we can also define the Dumont-Thomas numeration on $\mathbb{N}$. Let v be a one-sided fixed point of σ; we denote its first letter by v_0. We assume furthermore that $|\sigma(v_0)| \geq 2$, and that v_0 is a prefix of $\sigma(v_0)$. This numeration depends on this particular choice of a fixed point, and more precisely on the letter v_0. One checks ([74] Theorem 1.5) that every finite prefix of v can be uniquely expanded as

$$\sigma^n(p_n)\sigma^{n-1}(p_{n-1})\cdots p_0,$$

where $p_n \neq \varepsilon$, $\sigma(v_0) = p_na_ns_n$, and $(p_n, a_n, s_n)\cdots(p_0, a_0, s_0)$ is the sequence of labels of a path in the prefix-suffix automaton $\mathcal{M}_\sigma$ starting from the state v_0; one has for all i, $\sigma(p_i) = p_{i-1}a_{i-1}s_{i-1}$. Conversely, any path in $\mathcal{M}_\sigma$ starting from v_0 generates a finite prefix of v. This numeration works a priori on finite words but we can expand the nonnegative natural integer N as $N = |\sigma^n(p_n)| + \cdots + |p_0|$, where N stands for the length of the prefix $\sigma^n(p_n)\sigma^{n-1}(p_{n-1})\cdots p_0$ of v. One thus recovers a number system defined on $\mathbb{N}$.

If σ is a constant length substitution of length q, then one recovers the q-adic numeration. If σ is a β-substitution such as defined in Example 8, for β Parry number, then the expansion given in Theorem 5.1 with $a = 1$, coincides with the expansion provided by the β-numeration, up to a multiplication factor. Furthermore, the Dumont-Thomas numeration shares many properties with the β-numeration. In particular, when β is a Pisot number, then for every $a \in \mathcal{A}$, every element of $\mathbb{Q}(\beta) \cap [0, \delta_\sigma(a))$ admits an eventually periodic expansion, that is, the restriction to $\mathbb{Q}(\beta)$ yields a quasi-finite FNS. The proof can be conducted exactly in the same way as in [188].

We define $X^l_{\mathcal{N}}$ as the set of labels of infinite left-sided paths $(p_m, a_m, s_m)_{m\geq 0}$ in the prefix-suffix automaton; they satisfy $\sigma(a_{m+1}) = p_m a_m s_m$ for all $m \geq 0$. The subshifts $X^l_{\mathcal{N}}$ is a support of a subshift of finite type. The set $X^l_{\mathcal{N}}$ has an interesting dynamical interpretation with respect to the substitutive dynamical (X_σ, S). We follow here the approach and notation of [60, 61]. Let us recall that the substitution σ is assumed to be primitive. According to [165] and [36], every two-sided sequence $w \in X_\sigma$ has a unique decomposition $w = S^\nu(\sigma(v))$, with $v \in X_\sigma$ and $0 \leq \nu < |\sigma(v_0)|$, where v_0 is the 0-th coordinate of v, that is,

$$w = \dots \mid \underbrace{\dots}_{\sigma(v_{-1})} \mid \underbrace{w_{-\nu} \dots w_{-1}.w_0 \dots w_{\nu'}}_{\sigma(v_0)} \mid \underbrace{\dots}_{\sigma(v_1)} \mid \underbrace{\dots}_{\sigma(v_2)} \mid \dots$$

The two-sided sequence w is completely determined by the two-sided sequence $v \in X_\sigma$ and the value $(p, w_0, s) \in I$. The *desubstitution map* $\theta : X_\sigma \to X_\sigma$ is thus defined as the map that sends w to v. We then define $\gamma\colon X_\sigma \to I$ mapping w to (p, w_0, s). One checks that $(\theta^n(w))_{n\geq 0} \in X^l_{\mathcal{N}}$. The *prefix-suffix expansion* is then defined as the map $E_{\mathcal{N}} : X_\sigma \to X^l_{\mathcal{N}}$ which maps a word $w \in X_\sigma$ to the sequence $(\gamma(\theta^n w))_{n\geq 0}$, that is, the orbits of w through the desubstitution map according to the partition defined by γ.

Theorem 5.2. [60, 61, 104] *Let σ be a primitive substitution such that none of its periodic points is shift-periodic. The map $E_{\mathcal{N}}$ is continuous and onto the subshift of finite type $X^l_{\mathcal{N}}$; it is one-to-one except on the orbits under the shift S of the σ-periodic points of σ.*

In other words, the prefix-suffix expansion map $E_{\mathcal{N}}$ provides a measure-theoretic isomorphism between the shift map S on X_σ and an adic transformation on $X^l_{\mathcal{N}}$, considered as a Markov compactum, as defined in Section 6.4, when the set I is provided with a natural partial ordering coming from the substitution.

5.2. **Abstract numeration systems.** One essential feature in the construction of the previous section is that the dynamical system $X_{\mathcal{N}}$ is sofic, which means that the language $\mathcal{L}_{\mathcal{N}}$ is regular. Let us now extend this approach by starting directly with a regular language. According to [145], given an infinite regular language L over a totally ordered alphabet $(A, <)$, a so-called *abstract numeration system* $S = (L, A, <)$ is associated in the following way: enumerating the words of L by increasing genealogical order gives a one-to-one correspondence between $\mathbb{N}$ and L, the non-negative integer n is then represented by the $(n+1)$-th word of the ordered language L. Such an abstract numeration systems is a numeration system according to Definition 3.4, where $X = \mathbb{N}$, $I = A$, and φ is the (injective) map that sends

n to the $(n+1)$-th word of the ordered language L. Abstract numration systems thus include classical numeration systems like the q-adic numeration, β-numeration when β is a Parry number, as well as the Dumont-Thomas numeration associated to a substitution.

Moreover, these abstract systems have been extended to allow not only the representation of integers but also of real numbers [147]: a real number is represented by an infinite word which is the limit of a converging sequence of words in L. Under some extra hypotheses, we can describe such a representation thanks to a fibered number system defined as follows, according to [179, 39].

Let L be an infinite regular language over the totally ordered alphabet $(\Sigma, <)$. The trimmed minimal automaton of L is denoted by $\mathcal{M}_L = (Q, q_0, \Sigma, \delta, F)$ where Q is the set of states, q_0 is the initial state, $F \subseteq Q$ is the set of final states and $\delta : Q \times \Sigma \to Q$ is the (partial) transition function. We assime furthermore that $\mathcal{M}_L$ is such that $\mathcal{M}_L$ has a loop of label s_0 in the initial state q_0. For any state $q \in Q$, we denote by L_q the regular language accepted by $\mathcal{M}_L$ from state q, and by $\mathbf{u}_q(n)$ the number of words of length n in L_q.

The entry of index $(p,q) \in Q^2$ of the *adjacency matrix* $\mathbf{M}_L$ of the automaton $\mathcal{M}_L$ is given by the cardinality of the set of letters $s \in \Sigma$ such that $\delta(p,s) = q$. An abstract numeration system is said *primitive* if the matrix $\mathbf{M}_L$ is primitive, that is, there exists a nonnegative integer n such that $\mathbf{M}_L^n$ has only positive entries. Let $\beta > 1$ denote its dominating eigenvalue. We assume moreover that L is a language for which there exist $P \in \mathbb{R}[X]$, and some nonnegative real numbers a_q, $q \in Q$, which are not simultaneously equal to 0, such that for all state $q \in Q$

$$\lim_{n\to\infty} \frac{\mathbf{u}_q(n)}{P(n)\beta^n} = a_q. \tag{5.1}$$

The coefficients a_q are defined up to a scaling constant; in fact, the vector $(a_q)_{q\in Q}$ is an eigenvector of $\mathbf{M}_L$ [147]; by Perron-Frobenius theorem, all its entries a_q are positive; we normalize it so that $a_{q_0} = 1 - 1/\beta$, according to [179]. For $q \in Q$ and $s \in \Sigma$, set

$$\alpha_q(s) := \sum_{q'\in Q} a_{q'} \cdot \mathrm{Card}\{t < s \mid \delta(q,t) = q'\} = \sum_{\substack{t<s \\ (q,t)\in\mathrm{dom}(\delta)}} a_{\delta(q,t)}.$$

One has for all $q \in Q$, $0 \le \alpha_q(s) \le \beta a_q$, since $(a_q)_{q\in Q}$ is a positive eigenvector of $\mathbf{M}_L$. Notice also that if $s < t$, $s,t \in \Sigma$, then $\alpha_q(s) \le \alpha_q(t)$. We set for $x \in \mathbb{R}^+$,

$$\lfloor x \rfloor_q = \max\{\alpha_q(s) \mid s \in \Sigma,\ \alpha_q(s) \le x\}.$$

Since $(a_q)_{q\in Q}$ is an eigenvector of $\mathbf{M}_L$ of eigenvalue β, one has $\beta a_q = \sum_{r\in Q} a_r \cdot \mathrm{Card}\{s \in \Sigma\,;\, \delta(q,s) = r\}$, and one checks that for $x \in [0, a_q)$, then $\beta x - \lfloor \beta x \rfloor_q \in [0, a_{q'})$, with $\lfloor \beta x \rfloor_q = \alpha_q(s)$ and $\delta(q,s) = q'$. We define

$$\begin{array}{rcl} T : \bigcup_{q \in Q} [0, a_q] \times \{q\} & \longrightarrow & \bigcup_{q \in Q} [0, a_q] \times \{q\} \\ (x, q) & \longmapsto & (\beta x - \lfloor \beta x \rfloor_q, q'), \end{array}$$

where q' is determined as follows: let s be the largest letter such that $\alpha_q(s) = \lfloor \beta x \rfloor_q$; then $q' = \delta(q,s)$. One checks that $\mathcal{N} = (X, T, I, \varphi)$ is a fibred number system by setting $X = \cup_{q \in Q} [0, a_q] \times \{q\}$,

$$I = \{(s, q, q') \in \Sigma \times Q \times Q \,;\, q' = \delta(q, s)\}$$

and $\varepsilon(x, q) = (s, q, q')$, where s is the largest letter such that $\alpha_q(s) = \lfloor \beta x \rfloor_q$, and $q' = \delta(q, s)$. One checks furthermore that φ is injective.

We thus can expand any real number $x \in [0, a_{q_0}) = [0, 1 - 1/\beta)$ as follows: Let $(x_n, r_n)_{n \geq 1} = (T^n(x, q_0))_{n \geq 1} \in X^{\mathbb{N}^*}$; let $(w_0, r_0) := (s_0, q_0)$; for every $n \geq 1$, let us denote by w_n the first component of $\varepsilon_n = \varepsilon \circ T^{n-1}$ where $x_0 := x$; one has $x = \sum_{n=1}^{\infty} \alpha_{r_{n-1}}(w_n) \beta^{-n}$, according to [146].

Abstract numeration systems lead to the generalisation of various concepts related to the representation of integers like summatory functions of additive functions [95], or like the notion of odometer [40]. They also have been considered in [90].

5.3. **Rauzy fractals.** We have seen in Section 4.6 that it is possible, according to [4, 5, 7], to associate in a natural way to matrix number systems a central tile, also called generalised Rauzy fractal, in case β is a Pisot unit. Such a central tile is a compact subset of $\mathbb{R}^{d-1}$, where d stands for the degree of β; it is the closure of its interior, it has non-zero measure and a fractal boundary, and it is the attractor of some graph-directed Iterated Function System.

Rauzy fractals have first been introduced by G.Rauzy in [176] in the case of the Tribonacci substitution σ, and then in [210], in the case of the β-numeration associated to the Tribonacci number, that is, the real root of the polynomial $X^3 - X^2 - X - 1$. Rauzy fractals can more generally be associated to Pisot substitutions; see for instance [36, 60, 61, 114, 160, 161, 86, 198, 199]. One motivation for Rauzy's construction was to exhibit explicit factors of a substitutive dynamical system (X_σ, S), under the Pisot hypothesis, as rotations on compact abelian groups. Rauzy fractals are usually associated in the Pisot case to β-shifts and Pisot substitutions (see the survey [42]), but they also can be associated to abstract numeration systems [39], as well as to some automorphisms of the free group [22], namely the so-called irreducible with irreducible powers automorphisms [47].

There are several definitions associated to several methods of construction for Rauzy fractals. We detail here a construction based on formal power series in the substitutive case; this construction is inspired by the seminal paper [174], by [159, 160], and by [60, 61]. A different approach via Iterated

Function Systems and generalised substitutions has been developed following ideas from [111], and [23, 24]. Indeed, Rauzy fractals can be described as the attractor of some graph iterated function system (IFS), as in [105] where one can find a study of the Hausdorff dimension of various sets related to Rauzy fractals, and as in [200, 201, 202] with special focus on the self-similar properties of Rauzy fractals. Lastly, they can be defined in case σ is a Pisot substitution as the closure of the projection on the contracting plane of $\mathbf{M}_\sigma$ along its expanding direction of the abelianized of the prefixes of a σ-periodic point [86, 36, 114]; the abelianization map, also called Parihk map, is defined as $\mathbf{l}\colon \mathcal{A}^* \to \mathbb{N}^n$, defined as $\mathbf{l}(W) = (|W|_k)_{k=1,\dots,n} \in \mathbb{N}^n$. For more details on those approaches, see Chap. 7 and 8 of [86], and [42].

Let us describe how to associate a Rauzy fractal to a Pisot substitution, that is not necessarily unimodular, as a compact subset of a finite product of Euclidean and p-adic spaces following [198]. We thus consider a primitive substitution σ, that we assume furthermore Pisot. We then consider the FNS $\mathcal{N}$ provided by the Dumont-Thomas numeration, such as described in Section 5.1. We follow here [198, 41, 42]. Let us recall that the set $X^l_{\mathcal{N}}$ denotes the set of labels of infinite left-sided paths $(p_n, a_n, s_n) \in I^{\mathbb{N}}$ in the prefix-suffix automaton $\mathcal{M}_\sigma$, with the notation of Section 5.1. Let β stand for the dominating eigenvalue of the primitive substitution σ.

We first define the map Φ_X on $X^l_{\mathcal{N}}$ as

$$\Phi_X((p_n, a_n, s_n)_{n\geq 0}) = \sum_{n\geq 0} \delta_\sigma(p_n) X^n;$$

hence Φ_X takes its values in a finite extension of the ring of formal power series with coefficients in $\mathbb{Q}$; we recall that the coefficients $\delta_\sigma(p_n)$ take their values in a finite subset of $\mathbb{Q}(\beta)$.

Let us specialize these formal power series by giving to the indeterminate X the value β, and by considering all the archimedean and non-archimedean metrizable topologies on $\mathbb{Q}(\beta)$ in which all the series $\sum_{n\geq 0} \delta_\sigma p_n \beta^n$ would converge for $(p_n, a_n, s_n)_{n\geq 0} \in X^l_{\mathcal{N}}$.

We recall that β is a Pisot number of degree d, say. Let $\beta_2, \ldots, \beta_r$ be the real conjugates of β, and let $\beta_{r+1}, \overline{\beta_{r+1}}, \ldots, \beta_{r+s}, \overline{\beta_{r+s}}$ be its complex conjugates. For $2 \leq j \leq r$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{R}$, and for $r+1 \leq j \leq r+s$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{C}$ being endowed with the usual topology. Let $\mathcal{I}_1, \ldots, \mathcal{I}_\nu$ be the prime ideals in the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ that contain β, that is, $\beta\mathcal{O}_{\mathbb{Q}(\beta)} = \prod_{i=1}^{\nu} \mathcal{I}_i^{n_i}$. The field $\mathbb{K}_{\mathcal{I}}$ is a finite extension of the $p_{\mathcal{I}}$-adic field $\mathbb{Q}_{p_{\mathcal{I}}}$ where $\mathcal{I} \cap \mathbb{Z} = p_{\mathcal{I}}\mathbb{Z}$. The primes which appear as p-adic spaces are the prime factors of the norm of β. Let us denote by $\mathbb{K}_{\mathcal{I}}$ the completion of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology. One then defines the representation space of $X^l_{\mathcal{N}}$ as

$$\mathbb{K}_\beta = \mathbb{K}_{\beta_2} \times \ldots \mathbb{K}_{\beta_{r+s}} \times \mathbb{K}_{\mathcal{I}_1} \times \ldots \mathbb{K}_{\mathcal{I}_\nu} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{K}_{\mathcal{I}_1} \times \cdots \times \mathbb{K}_{\mathcal{I}_\nu}.$$

Endowed with the product of the topologies of each of its elements, $\mathbb{K}_\beta$ is a metric abelian group.

The canonical embedding of $\mathbb{Q}(\beta)$ into $\mathbb{K}_\beta$ is defined by the following morphism $\Phi_\beta\colon P(\beta) \in \mathbb{Q}(\beta) \mapsto (\underbrace{P(\beta_2)}_{\in\mathbb{K}_{\beta_2}}, \dots, \underbrace{P(\beta_{r+s})}_{\in\mathbb{K}_{\beta_{r+s}}}, \underbrace{P(\beta)}_{\in\mathbb{K}_{\mathcal{I}_1}}, \dots, \underbrace{P(\beta)}_{\in\mathbb{K}_{\mathcal{I}_\nu}}) \in \mathbb{K}_\beta$.

The topology on $\mathbb{K}_\beta$ has been chosen so that the series

$$\lim_{n\to+\infty} \Phi_\beta(\sum_{n=0}^{j} \delta_\sigma(p_n)\beta^n) = \sum_{n\geq 0} \Phi_\beta(\delta_\sigma(p_n)\beta^n)$$

are convergent in $\mathbb{K}_\beta$ for every $(p_n, a_n, s_n)_{n\geq 0} \in X_\beta^l$. One thus defines

$$\phi_\beta\colon X_\beta^l \to \mathbb{K}_\beta,\ (p_n, a_n, s_n)_{n\geq 0} \mapsto \Phi_\beta(\sum_{n\geq 0} \delta_\sigma(p_n)\beta^n).$$

We set $\mathcal{T}_\mathcal{N} = \phi_\beta(X_\mathcal{N}^l)$ and call it the *generalized Rauzy fractal* of $X_\mathcal{N}^l$. It can be divided into subpieces in a natural way: for every letter a in the alphabet $\mathcal{A}$ of the substitution, $\mathcal{T}_\mathcal{N}(a) = \phi_\beta(\{(p_n, a_n, s_n)_{n\geq 0} \in X_\mathcal{N}^l;\ (p_n, a_n, s_n)_{n\geq 0}$ is the label of an infinite left-sided path in $\mathcal{M}_\sigma$ arriving at state $a_0 = a\})$. The sets $\mathcal{T}_\mathcal{N}$ and $\mathcal{T}_\mathcal{N}(a)$, for every letter a, have a non-empty interior, hence they have non-zero measure [198], and they are the closure of their interior, according to [202]. For examples of Rauzy fractals, see Figure 5.1.

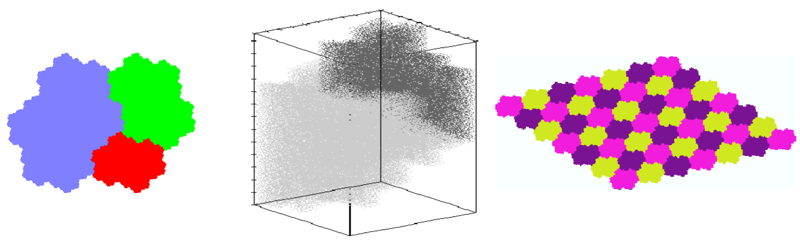

FIGURE 5.1. The Rauzy fractal for the Tribonacci substitution, for $1 \mapsto 1112, 2 \mapsto 12$, and the Rauzy lattice tiling.

Surprisingly enough, the sets $\mathcal{T}_\mathcal{N}(a)$ have disjoint interiors provided that the substitution σ satisfies a combinatorial condition, the so-called strong coincidence condition, according to [23] in the unimodular case, and [198], in the general case. Namely, a substitution is said to satisfy the *strong coincidence condition* if for any pair of letters (i, j), there exist two integers k, n such that $\sigma^n(i)$ and $\sigma^n(i)$ have the same k-th letter, and the prefixes of length $k-1$ of $\sigma^n(i)$ and $\sigma^n(j)$ have the same image under th. The strong coincidence condition has been introduced in [23], and inspired by Dekking's notion of coincidence [66] which yields a characterisation of constant length substitutions having discrete spectrum. It is conjectured that every Pisot substitution satisfies the strong coincidence condition; the conjecture holds in particular for two-letter substitutions [35]. For more details on the strong coincidence condition, see [86, 36, 114].

5.4. **The Pisot conjecture.** One of the main motivation for the introduction of the Rauzy fractal is the following result, due to [174]: The fixed point $(u_n)_{n\in\mathbb{N}}$ of the substitution $\sigma\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$ codes the orbit of the point 0 under the action of the translation

$$R_\beta\colon \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2) \tag{5.2}$$

with respect to the partition of the fundamental domain $\mathcal{T}_\mathcal{N}$ of $\mathbb{T}^2$ by the sets $(\mathcal{T}_\mathcal{N}(1), \mathcal{T}_\mathcal{N}(2), \mathcal{T}_\mathcal{N}(3))$, that is,

$$\forall N \in \mathbb{N},\ \forall i = 1, 2, 3,\ u_N = i \iff R_\beta^N(0) \in \mathcal{T}_\mathcal{N}(i).$$

This means that the symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the toral translation $(\mathbb{T}^2, R_\beta)$, which implies that (X_σ, S) has discrete spectrum, as recalled in Section 2.3.1. This result can also be restated in more geometrical terms: the Rauzy fractal generates a lattice tiling of the plane, as illustrated in Figure 5.1.

For any Pisot substitution the following equivalent conditions are conjectured to hold: (X_σ, S) is measure-theoretically isomorphic to a translation on the torus, equivalently, (X_σ, S) is measure-theoretically isomorphic to its maximal translation factor, (X_σ, S) has pure discrete spectrum, or else, the associated Rauzy fractal generates a lattice tiling. The equivalence between these conditions is due to [36]: indeed, the set of eigenvalues of (X_σ, S) is proved to coincide with the subgroup of $\mathbb{R}$ generated by the components of the eigenvector associated to the dominating eigenvalue β, normalized so that the sum of its coordinates is equal to one. A large literature is devoted to this question, which is surveyed in [86], Chap.7. See also for recent results, [42, 114, 36, 37, 28]. A similar conjecture holds for the β-numeration, for β Pisot, the action of the one-sided β-shift which plays in this latter case the role of the shift on X_σ being the odometer such as defined in Section 6, for abstract numeration systems [39]. Let us stress the fact that the irreducibility of the characteristic polynomial of the incidence matrix of the substitution is necessary for the pure discrete spectrum to hold (see Example 5.3 in [28]).

Let us note that one can perform a similar construction for the whole set $\widetilde{X_\mathcal{N}}$ of two-sided $\mathcal{N}$-representations, providing a geometric realisation of the natural extension of the transformation T of the FNS $\mathcal{N}$, such as defined in Section 2.2. For more details, see for instance [41]. This construction is used in [41] to characterise the numbers that have a purely periodic β-expansion, producing a kind of generalised Galois' theorem on classical continuous fractions, for β Pisot. This construction has also consequences for the effective construction of Markov partitions for toral automorphisms, the main eigenvalue of which is a Pisot number [112]. See also [44, 172, 37] in the β-numeration case. Based on the approach of [125, 216, 197, 196], an algebraic construction of symbolic representations of hyperbolic toral automorphisms as Markov partitions is similarly given in [189, 151], where homoclinic points are shown to play an essential rôle.

A sufficient condition for lattice tiling by the Rauzy fractal in the Pisot β-numeration case is the Property (F), such as defined in Example 2 and Section 4.4. Similar finiteness properties have been introduced in [42] for substitutive dynamical systems, see also [90]. There exist furthermore effective combinatorial characterisations for pure discrete spectrum based on graphs [199], see also [211], or inspired by the strong coincidence condition [36, 114].

6. G-scales and odometers

6.1. *G*-scales. Building the odometer. Fibred numeration systems consist in consecutive iterations of a transformation and give rise to infinite representations given by a sequence of digits. As indicated in Definition 3.6, natural compactification can be built considering the closure of the set of all those sequences. A simple generalisation is obtained by changing the transformation at any step. They are still numeration systems in the sense of Definition 3.4. Cantor (or radix) expansions are the most popular exemple in that direction:

Example 9. Let $(G_n)_n$ an increasing sequence of natural integers such that $G_0 = 1$ and $G_n|G_{n+1}$. Let $q_n = G_n/G_{n-1}$, with $T^{(n)}$ the transformation of Example 1-1 for $q = q_n$ and $\varepsilon^{(n)}$ the corresponding digit function. Take $n \in \mathbb{Z}$. Then $\varphi(n) = (\varepsilon^{(1)}(n), \varepsilon^{(2)}(T^{(1)}(n)), \varepsilon^{(3)}(T^{(2)} \circ T^{(1)}(n)), \ldots)$, which gives rise to an expansion $n = \sum_{j\geq 0} \varepsilon^{(j)}(T^{(j-1)} \circ \cdots \circ T^{(1)}(n))G_j$. This expansion makes sense in $\varprojlim \mathbb{Z}/G_m\mathbb{Z}$ for general n and in $\mathbb{N}$ for non-negative n, since the corresponding representation is finite (the digits are ultimately 0). Everything, including variants, is similar with Example 1.

To a certain extent, this kind of expansion is based on the divisibility order relation. A different approach is given by expansion of natural numbers, which are essentially based on the usual total ordering. It is called *greedy*, since it first looks for the most significant digit.[6]

Definition 6.1. *A G-scale is an increasing sequence of natural numbers $(G_n)_n$ with $G_0 = 1$.*

[6]The word *greedy* emphasises that at any step, the representation algorithm chooses in an appropriate sense the digit that gives the biggest contribution as possible. In the present situation, it is not fibred. On the opposite, what is called "greedy beta-representation" is fibred, since the greedyness is thought inside an imposed fibred framework: a FS is given (the beta-transformation on the unit interval). The digit is then choosen in this greedy way.

Given a G-scale, any natural integer n can be written in the form

$$n = \sum_{k \geq 0} e_k(n) G_k \tag{6.1}$$

avec $e_k(n) \in \mathbf{N}$. This representation is unique, provided that the following so-called Yaglom condition

$$\forall\, m \in \mathbf{N}\,,\ \sum_{k=0}^{m} e_k(n) G_k < G_{m+1} \tag{6.2}$$

is satisfied, what we always suppose in the sequel. The digits are obtained by the so-called *greedy algorithm*: let $N \in \mathbb{N}$.

- find the unique n such that $G_n \leq N < G_{n+1}$,
- compute $\varepsilon_n(N) = \lfloor N/G_n \rfloor$,
- $N \leftarrow N - \varepsilon_n(N) G_n$, go to the first step.

Formally, we get the expansion (6.1) by writing $\varepsilon_n(N) = 0$ for all values of n that have not been assigned during the performance of the greedy algorithm. In particular, this expansion is finite, in that sense that $\varepsilon_n(N) = 0$ for all but finitely many n.

The infinite word $J_G(n) := \varepsilon_0(n)\varepsilon_1(n)\varepsilon_2(n)\cdots$ is by definition the G-representation of n. In particular, $J_G(0) = 0^\omega$. Some examples and general properties are to be found [87]. First general elements about the association to a dynamical system are due to [94]. It is a numeration system according to Definition 3.4. Therefore, we may consider by Definition 3.6 its compactification. The usual notation in the litterature is $\mathcal{K}_G$ and we will use it. By Property (6.2) it is the set of sequences $e = e_0 e_1 e_2 \cdots$ belonging to the infinite product

$$\Pi(G) := \prod_{m=0}^{\infty} \{0, 1, \ldots, \lceil G_{m+1}/G_m \rceil - 1\},$$

satisfying (6.2). The natural integers $\mathbf{N}$ are embedded in $\mathcal{K}_G$ by the canonical injection $n \mapsto J_G(n)$, n and $J_G(n)$ being freely identified (except if it would be source of confusion). Their image form a dense subset of $\mathcal{K}_G$. The natural ordering on the natural integers yield a partial order, called *antipodal*, on $\mathcal{K}_G$ by $x \preccurlyeq y$ if $x_n = y_n$ for $n > n_0$ and $x_{n_0} < y_{n_0}$ or $x = y$. In particular, $n \mapsto J_G(n)$ is increasing with respect to the usual order on $\mathbb{N}$ and the antipodal order on $\mathcal{K}_G$.

From a topological point of view, the compact space $\mathcal{K}_G$ is almost always a Cantor set:

Proposition 6.1 ([33], Theorem 2)**.** *If the sequence $(G_{n+1} - G_n)_n$ is not bounded, then $\mathcal{K}_G$ is homeomorphic to the triadic Cantor space. Otherwise, it is homeomorphic to a countable initial segment of the ordinals.*

The addition of 1 extends naturally to $\mathcal{K}_G$ (Question 4):

$$\forall x = x_0 x_1 \cdots \in \mathcal{K}_G \; : \; \tau(x) = \lim_{n\to\infty} J(x_0 + x_1 G_1 + \cdots + x_n G_n + 1). \tag{6.3}$$

According to (6.2), this limit exists. It is 0^ω if and only if there are infinitely many integers n such that

$$x_0 + x_1 G_1 + \cdots + x_n G_n = G_{n+1} - 1 \tag{6.4}$$

Definition 6.2. *The dynamical system $(\mathcal{K}_G, \tau)$ is called odometer.*

There is no universal terminology, what an odometer is. From the common sense, the word "odometer" is concerned with counting from a dynamical point of view, especially how one goes from n to $n+1$. Several authors restrict this term to the Cantor case (for instance [71], or [1]). One encounters "adding machine" in the same sense.

Example 10. We continue Example 6. For the Zeckendorf representation, we have $G_{2n} - 1 = (01)^n$ and $G_{2n+1} - 1 = (10)^n 1$ (immediate verification by induction). Therefore, 0^ω possesses two preimages by τ, namely $(01)^\omega$ and $(10)^\omega$. This shows that there is no chance to extend the addition on $\mathbb{N}$ to a group on the compactification (as it is the case for the $q-$adic or even for the Cantor representation). Indeed, even the monoid law cannot be extended in a natural way: take $x = (01)^\omega$ and $y = (10)^\omega$. Then $(01)^n + (10)^n = (G_{2n} - 1) + (G_{2n-1} - 1) = G_{2n+1} - 2 = 0(01)^n$ but $(01)^n 0 + (10)^n 1 = (G_{2n} - 1) + (G_{2n+1} - 1) = G_{2n+2} - 2 = 10(01)^n$. We have two cluster points, which are the two elements of $\tau^{-2}\{0^\omega\}$. The same phenomenon appears for the general Ostrowski representation.

Example 11. G-scales arising from beta-numeration. Let $\beta > 1$ and $d^*_\beta(1) = (a_n)_{n\geq 0}$ (see Example 2). Then define $G_0 = 1$ and $G_{n+1} = a_0 G_n + a_1 G_{n-1} + \cdots + a_n G_0 + 1$. The sequence $(G_n)_n$ is a scale of numeration and the its compactification coincides (up to a miror symetry) with the compactification X_β [46]: the lexicographical condition defining the language is $(\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_0) <_{\text{lex}} (a_0, a_1, \ldots)$ for all $n \in \mathbb{N}$.

Furthermore, let be

$$\mathbb{Z}^+_\beta = \{w_m \beta^m + \cdots + w_0 \, ; \, m \in \mathbb{N}, w_m \cdots w_0 \in \mathcal{L}_\beta\}.$$

If $\mathcal{S}$ is the successor function $\mathcal{S}\colon \mathbb{Z}^+_\beta \to \mathbb{Z}^+_\beta$ given by $\mathcal{S}(x) = \min\{y; y > x\}$, and if $\varphi(\sum \varepsilon_n G_n) = \sum \varepsilon_n \beta^n$, then the diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\tau"] \arrow[d, "\varphi"'] & \mathbb{N} \arrow[d, "\varphi"] \\ \mathbb{Z}^+_\beta \arrow[r, "\mathcal{S}"'] & \mathbb{Z}^+_\beta \end{tikzcd}$$

is commutative and φ is bijective. See Section 7.3 for further connexions. It has been proved in [94] that the odometer is continuous if and only if the sequence $(a_n)n$ is purely periodic, that is if β is a simple Parry number. If the sequence is ultimately periodic with period $b_1 \cdots b_s$ (β is a Parry number), then

$$\omega(G) = \{(b_k b_{k+1} \cdots b_{k+s-1})^\omega; 1 \le k \le s-1\}.$$

The preimage of 0^ω is either empty if β is not a simple Parry number, or equal to $\omega(G)$ otherwise. If we have $a_0 = 2$ and if $a_1 a_2 \cdots$ is the Champernowne number in base two, then $\tau^{-1}(0^\omega) = \emptyset$ and $\omega(G) = \{0,1\}^{\mathbb{N}}$.

6.2. **Carries tree.** As it turned out from the paragraph above, the structure of the words $G_n - 1$ contains important information concerning the odometer. A tree of carries has been introduced in [33], that gives some visibility to this information. The nodes of the tree are $\mathbb{N} \cup \{-1\}$, and -1 is the root of the tree. The vertices are characterised by the relation

- $-1 \, \rho \, n$ if there is no $k < n$ such that $G_{k+1} - 1$ is a prefix of $G_{n+1} - 1$;
- for $0 \le n < m$, we have $n \, \rho \, m$ if $G_{n+1} - 1$ is a prefix of $G_{m+1} - 1$ and if n is the largest integer smaller than m with this property.

Giving the tree is equivalent with giving a *descent function* $D \colon \mathbb{N} \to \mathbb{N} \cup \{-1\}$ verifying $D(n) < n$ for all n and: $\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : D^k(n) = -1$. We have $D(n) = m$ if and only if $m \, \rho \, n$. Given a tree of that type, there exists infinitely many G-scales having this tree as carries tree. The smallest one with respect to the lexicographical order is unique and called a *low scale*. It is given by $G_{m+1} = G_m + G_{n+1}$ for $D(m) = n$. Therefore, one may imagine any type of tree one likes...

Example 12.
1. Linear tree. If $(G_n)_n$ is a Cantor scale, then $n \, \rho \, (n+1)$ for any node n.
2. Fibonacci tree. For the Zeckendorf expansion, we have $-1 \, \rho \, 0$ and $n \, \rho \, (n+2)$ for any node n.
3. Hedgehog tree. Let $q \ge 2$ and $G_{n+1} = qG_n + 1$. Then the language is the set of the words with letters in $\{0, 1, \ldots, q\}$ such that $x_n = q$ implies $x_j = 0$ for $j < n$. We have $-1 \, \rho \, n$ for all nodes $n \neq -1$.
4. Comb tree. It is given by $D(0) = -1$, and $D(2n+2) = D(2n+1) = 2n$ for any $n \ge 0$. The corresponding low scale is $G_{2n+j} = 2^j 3^n$ for $n \ge 0$ and $j = 1, 2$.

The preimages of 0^ω correspond to the infinite branches of the carries tree: to an infinite branch $(n_0, n_1, n_2, \ldots)$, where $n_0 = -1$ and $D(n_{k+1}) = n_k$ correspond $x = \lim(G_{n_k+1} - 1)$. In particular, for the scale with hegehog tree, the preimage of 0^ω is empty. The following proposition indicates a further property of the odometer which can be read on the carries tree.

Proposition 6.2. *A tree is of finite type if all nodes have finitely many neighboors. For carries trees, that means:*

$$\forall n \in \mathbb{N} \cup \{-1\} : \ \{m > n\,;\, n\,\rho\, m\} \text{ is finite.}$$

Then the set of discontinuity points of τ is $\omega(G) \setminus \tau^{-1}(0^\omega)$; where the omega limit set $\omega(G)$ is the set of limit points in $\mathcal{K}_G$ of the sequence $(G_n - 1)_n$. Furthermore, the carries tree is of finite type if and only if τ is continuous.

Proof. By construction, $\omega(G)$ is not empty and compact. For $x \in \mathcal{K}_G$, let $m(x) = \max\{k\,;\, G_{k+1} - 1 \text{ is a prefix of } x\} \in \{-1, 0, \ldots, +\infty\}$. Clearly, τ is continuous at any point $x \in \tau^{-1}\{0^\omega\}$. If $x \notin \omega(G)$, then there is a cylinder C containing x and not intersecting $\omega(G)$, hence $m(x)$ is bounded on C. Therefore, τ is continous at x. If $x \in \omega(G)$, then $x = \lim(G_{n_k} - 1)$. If $\tau(x) \neq 0^\omega$, then τ is not continous at x, since $\tau(G_{n_k} - 1) = G_{n_k}$, which tends to zero in $\mathcal{K}_G$.

Let $x = \lim(G_{n_{k+1}} - 1)$. Then the characterisation follows from the equivalence of the following propositions:
• $G_{m(x)+1} - 1$ is a prefix of $G_{n_{k+1}} - 1$ for k large enough and $\lim D(n_k) = \infty$;
• $m(x)$ is finite;
• x is a point of discontinuity of τ. □

For a low scale, the points of discontinuity are exactly the $G_{n+1} - 1$ such that the node n is not of finite type.

6.3. **Metric properties.** ***Da capo al fine*** **subshifts.** If $(G_n)_n$ is a Cantor scale, the odometer is a translation on a compact group for which all orbits are dense. In particular, $(\mathcal{K}_G, \tau)$ is uniquely ergodic and minimal. In general, a natural question is whether there exists at least one τ-invariant measure on $\mathcal{K}_G$ and whether it is unique. Since $\mathcal{K}_G$ is compact, the theorem of Krylov-Bogoliubov asserts that there exists an invariant measure, provided that τ is continuous. But the question remains open without this assumption.

Results on invariant measures concerned with special families can be found in [216] for Ostrowski expansions (as in Example 6) and in [94] for linear recurrent numeration systems arrising from a simple beta-number (as in Example 6.1). For Ostrowski scales $(G_n)_n$ it is proved that the odometer is metrically isomorphic to a rotation, hence in particular uniquely ergodic. Furthermore, Vershik and Sidorov give the distribution of the coordinates and show that they form a non-homogenous Markov chain with explicit transitions.

For linear recurrent numeration systems, Grabner, Liardet and Tichy use the following characterisation of unique ergodicity: the means

$$(N^{-1} \sum_{m \leq n < m+N} f \circ \tau^n)_N$$

converge uniformly w.r.t. m for N tending to infinity for any continuous function $f\colon \mathcal{K}_G \to \mathbb{C}$. A standard application of Stone-Weierstraß theorem allows to consider only G-multiplicative functions f depending on finitely many coordinates (see Section 7.1 for the definition). A technical lemma reduces the problem again to the study of convergence of $(G_n^{-1}\sum_{k<G_n} f(k))_n$. However, $(F_n = \sum_{k<G_n} f(k))_n$ ultimately satisfies the same recurrence relation as $(G_n)_n$, from which one derives the desired convergence. The unique invariant probability measure is explicitely given.

Example 13. We consider the Zeckendorf expansion again (continuation of Example 10). The golden ratio is denoted by $\rho = (1+\sqrt{5})/2$, the unique τ invarariant measure on $\mathcal{K}_G$ is $\mathbf{P}$. By unique ergodicity of the odometer, if X_n is the n-th projection on the compactification, $X_n\colon \mathcal{K}_G \to \{0,1\}$, $X_n(x_0, x_1, \ldots) = x_n$, then

$$\begin{aligned}\mathbf{P}(X_n = 0) &= \lim_{s\to\infty} \frac{1}{F_{n+s}} \#\{k < F_{n+s}\,;\, \varepsilon_n(k) = 0\}\\ &= \lim_{s\to\infty} \frac{F_n F_{s-1}}{F_{n+s}} = \frac{F_n}{\rho^{n+1}}.\end{aligned}$$

Similarily (or by computing $1 - \mathbf{P}(X_n = 0)$), one finds $\mathbf{P}(X_n = 0) = F_{n-1}\rho^{-n-2}$ and the transition matrix is $\begin{pmatrix} 1/\rho & 1 \\ 1/\rho^2 & 0 \end{pmatrix}$. In particular, the sequence $(X_n)_n$ is an homogenous Markov chain. For the most general case of scales arising from a simple Parry beta-number of degree d, one gets an homogenous Markov chain of order $d-1$. We refer to [73] and [207] for more information, especially application to asympototic study of related arithmetical functions.

The arguments of [94] can be extended with technical difficulties to more general odometers. However, a quite different approach turned out to be more powerful. We expose it in the sequel.

From now on, $(G_n)_n$ is a G-scale and $(\mathcal{K}_G, \tau)$ the associated odometer. For $x \in \mathcal{K}_G$, we define a "valuation" $\nu(x) = \nu_G(x) = \min\{k; x_k \neq 0\}$ if $x \neq 0^\omega$ and $\nu(0^\omega) = \omega$. Note $\Lambda = \mathbb{N} \cup \{\omega\}$ the one point compactification of $\mathbb{N}$. The valuation yields a map $A_G\colon \mathcal{K}_G \to \Lambda^{\mathbb{N}}$ defined by $A_G(x) = (\nu(\tau^n x))_{n\geq 0}$ (for instance, $\nu(G_m) = m$). Let be $A_m = \nu(1)\nu(2)\cdots\nu(G_m - 1)$ and A the infinite word defined by the concatenation of the sequence $A_G(1)$. Then, for $n = \sum_{k\leq \ell} \varepsilon_k(n) G_n$, the prefix of length n of A is

$$(A_\ell \ell)^{\varepsilon_\ell(n)} (A_{\ell-1}(\ell-1))^{\varepsilon_{\ell-1}(n)} \cdots (A_0 0)^{\varepsilon_0(n)}. \tag{6.5}$$

Let (X_G, S) be the subshift associated to A and $X_G^{(0)} = X_G \cap \mathbb{N}^{\mathbb{N}}$.

Proposition 6.3 ([34], Proposition 2)**.** *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{K}_G & \xrightarrow{\tau} & \mathcal{K}_G \\ A_G \big\downarrow & & \big\downarrow A_G \\ X_G & \xrightarrow[S]{} & X_G \end{array} \tag{6.6}$$

The map A_G is borelian. It is continuous if and only if τ is. It induces a bijection between $\mathcal{K}_G^\infty = \mathcal{K}_G \setminus \mathcal{O}_{\mathbb{Z}}(0^\omega)$ and $X_G^{(0)}$, whose inverse map is continuous. If τ is continous, this bijection is an homeomorphism ($\mathcal{O}_{\mathbb{Z}}(0^\omega)$ is the two-sided orbit of 0^ω.)

The precise study of A_G is not simple. Some elements can be found in [34]. For example, the equality $A_G(\mathcal{K}_G)$ holds if and only if $\mathbb{N} \cap \omega(G) = \emptyset$.

The latter proposition has important consequences.

Corollary 6.1. *The quadruples $(\mathbb{N} \setminus \{0\}), \tau, \mathbb{N}, A_G)$ and $(\mathcal{K}_G^\infty, \tau, \mathbb{N}, A_G)$ are fibred numeration systems. In the sense of Definition 3.6 they own the same compactification X_G on which the shift operator acts. The dynamical systems $(\mathcal{K}_G, \tau)$ and (X_G, S) are metrically conjugated. In particular, A_G transports shift-invariant measures supported by $X_G^{(0)}$ to τ-invariant measures on $\mathcal{K}_G$.*

The subshift (X_G, S) is called the *valumeter*. It is a way to see as fibred something that is not fibred (the odometer). Somehow, the dynamical study of the odometer reduces to that of the valumeter, which is a more pleasing object - for instance, the shift operator is always continuous, even if the addition is not.

Proposition 6.4. *The following statements are equivalent:*
- *the word A is recurrent (each factor occurs infinitely often);*
- *the word A has a preimage in X_G by the shift;*
- *$X_G^{(0)}$ is not countable;*
- *$\mathcal{K}_G$ is a Cantor space.*

Furthermore, the valumeter is minimal if and only if the letter ω appears infinitely many times with bounded gaps in no element of X_G.

Example 14. If G is the scale of Example 12-3 (see also Example 17, *infra*), X_G contains the element $(\omega, \omega, \ldots)$. Then the valumeter is not minimal.

The most important difficulty in proving theorems on invariant measures on (X_G, S) comes from the fact that Λ is not discrete, but possesses one non-isolated point - ω. The usual techniques lie on $*$-weak compactness of the set of probability measures. Indeed, $*$-weak convergence of a sequence $(\mu_n)_n$ of probability measures defined on X_G expresses the convergence of the sequence $(\mu_n(U))_n$ for cylinders of the type $U = [a_1, \ldots, a_n]$, with $a_n \in \mathbb{N}$ or $a_n = \{m \in \Lambda\,;\, m \geq n_0\}$. But the relevant notion of convergence in that context takes in account cylinders $U = [a_1, \ldots, a_n]$, with $a_n \in \Lambda$. Hence one has to introduce a so-called *soft topology*, which is finer as the usual $*$-weak topology. However, the following results can be proved.

Theorem 6.1 ([34], Theorems 7& 8)**.** *1. If the series $\sum G_n^{-1}$ converges, then there exists a shift-invariant probability measure on X_G supported by $X_G^{(0)}$.*
2. If the sequence $(G_{n+1} - G_n)$ tends to infinity and if the sequence $m \mapsto G_m \sum_{k \geq m} G_k^{-1}$ is bounded, then $(X_G^{(0)}, S)$ is uniqueley ergodic, and $(\mathcal{K}_G, \tau)$ as well.
3. The odometer $(\mathcal{K}_G, \tau)$ has zero metric entropy with respect to any invariant measure. If τ is continous, it has zero topological entropy.

For instance, G-scales satisfying $1 < a < G_{n+1}/G_n < b < \infty$ for all n satisfy the second condition of Theorem 6.1. Example 9 of [34] gives an example of a continous odometer with several invariant measures. The construction below is not continuous, but more simple.

Example 15. Suppose that I, J and K realise a partition of $\mathbb{N}$, and assume that I and K have upper-density one. Define $G_{n+1} = G_n + 1$ if $n \in I$, $G_{n+1} = G_n + 2$ if $n \in K$ and $G_{n+1} = a_n G_n + 1$ for $n \in J$, where the a_n are choosen to make the seires $\sum G_n^{-1}$ convergent. Then $\omega(G) = \{0, 1\}$, and τ is discontinous at those both points. Furthermore, $N^{-1} \sum_{n<N} \delta_0 \circ \tau^n$ has at least two accumulation points. Hence there exist at least two τ-invariant measures.

Remark 6.1. Downarowicz calls (X_G, S) *da capo al fine* subshift.[7] Consider a triangular array of natural integers $(\varepsilon_j^{(m)})_{0 \leq j \leq m}$ such that $\varepsilon_j^{(j)} \geq 1$ for all j and

$$\varepsilon_j^{(m)} \varepsilon_{j-1}^{(m)} \cdots \varepsilon_0^{(m)} \leq_{\text{lex}} \varepsilon_j^{(j)} \varepsilon_{j-1}^{(j)} \cdots \varepsilon_0^{(j)}$$

for all $j \leq m$. Define recursively A_0 to be the empty word and

$$A_{m+1} = (A_m m)^{\varepsilon_m^{(m)}} \cdots (A_0 0)^{\varepsilon_0^{(m)}}.$$

The sequence of words $(A_m)_m$ converges to an infinite word A. This sequence is associated to the G-scale $(G_n)_n$ constructed recursively by $G_0 = 1$ and $G_{m+1} - 1 = \sum_{j \leq m} \varepsilon_j^{(m)} G_j$. The last two equations express (6.2) and (6.5) respectively. At each step, the song is played *da capo*, where the mark *fine* is set at the position $G_{m+1} - G_m - 1$. If this number is larger than G_m, the above formula instructs us to repeat periodically the entire song until the position $G_{m+1} - 1$ is reached (usually the last repetition is incomplete). In all cases, the note $m + 1$ is added at the end.

[7] "The expression *da capo al fine* is taken from the terminology of music. Having played the entire song the musicians must play it again from start (*da capo*) to a certain spot marked in the score as *fine*. [...] Following the musical convention, the elements of Λ will be called *notes*". [70]

6.4. **Markov's compacta.** Let $(r_n)_n$ a sequence of natural integers, $r_n \geq 2$ for all n, and a sequence of $0-1$ matrices $(M^{(n)})_n$, where $M^{(n)}$ is a $r_n \times r_{n+1}$ matrix. Build the Markov compactum

$$\mathcal{K}(M) = \left\{(x_0, x_1, \dots) \in \Pi(G)\,;\, \forall n \in \mathbb{N} : M^{(n)}_{x_n, x_{n+1}} = 1\right\}. \tag{6.7}$$

$\mathcal{K}(M)$ is an analog of non-stationary topological Markov chain. According to Vershik, the *adic transformation* $\mathcal{S}$ associates to $x \in \mathcal{K}(M)$ its successor with respect to the antipodal order. Then Vershik has proved in [215] the following theorem (see also [79] for related results)

Theorem 6.2. *Any ergodic automorphism of a Lebesgue space is metrically isomorphic to some adic transformation.*

Unfortunately, the isomorphism is not explicit. If $(G_n)_n$ is a Cantor scale (with notation of Example 9), then the odometer $(\mathcal{K}_G, \tau)$ is an adic transformation, where $M(n)$ is the $q_n \times q_{n+1}$ matrix containing only 1's (lines and columns being indexed from 0 included). Odometers that are adic transformations are not difficult to characterise: they correspond to scales $(G_n)_n$ where the expansions of $G_n - 1$ satisfy

$$G_{n+2} - 1 = \begin{cases} a_{n+1}G_{n+1} + a'nG_n + (G_n - 1) & \text{if } a'_n < a_n; \\ a_{n+1}G_{n+1} + (G_{n+1} - 1) & \text{if } a'_n = a_n, \end{cases}$$

with initial condition $G_1 = a_0 + 1$. The transition matrices $M^{(n)}$ have $a_n + 1$ rows and $a_{n+1} + 1$ columns and have zero coefficients $m_{i,j}$ if and only if $j = a_{n+1}$ and $i > a'_n$.

Assume now that the odometer does not coincide with an adic transformation. In some simple cases (Multinacci for instance), there is a simple isomorphism with such a dynamical system.

Example 16. Consider the scale of Example 12-3. The Markov compactum $\mathcal{K}(M)$ build form the square $q+1$-dimensional matrices with $m_{i,j} = 0$ if $j = q$ and $i \neq q$ is formed by sequences $q^k \varepsilon_k \varepsilon_{k+1} \cdots$ with $0 \leq \varepsilon_j < q$. The map $\psi\colon \mathcal{K}(M) \to \mathcal{K}_G$ defined by

$$\psi(0^\omega) = 0^\omega \quad \text{and} \quad \psi(q^k \varepsilon_k \varepsilon_{k+1} \cdots) = 0^{k-1} q \varepsilon_k \varepsilon_{k+1} \cdots$$

realises an isomorphism between the odometer and the adic system.

6.5. **Spectral properties.** In general , the spectral structure of an odometer associated to a given scale $G = (G_n)_{n \geq 0}$, is far from to be elucidated. We present in this sub-section both old and recent results. As above, let $(\mathcal{K}_G, \tau)$ be the G-odometer. We assume that the sequence $n \to G_{n+1} - G_n$ is unbounded, so that K_G is a Cantor set according to Proposition 6.1 . The first step is to identify odometers that admit an invariant measure μ. Results are recalled in Theorem 6.1. The next question is to characterise G-odometers which have a non-trivial eigenvalue and we keep in mind a basic result due to Halmos which asserts that the family of ergodic dynamical systems with discrete spectrum coincides with the family of dynamical systems with are, up to an isomorphism, translations on compact abelian group with a dense orbit (and hence are ergodic).2.3.1. We look at examples.

In base q (see 1), the odometer corresponds to the scale $G_n = q^n$ and is nothing but the translation $x \mapsto x+1$ on the compact group of q-adic integers. The situation is analogous for the Cantor scales (see 9) for which the q-adic integers is replaced by very similar groups exposed *supra*.

Example 17. The scale given by $G_{n+1} = qG_n - 1$ (q integer) is the first example of family of weak mixing odometer. Furthermore, those odometers are measure-theoretically isomorphic to a rank one transformation of the unit interval, the transformation being constructed by a cutting-stacking method [69].

In case $(G_n)_n$ is given by a finite homogeneous linear recurrence coming from a simple Parry number as in Example 6.1 the odometer is continuous, uniquely ergodic but a few is known about its spectral properties. The following sufficient condition is given in [94]. Let us say that the odometer satisfies hypothesis (B) if there exists an integer $b > 0$ such that for all k and integer N with G-expansion

$$\varepsilon_0(N) \cdots \varepsilon_k(N) 0^{b+1} \varepsilon_{k+b+2}(N) \cdots ,$$

addition of G_m to N (with $m \geq k+b+2$) does not change the digits $\varepsilon_0(N), \ldots, \varepsilon_k(N)$. Then the odometer is measure-theoretically isomorphic to a group rotation whose pure discret spectrum is the group

$$\{z \in \mathbb{C}\,;\ \lim_n z^{G_n} = 1\}.$$

This result applies in particular to the Multinacci scale. For the Fibonacci numeration system, the conjugation map which turn the odometer into the translation on $\mathbb{T}$ by the golden number is exhibited in (see [216, 32]).

Example 18. The study of wild attractors involves some nice odometers. In particular the following scales $G_{n+1} = G_n + G_{n-d}$ are investigated in [54]. where it is proved that for $d \geq 4$, the odometer is weakly mixing, but not mixing (Theorem 3). Using results of Host and Mauduit, the authors prove that the non trivial eigenvalues $e^{2\pi i\rho}$ (if they were any) are such that ρ is irrational or would belong to $\mathbb{Q}(\lambda)$ for any root λ of the characteristic polynomial P of the recurrence with modulus at least 1. They first treat the case $d \equiv 4 \pmod 6$, for which $(x^2 - x + 1) \mid P$, hence a contradiction. The case $d \not\equiv 4 \pmod 6$ is more complicated, since P is then irreducible. But the Galois group of P is the whole symetric group $\mathfrak{S}_{d+1}$ in that case and they can apply an argument of [205].

7. Applications

7.1. **Sums of digits.** For a numeration system $G = (G_n)_n$ and the corresponding digits ϵ_n, an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called G-additive (and $g : \mathbb{N} \to \mathbb{C}$ is called G-multiplicative), if $f(0) = 0$, $(g(0) = 1)$, and if
(7.1)
$$f\left(\sum_{k=0}^{\infty} \varepsilon_k(n) G_k\right) = \sum_{k=0}^{\infty} f\left(\varepsilon_k(n) G_k\right); \quad g\left(\sum_{k=0}^{\infty} \varepsilon_k(n) G_k\right) = \prod_{k=0}^{\infty} g\left(\varepsilon_k(n) G_k\right).$$

The most popular additive function is the sum-of-digits defined by $s_G(n)(n) = \sum_k \varepsilon_k(n)$. A less immediate example of q-multiplicative function is given by the Walsh functions (for $G_n = q^n$): for $x \in \mathbb{Z}_q$, $w_x(n) = \prod_k e(q^{-1} x_k \varepsilon_k(n))$, where $e(x) = \exp(2\pi i x)$. In fact, those functions w_x are the characters of the additive group $\mathbb{N}_q$ where the addition is done in base q ignoring carries. The following result is due to Mendès-France [158].

Proposition 7.1. *If $x \in \mathbb{Z}_q \setminus \mathbb{N}$, the Walsh character w_x is pseudo-random (that is, its spectral measure is continuous), but it is not pseudo-random in the sense of Bass (that is its correlation function do not converge to 0 at infinity).*

The sum-of-digits function has been extensively studied and it is not our purpose to expose the results on it. We only mention two results and then give a few examples where the dynamics plays a rôle. In 1948, Bellman & Shapiro [38] proved the asymptotics
$$\sum_{n<N} s_q(n) \sim \frac{q-1}{2} N \log_q N.$$
Later, Delange expressed the error term as $NF(\log_q N)$, where F is a 1-periodic function, continous and nowhere differentiable, and expressed its Fourier coefficients in terms of the Riemann zeta function.

Very recently, Mauduit & Rivat [157] solved a long standing conjecture of Gelfond by proving that the sum-of-digit s_q is uniformly distributed in the residue classes mod m, $(m, q) = 1$ along the primes.

In [67] Delange proved that a real-valued q-additive function f (that is for $G_n = q^n$) admits an asymptotic distribution function (that is: the sequence of measures $(N^{-1} \sum_{n<N} \delta_{f(n)})_N$ converges weakly to a probability measure), if and only if the two series
$$\sum_{j=0}^{\infty} \left(\sum_{\varepsilon=0}^{q-1} f(\varepsilon q^j)\right) \quad \text{and} \quad \sum_{j=0}^{\infty} \sum_{\varepsilon=0}^{q-1} f(\varepsilon q^j)^2 \tag{7.2}$$
converge. Delange used the characterisation of weak convergence due to Lévy: the sequence of characteristic functions converges pointwise to a function, which has to be continuous at 0. Therefore, the most important part

of the proof deals with estimates of means of q-multiplicative functions g. Let $M_N(g) = N^{-1}\sum_{n<N} g(n)$. Then

$$M_{q^\ell}(g) = \prod_{j<\ell}\sum_{\varepsilon<q} f(\varepsilon q^k).$$

A typical result proved by Delange is $M_N(g) - M_{q^n}(g) = o(1)$ for $q^n \leq N < q^{n+1}$. This has been generalised to further systems of numeration by Coquet and others. For Cantor numeration systems, this result is not true in general. For example, if $G_n = n^2 G_{n-1}$ and $g(\varepsilon G_n) = -1$ whenever $\varepsilon = 1$ and 1 otherwise, one checks that $(M_{G_n})_n$ converges to some positive constant although $(M_{2G_n})_n$ tends to zero. Using a clever martingale argument, Mauclaire proved the following:

Theorem 7.1. [156] *Assume $(G_n)_n$ is a Cantor numeration system with compactification $\mathcal{K}_G$. Recall that $\mathcal{K}_G = \mathbb{Z}_G$ is a profinite group. For $x = \sum_{k=0}^{\infty} x_k G_k \in \mathbb{Z}_G$, let $x_n = \sum_{k=0}^{n} x_k G_k$. Then*

$$\frac{1}{x_n}\sum_{k=0}^{x_n-1} g(k) - \frac{1}{G_n}\sum_{k=0}^{G_n-1} g(k) = o(1).$$

holds for almost all $x \in \mathbb{Z}_G$ with respect to Haar measure, when n tends to infinity.

Barat and Grabner[31] observed that if f is a real-valued q-additive function and $f_n\colon \mathbb{Z}_q \to \mathbb{R}$ defined by $f_n(\sum x_k q^k) = f(x_n q^n)$, then the conditions (7.2) can be rewritten as the convergence of $\sum \mathbb{E}(f_n)$ and $\sum \mathbb{E}(f_n^2)$, which is indeed equivalent to the convergence of $\sum \mathbb{E}(f_n)$ and $\sum \sigma^2(f_n^2)$ by $f_n(0) = 0$ and Cauchy-Schartz inequality. Since the random variables f_n are independant and bounded, further equivalent conditions to (7.2) are almost sure convergence of the series $\sum f_n$ (Kolmogorov's three series theorem) and convergence in distribution of the same series. Finitely, convergence in distribution of $\sum f_n$ is by definition the weak convergence of the sequence $(q^{-N}\sum_{n<q^N}\delta_{f(n)})_N$ to a probability measure.

After this analysis of the problem, one of the implication in Delange's theorem is trivial. The convers assumes the almost sure convergence of $f = \sum f_n$; it uses basic ergodic theory aroung Birkhoff's theorem an is about 10 times shorter than the original argument. The whole procedure applies to more general numeration systems, even if the lack of independance for the f_n makes the work more involved. In the same direction, Manstavicius developped in [155] a Kubilius model for G-additive functions w.r.t. Cantor numeration systems.

Given a numeration scale $(G_n)_n$ and a unimodular G-multiplicative function g, several authors have investigated the subshift associated with the sequence $(g(n))_n$, that we denote $\mathcal{F}(g)$. The first results were concerned with q-adic numeration systems and functions of the type $g(n) = e(\alpha s_q(n))$ and are due to Kamae.

Recently, Barat and Liardet[32] considered the case of the Ostrowski numeration and arbitrary G-multiplicative functions with values in the unit circle $\mathbb{U} \subset \mathbb{C}$. The odometer plays a key rôle in the whole study. It is first proved that if $\Delta g(n) = g(n+1)g(n)^{-1}$ is the first backward difference sequence, then the subshift $\mathcal{F}(\Delta g)$ is either constant or almost topological isomorph to the odometer $(\mathcal{K}_\alpha, \tau)$. An useful proper ty of the sequence Δg is that it extends continuously (up to a countable set) to the whole compact space $\mathcal{K}_\alpha$. Furthermore, the subset $G_1(g)$ of $\mathbb{U}$ of topological essential values is defined as the decreasing intersection of the sets $\overline{g([0^n])}$. It turns out that $G_1(g)$ is a group containing the group of the essential values $E(\Delta g)$ of Klaus Schmidt and that $E(\Delta g) = G_1(g)$ if and only if $\mathcal{F}(g)$ is uniquely ergodic. The topological essential values are also characterised in terms of the compactification: for a character $\chi \in \hat{\mathbb{U}}$, the restriction of χ to $G_1(g)$ is trivial if and only if $\chi \circ g$ extends continuously to $\mathcal{K}_\alpha$. In general, $\mathcal{F}(g)$ is topological isomorph to a skew product $\mathcal{F}(\Delta g) \times G_1(\zeta)$. In case of unique ergodicity, a consequence is the well uniformly distribution of the sequence $(g(n))_n$ in $\mathbb{U}$. More precise results had been obtained by Lesigne and Mauduit in the case of q-adic numeration.[149]

Dumont-Thomas numeration has been introduced in [74, 75] in order to get asymptotic estimations of summatory functions of the form $\sum_{1\le n\le N} f(u_n)$, where (u_n) is a one-sided fixed point of a substitution over the finite alphabet $\mathcal{A}$, and f is a map defined on $\mathcal{A}$ with values in $\mathbb{R}$. These estimates are deduced from the self-similarity properties of the substitution via the Dumont-Thomas numeration, and are shown to behave like sum-of-digits functions with weights provided by the derivative of f. For some particular cases of substitutions, such as constant length substitutions, one recovers classical summatory functions associated, e.g., to the number of 1's in the binary expansion of n (consider the Thue-Morse substitution), or the number of 11's (consider the Rudin-Shapiro substitution). For more details, see [76] and the references in [74, 18].

In the same flavour, [177] studies the asymptotic behaviour of

$$\sum_{1\le n\le N} \mathbf{1}_{[0,1/2]}(\{n\alpha\}),$$

where $\mathbf{1}_{[0,1/2]}$ is the indicator function of the interval $[0, 1/2]$, for algebraic/quadratic values of α. The strategy consists in introducing a fixed point of a substitution and considering orbits in some dynamical systems.

7.2. **Diophantine approximation.** We review here some applications of Rauzy fractals (see Section 5.3) associated to Pisot β-numeration and Pisot substitutions. They have indeed many applications in arithmetics; this was one of the motivations for their introduction by G. Rauzy [174, 176, 177].

A subset A of the d-dimensional torus $\mathbb{T}^d$ with (Lebesgue) measure $\mu(A)$ is said to be a *bounded remainder set* for the minimal translation $R_\alpha : x \mapsto x + \alpha$, defined on $\mathbb{T}^d$, if there exists $C > 0$ such that

$$\forall N \in \mathbb{N}, \quad |\mathrm{Card}\{0 \leq n < N;\ n\alpha \in A\} - N\mu(A)| \leq C.$$

When $d = 1$, an interval of $\mathbb{R}/\mathbb{Z}$ is a bounded remainder set if and only its length belongs to $\alpha\mathbb{Z}+\mathbb{Z}$ [98]. In the higher-dimensional case, it is proved in [150] that there are no nontrivial rectangles which are bounded remainder sets for ergodic translations on the torus [150]. Rauzy fractals associated either to a Pisot unimodular substitution or to a Pisot unit β-numeration provide efficient ways to construct bounded remainder sets for toral translations, provided that we have discrete spectrum. Assume for instance that $\mathcal{N}$ is the FNS based on Dumont-Thomas numeration, for σ Pisot unimodular substitution, and that $\mathcal{T}_\mathcal{N}$ is a fundamental domain of the torus. The idea is that the subpieces $\mathcal{T}_\mathcal{N}(a)$, for $a \in \mathcal{A}$, of the Rauzy fractal satisfy the following: the first return map of the translation R_α on the pieces $\mathcal{R}_\mathcal{N}(a)$ is again a rotation, by self-similarity of the subtsitution. Then, according to [84, 175], if the induced map of a translation on a set A is still a translation, then this set A is a bounded remainder set.

A second application in Diophantine approximation consists in exhibiting sequences of best approximations. Let β denote the Tribonacci number, that is, the real root of $X^3 - X^2 - X - 1$. The Tribonacci sequence $(T_n)_{n\in\mathbb{N}}$ is defined as: $T_0 = 1, T_1 = 2, T_2 = 4$ and for all $n \in \mathbb{N}$, $T_{n+3} = T_{n+2}+T_{n+1}+T_n$. It is proved in [62] (though this was probably already known to Rauzy) that the rational numbers $(T_n/T_{n+1}, T_{n-1}/T_{n+1})$ provide the best possible simultaneous approximation of $(1/\beta, 1/\beta^2)$ if we use the distance to the nearest integer defined by a particular norm, the so-called Rauzy norm; recall that if $\mathbb{R}^d$ is endowed with the norm $||\,.||$, and if $\theta \in \mathbb{T}^d$, then an integer $q \geq 1$ is a best approximation of θ if $|||q\theta||| < |||k\theta|||$ for all $1 \leq k \leq q-1$. Furthermore, the best possible constant

$$\inf\{c\,;\ q^{1/2}|||q\left(1/\beta, 1/\beta^2\right)||| < c \quad \text{for infinitely many} \quad q\}$$

is proved in [62] to be equal to $(\beta^2 + 2\beta + 3)^{-1/2}$. See also [110] for closely related results on a class of cubic parameters.

Let α irrational in $\mathbb{R}$. The *local star discrepancy* for the Kronecker sequence $(n\alpha)_{n\in\mathbb{N}}$ is defined as $\Delta_N^*(\alpha, \beta) = |\sum_{n=0}^{N-1} \chi_{[0,\beta[}(\{n\alpha\}) - N\beta|$, whereas the *star discrepancy* is defined as $D_N^*(\alpha) = \sup_{0<\beta<1} \Delta_N^*(\alpha, \beta)$. Most of the discrepancy results concerning Kronecker sequences were obtained by using the Ostrowski numeration system (see Example 6); for more details and references, we refer to [139] and [72]. A similar approach has been developed in

[2] where an algorithm is proposed, based on Dumont-Thomas numeration, which computes $\limsup \frac{\Delta_n^*(\alpha,\beta)}{\log n}$ when α is a quadratic number and $\beta \in \mathbb{Q}(\alpha)$

7.3. **Applications in physics: Rauzy fractals and quasicrystals.** An important issue in β-numeration deals with topological properties of the set $\mathbb{Z}_\beta = \{\pm w_M \beta^M + \cdots + w_0;\ M \in \mathbb{N},\ (w_M \cdots w_0) \in \mathcal{L}\}$, where $\mathcal{L}$ denotes the language of the β-numeration.

A set $X \subset \mathbb{R}^n$ is said to be *uniformly discrete* if there exists a positive real number r such that for any $x \in X$, the open ball located at x of radius r contains *at most* one point of X; a set $X \subset \mathbb{R}^n$ is said *relatively dense* it there exists a positive real number R such that for any x in $\mathbb{R}^n$, the open ball located at x of radius R contains *at least* one point of X. A subset of $\mathbb{R}^n$ is a *Delaunay set* if it is uniformly discrete and relatively dense. A Delaunay set is a *Meyer set* if $X - X$ is also a Delaunay set if there exists a finite set F such that $X - X \subset X + F$ [162, 163]. This endows a Meyer set with a structure of "quasi-lattice": Meyer sets play indeed the role of the lattices in the theory of crystalline structure: A Meyer set [162, 163] is in fact a mathematical model for quasicrystals [164, 27].

If β is a Parry number, then $\mathbb{Z}_\beta$ is a Delaunay set [210]. More can be said when β is a Pisot number. Indeed it is proved in [59], that β is a Pisot number, then $\mathbb{Z}_\beta$ is a Meyer set. For some families of β (mainly Pisot quadratic units), an internal law can even be produced formalizing the quasi-stability of $\mathbb{Z}_\beta$ under subtraction and multiplication [59]. Beta-numeration reveals itself as a very efficient and promising tool for the modeling of families of quasicrystals thanks to beta-grids [59, 58, 214].

An important issue is to characterise those β for which $\mathbb{Z}_\beta$ is uniformly discrete or even a Meyer set. Observe that $\mathbb{Z}_\beta$ is at least always a discrete set. It can easily be seen that $\mathbb{Z}_\beta$ is uniformly discrete if and only if the β-shift X_β is specified, that is, if the strings of zeros in $d_\beta(1)$ have bounded lengths; let us observe that the set of specified real numbers $\beta > 1$ with a noneventually periodic $d_\beta(1)$ has Hausdorff dimension 1 according to [183]; for more details, see for instance [50] and the discussion in [214]. Let us note that if $\mathbb{Z}_\beta$ is a Meyer set, then β is a Pisot or a Salem number [163].

Let us note that a proof of β is Pisot number implies $\mathbb{Z}_\beta$ is a Meyer set is given in [214] by exhibiting a cut-and-project scheme. A *cut and project scheme* consists of a direct product $\mathbb{R}^k \times H$, $k \geq 1$, where H is a locally compact abelian group, and a lattice D in $\mathbb{R}^k \times H$, such that with respect to the natural projections $p_0 :\ \mathbb{R}^k \times H \to H$ and $p_1 :\ \mathbb{R}^k \times H \to \mathbb{R}^k$:

(1) $p_0(D)$ is dense in H;
(2) p_1 restricted to D is one-to-one onto its image $p_1(D)$.

This cut and project scheme is denoted by $(\mathbb{R}^k \times H, D)$. A subset Γ of $\mathbb{R}^k$ is a *model set* if there exists a cut and project scheme $(\mathbb{R}^k \times H, D)$ and a relatively compact set (i.e., a set the closure of which is compact) Ω of H

with non-empty interior such that

$$\Gamma = \{p_1(P);\ P \in D,\ p_0(P) \in \Omega\}.$$

The set Γ is called the *acceptance window* of the cut and project sheme. Meyer sets are proved to be the subsets of model set of $\mathbb{R}^k$, for some $k \geq 1$, that are relatively dense [162, 163, 164]. For more details, see for instance [27, 148, 144, 194].

There are close connections between such a generation process for quasicrystals and lattice tilings for Pisot unimodular substitutions; see for instance [217, 218]. We assume that σ is a unimodular Pisot substitution over a d-letter alphabet. Let π' stands for the projection in $\mathbb{R}^d$ on the expanding line generated of the incidence matrix $\mathbf{M}_\sigma$ along its contracting space $\mathbb{H}_c$, whereas we denote by π the projection on the contracting space $\mathbb{H}_c$ along the expanding line. Let u be a periodic point of σ. The broken line associated to u is defined as $\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\}$, where $\mathbf{l}$ stands for the abelianization map. We assume that the projections by π of the vertices of the broken line belong to the interior of the Rauzy fractal. The subset $\pi'(\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\})$ of the expanding eigenline obtained by projecting the vertices of the broken line given by a periodic point of the substitution σ is then a Meyer set associated with the cut and project scheme $(\mathbb{R} \times \mathbb{R}^{n-1}, \mathbb{Z}^n)$, whose acceptance window is the interior of the Rauzy fractal $\mathcal{T}_\mathcal{N}$. In other words,

$$\begin{aligned}&\{\mathbf{l}(u_0 \cdots u_{N-1});\ N \in \mathbb{N}\} = \\ &\qquad = \{P = (x_1, \cdots, x_n) \in \mathbb{Z}^n;\ \sum_{1 \leq i \leq n} x_i \geq 0; \pi(P) \in \mathrm{Int}(\mathcal{T}_\mathcal{N})\},\end{aligned}$$

that is, the vertices of the broken line are exactly the points of $\mathbb{Z}^n$ selected by shifting the central tile $\mathcal{T}_\mathcal{N}$ (considered as an acceptance window), along the expanding eigendirection [42].

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