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# The Categorical Product of two 5-chromatic digraphs can be 3-chromatic.

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## Abstract

We provide an example of a 5-chromatic oriented graph  $D$  such that the categorical product of  $D$  and  $TT_5$  is 3-chromatic, where  $TT_5$  is the transitive tournament on 5 vertices.

The transitive tournament  $TT_5$  is the digraph on the vertex set  $\{1, 2, 3, 4, 5\}$  and edge set  $\{(i, j) : i < j\}$ . Let  $D_1 = (V_1, E_1)$  and  $D_2 = (V_2, E_2)$  be two digraphs. The *categorical product* of  $D_1$  and  $D_2$  is the digraph  $D_1 \times D_2 = (V_1 \times V_2, E)$  where  $E = \{((x, y), (z, t)) : (x, z) \in E_1 \text{ and } (y, t) \in E_2\}$ . Motivated by Hedetniemi's conjecture (see Sauer [5] for a survey), the following function was defined:

$$\delta(n) := \min\{\chi(D_1 \times D_2) : \chi(D_1) = \chi(D_2) = n \text{ and } D_1, D_2 \in \mathcal{D}\}$$

where  $\mathcal{D}$  is the class of finite digraphs and  $\chi(D)$  is the chromatic number of  $D$ . The first striking result concerning this function was proved by Poljak and Rödl [4]: either  $\delta$  is bounded by 4 or it tends to infinity. Later on, Poljak [3], and independently Zhu [6] proved that 4 can be replaced by 3. When  $\delta$  is restricted to undirected graphs, Hedetniemi [2] conjectured that  $\delta(n) = n$ . El-Zahar and Sauer proved in [1] that  $\delta(4) = 4$  for undirected graphs. For directed graphs, it is known that  $\delta(4) = \delta(3) = 3$ .

**Theorem 1**  $\delta(5) = 3$

**Proof.** The oriented graph  $D$  depicted in Fig. 1 is 5-chromatic. Observe for this that the twelve vertices with degree 8 induce a graph  $G$  which is uniquely 4 colorable up to a rotation. Indeed the stability of  $G$  is 3 and the color classes must consist of three consecutive vertices in the cyclic order. Now, one of the three vertices with degree 4 has to have one neighbour in each of the four color classes. Thus  $D$  is not 4-colorable. The 3-coloration  $f$  of  $D \times TT_5$  is given by the label of the vertices of  $D$ : when a vertex  $x$  of  $D$  is labelled by  $c^3a^2$ , for instance, we mean that  $f(x, 1) = c, f(x, 2) = c, f(x, 3) = c, f(x, 4) = a, f(x, 5) = a$ . To see that  $f$  is a good coloration, note that when there is an edge from  $x$  to  $y$  (both seen as words of length 5 on the alphabet  $\{a, b, c\}$ ), the  $i^{\text{th}}$  letter of  $x$  is not equal to the  $j^{\text{th}}$  letter of  $y$  for every  $j > i$ .  $\square$

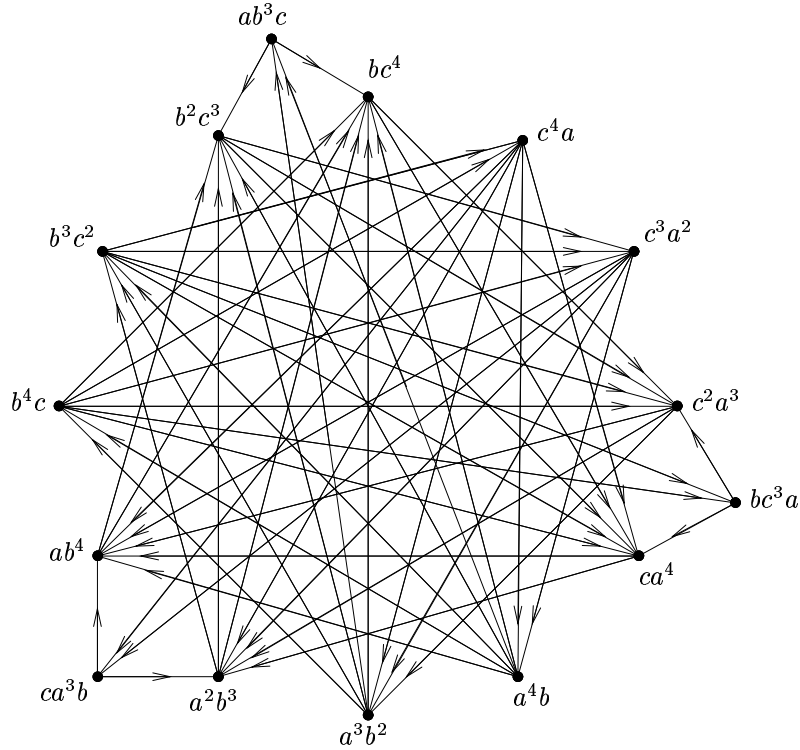


Fig. 1

This example is a subgraph of the graph  $K_3^{TT^5}$  which has chromatic number 5 (this is not hard to check). A short analysis also gives that  $\chi(K_3^{TT^4}) = 6$  and  $\chi(K_3^{TT^k}) = 4$  for all  $k > 5$ . We were not able to find an infinite family of 5-chromatic digraphs  $(D_1, D_2)$  such that  $\chi(D_1 \times D_2) = 3$ .

## References

- [1] E. El-Zahar and N. Sauer, The chromatic number of the product of two 4-chromatic graphs is 4, *Combinatorica*, **5** (1985), 121-126.
- [2] S.T. Hedetniemi, Homomorphisms and graph automata, University of Michigan Technical Report, **03105-44-T** (1966).
- [3] S. Poljak, Coloring digraphs by iterated antichains. *Comment. Math. Univ. Carolin.*, **32** (1991), 209-212.
- [4] S. Poljak and V. Rödl, On the arc-chromatic number of a digraph, *J. Combin. Theory Ser. B*, **31** (1981), 190-198.
- [5] N. Sauer, Hedetniemi's conjecture - a survey, preprint.
- [6] X. Zhu, Multiplicative Structures, Ph. D. thesis (1990), The university of Calgary.