Local Rule Substitutions and Stepped Surfaces

Thomas Fernique

Local Rule Substitutions and Stepped Surfaces

Thomas Fernique

LIRMM CNRS-UMR 5506 and Université Montpellier II, 161 rue Ada, 34392 Montpellier Cedex 05, France

Abstract

Substitutions on words, *i.e.*, non-erasing morphisms of the free monoid, are simple combinatorial objects which produce infinite words by replacing iteratively letters by words. This paper introduces a notion of substitution acting on multi-dimensional words, namely the *local rule substitutions*. Roughly speaking, *local rules* play for multi-dimensional words the role played by the concatenation product for substitutions on words. We then particularly focus on the local rule substitutions which act on the 2-dimensional words coding *stepped surfaces*, and we show that a wide class of them can be derived from *generalized substitutions*.

Key words: Multi-dimensional word, generalized substitution, stepped surface.

Introduction

A substitution acts on a word in this way: the image of each letter is a word, and the image of the whole word is the concatenation of images of its letters. Substitutions are powerful combinatorical tools and have natural interactions with automata theory, language theory, number theory *etc.* (see [12] and references inside). An extension of the notion of substitution to the more general framework of multi-dimensional words, *i.e.*, words with letters indexed by $\mathbb{Z}^d$ instead of $\mathbb{N}$ for classic words, would be interesting, in particular in regards to tilings, quasicrystals or discrete geometry.

One could define a map which does not act on letters of a multi-dimensional word but on the *boundaries* of this word, as done *e.g.*, in [5–7], but here we

Email address: `fernique@lirmm.fr` (Thomas Fernique).

would like to directly deal with letters. More precisely, we would like to define a map from multi-dimensional letters to multidimensional words, such that this map can be extended to multi-dimensional words. However, contrarily to the case of classic substitutions on words, we cannot map a letter L' "after" a letter L onto a multi-dimensional word $\sigma(L')$ "after" the multi-dimensional word $\sigma(L)$, since the term "after" does not make any more sense. For this reason, we rely in this paper on so-called *local rules*. Local rules were introduced on an example in [3], and are closely related to the notion of *combinatorial substitution* defined in terms of graphs in [11], which extends the classic notions of substitution rules used by physicists. A set of local rules defines the way the images of two adjacent letters (*i.e.*, multi-dimensional words) are placed each relatively to the other. This is a local definition, which naturally yields a corresponding algorithm, and is thus convenient for effective computations. However, the main problem is that such a local definition does not clearly yield a *consistent* global definition, since given local rules can be used in different ways for computing the image of a word; it is not obvious to find non-trivial examples of local rules acting on large sets of multi-dimensional words.

Our two main results (besides all the formalism introduced for local rules) are the following. First, Theorem 1 provides a way to obtain consistent local rules, by derivation from a global rule. Second, Theorem 2 shows that *generalized substitutions*, introduced in [4] and studied *e.g.* in [2,3,8,9,12], can be seen as global rules on multi-dimensional words coding *stepped surfaces*. This provides in particular a wide class of consistent local rules, by Theorem 1.

The rest of the paper is organized as follows. Section 1 introduces the formalism, particularly local rules and the way they defines, when they are consistent, a *local rule substitution*. Section 2 then gives two examples, namely classic substitutions on words and *uniform-shape substitutions*, where consistent local rules can be easily obtained. Then, Section 3 is more specifically devoted to the problem of consistency of local rules. We define global rules and prove Theorem 1. Last, we briefly review in Section 4 the notions of *stepped surfaces* and *generalized substitutions*, and we prove Theorem 2 in Section 5.

1 Local rule substitutions

1.1 Multi-dimensional pointed and non-pointed words

Let $\mathcal{A}$ be a finite alphabet. A *d-dimensional pointed letter* is an element $L = (\vec{x}, l)$ of $\mathbb{Z}^d \times \mathcal{A}$, with $\vec{x}$ being the *support* of the letter l. The set of

d-dimensional pointed letters is denoted by $\mathcal{L}_d$. A *d-dimensional pointed word* P is a set of d-dimensional pointed letters with distinct supports, and the *support* of P is defined as the set of supports of its letters. The set of d-dimensional pointed words is denoted by $\mathcal{P}_d$.

The lattice $\mathbb{Z}^d$ acts by translation on the supports of d-dimensional pointed letters and words. Cosets of this action are called *d-dimensional non-pointed letters and words*, and respectively denoted by $\overline{\mathcal{L}}_d$ and $\overline{\mathcal{P}}_d$, with the coset of a d-dimensional pointed letter L (resp. word P) being denoted by $\overline{L}$ (resp. $\overline{P}$). One thus has:

$$\forall P,\ P' \in \mathcal{P}_d, \qquad \overline{P} = \overline{P'} \ \Leftrightarrow\ \exists \vec{x} \in \mathbb{Z}^d \text{ such that } P' = \vec{x} + P,$$

and in such a case, we denote the vector $\vec{x}$ by $\vec{v}(P, P')$.

In all that follows, we omit to mention the dimension d when it is not necessary. Note that d-dimensional words are not necessarily connected, although we will further introduce a similar condition (see Def. 7).

1.2 Local rules

A classic substitution on words σ is defined on letters of a finite alphabet $\mathcal{A}$: it maps them onto non-empty words. Then, σ acts on words over $\mathcal{A}$ according to the rule $\sigma(u\cdot v) = \sigma(u)\cdot \sigma(v)$, where $\cdot$ is the concatenation product.

By analogy, a "d-dimensional substitution" should be defined on d-dimensional letters of a finite alphabet and should map them onto non-empty d-dimensional words over this alphabet. Then, it should act on d-dimensional words according to a rule which would play the role of a "d-dimensional concatenation product". The aim of this section is to define such d-dimensional substitutions.

We first define our d-dimensional substitutions, which map non-pointed letters onto non-empty non-pointed words:

Definition 1 Non-pointed substitutions *are non-erasing maps from $\overline{\mathcal{L}}_d$ to $\overline{\mathcal{P}}_d$.*

We then introduce *local rules*:

Definition 2 Local rules *for a non-pointed substitution $\overline{\sigma}$ are of two types:*

(1) an initial rule *λ^* maps a pointed letter $I(\lambda^*) = L$ onto a pointed word*

such that $\overline{\lambda^*(L)} = \overline{\sigma}(\overline{L})$*;*

(2) an extension rule λ *is defined on a set* $E(\lambda) = \{L, L'\}$ *of two pointed letters with distinct supports, which are mapped onto pointed words with distinct supports, such that* $\overline{\lambda(L)} = \overline{\sigma}(\overline{L})$ *and* $\overline{\lambda(L')} = \overline{\sigma}(\overline{L'})$*.*

Roughly speaking, an initial rule tells how to map, out of context, a first pointed letter, and extension rules then tell how to map, one relatively to the other, the following pointed letters (see Fig. 1).

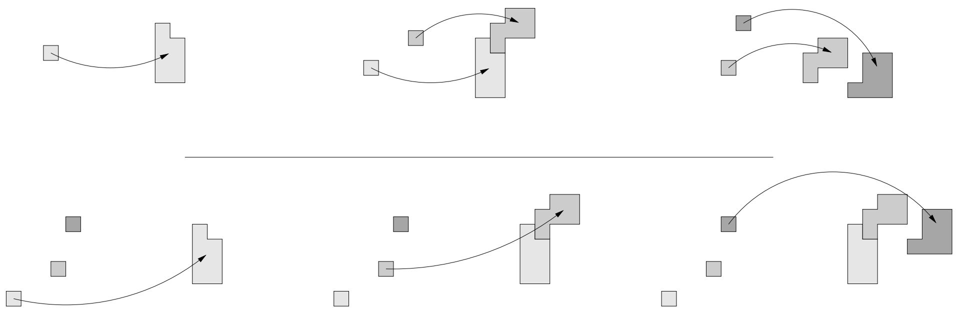

Fig. 1. An initial rule and two extension rules (top, from left to right); the computation, step by step, of the image of a pointed word by these three local rules (bottom, from left to right).

More precisely:

Definition 3 (Local rule substitution) *Let* P *be a pointed word and* Λ *be a finite set of local rules for a non-pointed substitution* $\overline{\sigma}$*.*
Let (P_n) *be the sequence of pointed words defined by:*

$$P_0 = \{L \in P \ s.t.\ \exists \lambda^* \in \Lambda,\ I(\lambda^*) = L\},$$

$$P_{n+1} = \{L' \in P \ s.t.\ \exists (L, \vec{x}, \lambda) \in P_n \times \mathbb{Z}^d \times \Lambda,\ E(\lambda) = \{\vec{x} + L, \vec{x} + L'\}\}.$$

We then define an action of $\overline{\sigma}$ *endowed by* Λ*, denoted by* $(\overline{\sigma}, \Lambda)$*, which maps the pointed letters of the* P_n*'s to pointed words as follows:*

$$\forall L \in P_0,\ (\overline{\sigma}, \Lambda)(L) = \lambda^*(L),$$

where $\lambda^* \in \Lambda$ *and* $I(\lambda^*) = \{L\}$*,*

$$\forall L' \in P_{n+1},\ (\overline{\sigma}, \Lambda)(L') = \lambda(\vec{x} + L') + \vec{v}(\lambda(\vec{x} + L), (\overline{\sigma}, \Lambda)(L)),$$

where $(L, \vec{x}, \lambda^*) \in P_n \times \mathbb{Z}^d \times \Lambda$ *and* $E(\lambda) = \{\vec{x} + L, \vec{x} + L'\}$*.*

Note, however, that two problems can arise in the definition of $(\overline{\sigma}, \Lambda)$. First, although $P_n \subset P$ for any n, it is not ensured that any pointed letter of P eventually belongs to a P_n. In other words, $(\overline{\sigma}, \Lambda)$ is not necessarily defined on the whole P. Second, more than one triplet $(L, \vec{x}, \lambda) \in P_n \times \mathbb{Z}^d \times \Lambda$, which is used to define the image of $L' \in P_{n+1}$ relatively to the image of L, can exist, and do not necessarily lead to the same definition. We thus introduce the following definition:

Definition 4 (Consistency) *A finite set Λ of local rules for a non-pointed substitution $\overline{\sigma}$ is said to be* consistent *on a pointed word P if the map $(\overline{\sigma}, \Lambda)$ is unambiguously defined (Def. 3) over all the pointed letters of P.*

Then, it is especially interesting when Λ is consistent over a set $\mathcal{W}$ of pointed words, such that $(\overline{\sigma}, \Lambda)(\mathcal{W}) \subset \mathcal{W}$. Indeed, this allows to iterate $(\overline{\sigma}, \Lambda)$ on pointed words. This yields natural interactions with ergodic theory and spectral theory (see *e.g.* [12] and references inside). Before discussing more carefully the problem of consistency of local rules, we provide in the next section some simple examples.

2 Two simple examples

2.1 Substitutions on words

Here, we show that classic substitutions on words are local rule substitutions. Let, for example, σ be the classic Fibonacci substitution on words defined on the alphabet $\mathcal{A} = \{a, b\}$ by $\sigma(a) = ab$ and $\sigma(b) = a$. We define [1] a non-pointed substitution $\overline{\sigma}$ by:

$$\overline{\sigma} \; : \; \begin{cases} \overline{(0,a)} & \rightarrow \; \overline{\{(0,a),(1,b)\}} \\ \overline{(0,b)} & \rightarrow \; \overline{\{(0,a)\}} \end{cases}$$

We then define a set Λ of local rules for $\overline{\sigma}$ which consists of the following two initial rules:

$$\lambda_1^* \; : \; (0,a) \; \rightarrow \; \{(0,a),(1,b)\} \qquad \lambda_2^* \; : \; (0,b) \; \rightarrow \; \{(0,a)\}$$

and of the following four extension rules:

$$\lambda_{aa} \; : \; \begin{cases} (0,a) & \rightarrow \; \{(0,a),(1,b)\} \\ (1,a) & \rightarrow \; \{(2,a),(3,b)\} \end{cases} \qquad \lambda_{ab} \; : \; \begin{cases} (0,a) & \rightarrow \; \{(0,a),(1,b)\} \\ (1,b) & \rightarrow \; \{(2,a)\} \end{cases}$$

$$\lambda_{ba} \; : \; \begin{cases} (0,b) & \rightarrow \; \{(0,a)\} \\ (1,a) & \rightarrow \; \{(1,a),(2,b)\} \end{cases} \qquad \lambda_{bb} \; : \; \begin{cases} (0,b) & \rightarrow \; \{(0,a)\} \\ (1,b) & \rightarrow \; \{(1,a)\} \end{cases}$$

It is convenient to represent these local rules as depicted on Fig. 2.

[1] Recall that a letter l with support $\vec{x}$ is denoted by $(\vec{x}, l)$.

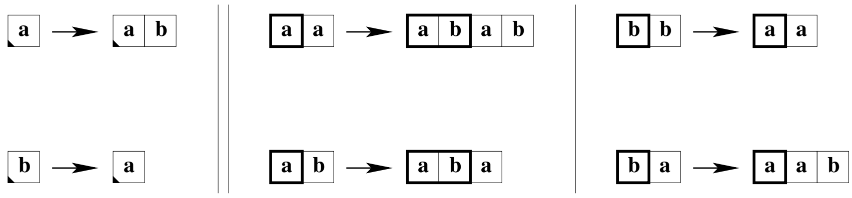

Fig. 2. A set of local rules: two initial rules (leftmost column) and four extension rules (right). For initial rules, a black corner highlights the pointed letters with support 0. For extension rules, framed pointed letters are mapped onto framed pointed words.

Then, it is not hard to see that Λ is consistent over the set $\mathcal{W}_1$ of 1-dimensional pointed words whose supports are intervals of $\mathbb{Z}$. Moreover, $(\overline{\sigma}, \Lambda)(\mathcal{W}_1) \subset \mathcal{W}_1$, so that $(\overline{\sigma}, \Lambda)$ can be iterated on $\mathcal{W}_1$. Note that there is nothing astonishing- we just retrieved the action of classic substitutions on words (see Fig. 3).

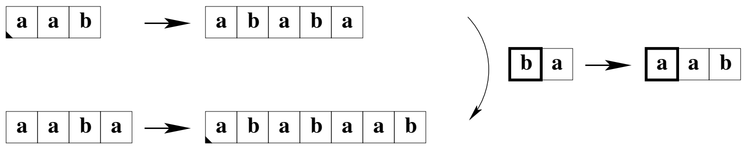

Fig. 3. Knowing the image of a pointed word (top-left) allows us to compute the image of a larger pointed word (bottom-left) by a suitable extension rule (right).

2.2 Uniform-shape substitutions

We here introduce *uniform-shape substitutions*, and we show that they are local rule substitutions.

Definition 5 (Uniform-shape substitution) *Let* $U = (\vec{u}_1, \ldots, \vec{u}_d)$*,* $u_k \in \mathbb{Z}^d$*, and* $S \subset \mathbb{Z}^d$ *(the "*uniform-shape*"), such that* $(S_{\vec{x}})_{\vec{x} \in \mathbb{Z}^d}$ *is a partition of* $\mathbb{Z}^d$*, where:*

$$S_{(x_1,\ldots,x_d)} = \sum_{1 \le k \le d} x_k \vec{u}_k + S.$$

Then, the d*-dimensional* uniform-shape substitution $\mu_{U,S} : \mathcal{L}_d \to \mathcal{P}_d$ *maps a pointed letter* $L = (\vec{x}, l)$ *onto a pointed word* $\mu_{U,S}(L)$*, with support* $S_{\vec{x}}$ *and such that* $\overline{\mu_{U,S}(L)}$ *depends only on the letter* l*.*

Note that the support S thus provides a periodic tiling of $\mathbb{Z}^2$. For example, the following case corresponds to the substitutions studied in [1,13], which map letters to rectangular words:

$$a,\ b \in \mathbb{N}, \qquad U = ((a,0),(0,b)), \qquad S = \{(i,j),\ 0 \le i < a,\ 0 \le j < b\}.$$

Here, the framework is more general, and includes, for example, the following two-dimensional case (see Fig. 4):

$$U = ((2,1),(1,-2)), \qquad S = \{(0,0),(1,0),(0,1),(1,1),(0,2)\}.$$

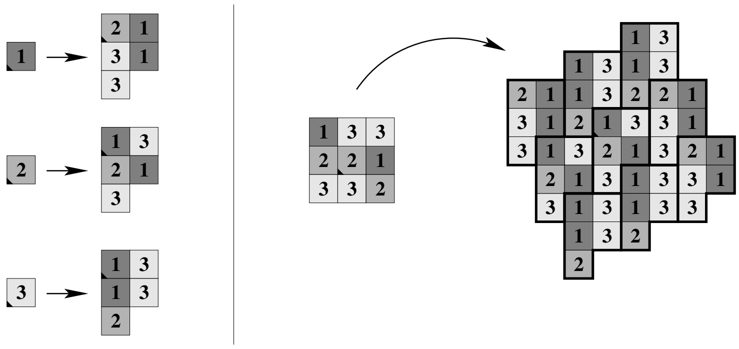

Fig. 4. Example of a uniform-shape substitution acting on pointed letters (left) and on a pointed word (right).

Given a uniform-shape substitution $\mu_{U,S}$, let $\overline{\sigma}$ be the non-pointed substitution defined by $\overline{\sigma}(\overline{L}) = \overline{\mu_{U,S}(L)}$, and Λ be the finite set of local rules for $\overline{\sigma}$ consisting of the initial rules:

$$\lambda_l^* \ : \ (\vec{0}, l) \to \mu_{U,S}(\vec{0}, l), \quad l \in \mathcal{A},$$

and of the extension rules:

$$\lambda_{l,l'} \ : \ \begin{cases} (\vec{0}, l) & \to \ \mu_{U,S}(\vec{0}, l) \\ (\vec{d_k}, l') & \to \ \mu_{U,S}(\vec{d_k}, l') \end{cases}, \quad l,\ l' \in \mathcal{A},\ 1 \le k \le d,$$

where $\vec{d_k}$ is the vector whose k-th entry is a 1 and the others are 0's.

For example, the extension rules associated with the uniform-shape substitution of Fig. 4 are depicted on Fig. 5.

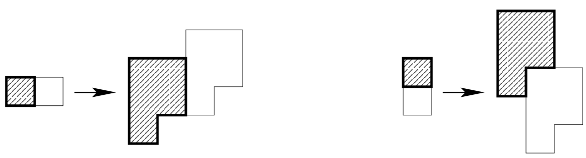

Fig. 5. The extension rules associated with the uniform-shape substitution of Fig. 4 are all on the same model; framed pointed letters are mapped onto framed pointed words.

It is not hard to see that Λ is consistent over the set $\mathcal{W}_d$ of d-dimensional pointed words whose supports are *connected*, *i.e.*, such that for two letters L

and L', there is a sequence of pointed letters $L = L_1, \ldots, L_k = L'$ where the support $\vec{x}_i$ of L_{i+1} is obtained from the support $\vec{x}_{i+1}$ of L_{i+1} by adding or removing 1 to exactly one of the entries of $\vec{x}_i$. Moreover, $(\overline{\sigma}, \Lambda)(\mathcal{W}_d) \subset \mathcal{W}_d$, so that $(\overline{\sigma}, \Lambda)$ can be iterated on $\mathcal{W}_d$.

3 Consistency of local rules

3.1 A more complex example

The previous section provided an example where the consistency of local rules is obvious. Let us now consider the local rules of Fig. 6.

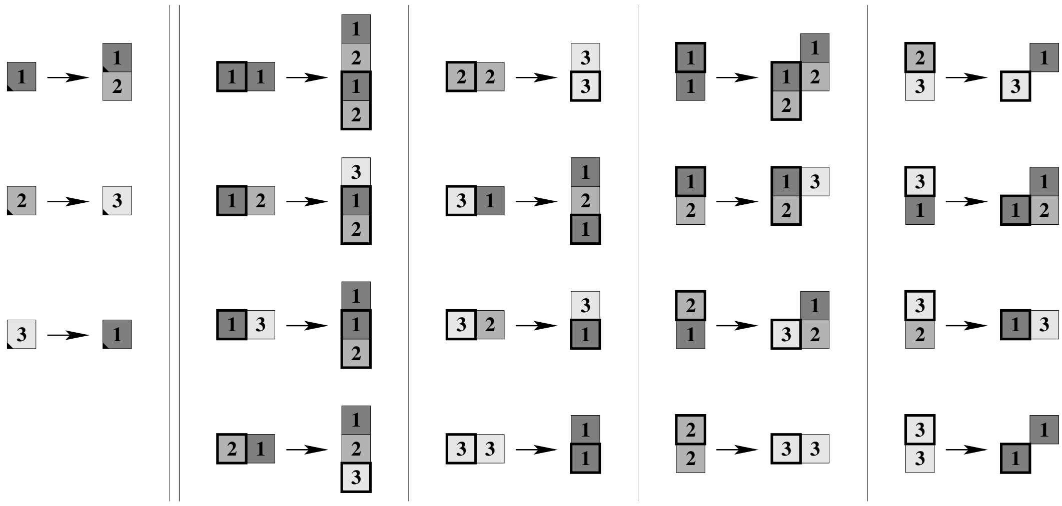

Fig. 6. A set of three initial rules (leftmost column) and 16 extension rules.

This set of local rules turns out to be consistent on the pointed words depicted on Fig. 7. However, Fig. 8 shows that these local rules are not consistent over all 2-dimensional pointed words.

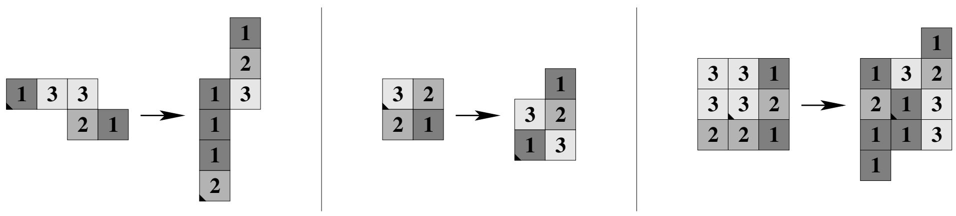

Fig. 7. Three examples of the action of the local rules of Fig. 6, which are rather easily computed.

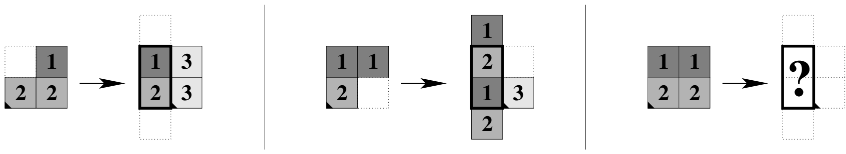

Fig. 8. The local rules of Fig. 6 turn out to be consistent on the first two pointed words here depicted (leftmost). Note that the images of these two pointed words contain different pointed letters sharing the same support. Thus, these local rules are not consistent on the last pointed word (rightmost).

Characterizing the pointed words on which the previous local rules are consistent does not seem obvious. This is a motivation for the *global rules* introduced hereafter. Indeed, it is worth being able, either to obtain consistent local rules, or to ensure the consistency of given local rules.

3.2 Local rules derived from global rules

Let us define *global rules*:

Definition 6 (Global rule) *Let P be a pointed word and $\overline{\sigma}$ be a non-pointed substitution. A* global rule *on P, for $\overline{\sigma}$, is a map Γ, mapping the pointed letters of P to pointed words, which satisfies:*

- *for any pointed letter $L \in P$, $\overline{\Gamma(L)} = \overline{\sigma}(\overline{L})$;*
- *distinct pointed letters are mapped onto pointed words with disjoint supports;*
- *if $L \in P$, $L' \in P$ and $\vec{x} \in \mathbb{Z}^d$ are such that $\vec{x}+L \in P$ and $\vec{x}+L' \in P$, then:*

$$\vec{v}(\Gamma(\vec{x}+L), \Gamma(\vec{x}+L')) = \vec{v}(\Gamma(L), \Gamma(L')).$$

A global rule tells us where to place the image of *each* pointed letter. In fact, this is exactly what we would like to do with local rule substitutions. Intuitively, the aim of local rules is to provide finite and local descriptions of global rules. For example, in Section 2, we have defined a uniform-shape substitution by a map $\mu_{U,S}$, which is easily checked to be a global rule. We then have provided an equivalent local rule substitution, *i.e.*, a local description of the map $\mu_{U,S}$. We are here interested in the general case. We first define a notion of weak connectivity:

Definition 7 (Σ-connectivity) *Let Σ be a finite set of pairs of pointed letters. A pointed word P is said to be* Σ-connected *if, for two pointed letters L and L' of P, there are a sequence $((A_1, B_1), \ldots, (A_k, B_k))$ of pairs of Σ and a sequence $(\vec{x}_i, \ldots, \vec{x}_k)$ of vectors of $\mathbb{Z}^d$, such that $\vec{x}_1 + A_1 = L$, $\vec{x}_k + B_k = L'$ and $\vec{x}_i + B_i = \vec{x}_{i+1} + A_{i+1}$ for $1 \leq i < k$.*

Then, a set of Σ-connected pointed words is said to be Σ-connected. This notion turns out to be meaningful for computing consistent sets of local rules:

Theorem 1 (Derivation) *Let Γ be a global rule on a pointed word P, for a non-pointed substitution $\overline{\sigma}$. If P is Σ-connected, then one can effectively derive from Σ a finite set Λ of local rules for $\overline{\sigma}$, such that $\Gamma = (\overline{\sigma}, \Lambda)$ on P. In particular, Λ is consistent on P.*

PROOF. Let Λ be the finite set consisting of one initial rule $\lambda^* : L_0 \to \Gamma(L_0)$, for one specific pointed letter $L_0 \in P$, and of the extension rule $\lambda : A \to \Gamma(A), B \to \Gamma(B)$, for each pair $(A, B) \in \Sigma$. We prove, by induction on n, that $\Gamma = (\overline{\sigma}, \Lambda)$ on the pointed letters of the P_n's (with the notations of Def. 3):

- one has $P_0 = \{L_0\}$, and $(\overline{\sigma}, \Lambda)(L_0) = \Gamma(L_0)$;
- suppose now that $\Gamma = (\overline{\sigma}, \Lambda)$ on the pointed letters of P_n. Let $L' \in P_{n+1}$ and $(\lambda, \vec{x}, L) \in \Lambda \times \mathbb{Z}^d \times P_n$ such that $E(\lambda) = \{\vec{x} + L, \vec{x} + L'\}$. First, $(\overline{\sigma}, \Lambda)(L) = \Gamma(L)$ by induction. Second, $\lambda(\vec{x}+L) = \Gamma(\vec{x}+L)$ and $\lambda(\vec{x}+L') = \Gamma(\vec{x} + L')$, by definition of Λ. Third, the fact that Γ is a global rule yields $\Gamma(\vec{x} + L') + \vec{v}(\Gamma(\vec{x} + L), \Gamma(L)) = \Gamma(L')$. Thus, one computes:

$$\begin{aligned}(\overline{\sigma}, \Lambda)(L') &= \lambda(\vec{x} + L') + \vec{v}(\lambda(\vec{x} + L), (\overline{\sigma}, \Lambda)(L))\\ &= \lambda(\vec{x} + L') + \vec{v}(\lambda(\vec{x} + L), \Gamma(L))\\ &= \Gamma(\vec{x} + L') + \vec{v}(\Gamma(\vec{x} + L), \Gamma(L))\\ &= \Gamma(L').\end{aligned}$$

Last, to complete the proof, note that $\cup_n P_n = P$ by Σ-connectivity of P. ■

Note that the previous theorem also allows us to derive a consistent set of local rules from a global rule Γ on a Σ-connected set of pointed words $\mathcal{W} \subset \mathcal{P}_d$. Indeed, the derived set of local rules depends on Γ and Σ. When $\Gamma(\mathcal{W}) \subset \mathcal{W}$, this thus allows us to iterate the obtained local rule substitution.

The previous theorem showed that consistent local rules can be derived from a given global rule. So, a natural question is: can we *directly* (and effectively) ensure the consistency of a given set of local rules (*i.e.*, without global rules)? Note that checking the consistency of a set of local rules on a finite pointed word can be easily done, trying to compute the image of this word. The interesting case is the one of infinite pointed words or of an infinite set of pointed words. We thus can adress the following question (whose answer seems to be positive, although a proof of this remains to be written):

Question 1 *Let $F_1, \ldots, F_k$ be finite pointed words and Λ be a finite set of local rules. Is the consistency of Λ, on pointed words without factor F_i, computable?*

4 Stepped surfaces and generalized substitutions

Notions of *stepped surfaces*, *stepped planes* and *generalized substitutions*, as well as known results concerning them, are here briefly reviewed. A more detailed exposition can be found in [2,8] (see also references inside). These notions and results are then used, in the next section, to provide a large class of consistent local rules.

4.1 Stepped surfaces and stepped planes

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be the canonical basis of $\mathbb{R}^3$. The *face* of type $i \in \{1,2,3\}$ and located at $\vec{x} \in \mathbb{Z}^3$, denoted by $(\vec{x}, i^*)$, is the subset of $\mathbb{R}^3$ defined by:

$$(\vec{x}, i^*) = \{\vec{x} + \sum_{j \neq i} \lambda_j \vec{e}_j,\ 0 \leq \lambda_j \leq 1\}.$$

The set of faces is denoted by $\mathcal{F}$. Fig. 9 represents some faces.

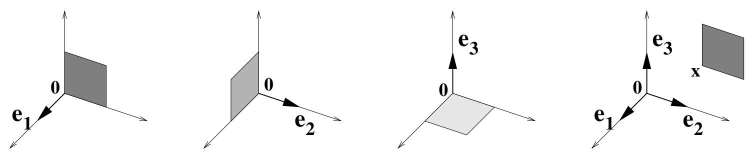

Fig. 9. The faces $(\vec{0}, 1^*)$, $(\vec{0}, 2^*)$, $(\vec{0}, 3^*)$ and $(\vec{x}, 1^*)$ (from left to right).

Let π be the orthogonal projection onto Δ, the real plane of normal vector $\vec{e}_1 + \vec{e}_2 + \vec{e}_3$. One defines *stepped surfaces* as follows:

Definition 8 *A* stepped surface *is a set of faces $\mathcal{S}$, whose union is homeomorphic, by π, to the real plane Δ.*

Moreover, stepped surfaces corresponding to discrete approximations of real planes are called *stepped planes*:

Definition 9 *Let $\rho \in \mathbb{R}$ and $\vec{\alpha}$ be a vector of $\mathbb{R}^3$ with positive entries. The* stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ *is the set of faces defined by:*

$$\mathcal{P}_{\vec{\alpha},\rho} = \{(\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle + \rho < 0 \leq \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle + \rho\},$$

where $\langle .,. \rangle$ is the canonical scalar product.

Fig. 10 illustrates both Def. 8 and 9.

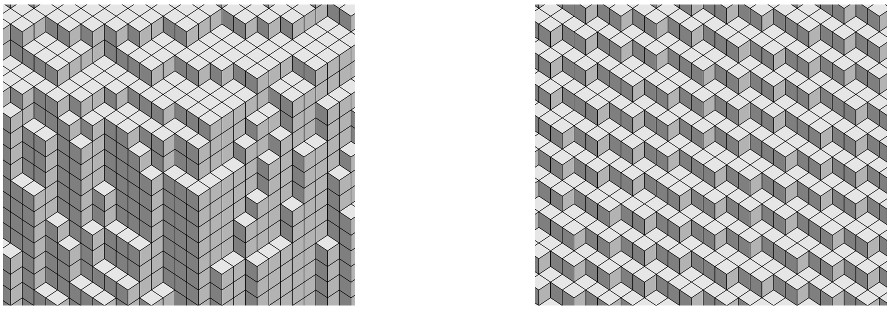

Fig. 10. A stepped surface (left) and a stepped plane (right). In both cases, the union of their faces is homeomorphic to the real plane of normal vector $\vec{e}_1 + \vec{e}_2 + \vec{e}_3$, by the orthogonal projection onto this plane. Note that, besides the three dimensional viewpoint, there is also a two-dimensional viewpoint as lozenge tilings of the plane.

Let us now see how to code stepped surfaces by 2-dimensional words. The projection, by π on Δ, of the faces of a stepped surface, provides a tiling of Δ by lozenges of three types, whose vertices belong to a lattice of rank 2. It turns out that this tiling admits a 2-dimensional coding. Indeed, let v be the map defined on faces by:

$$v(\vec{x}, i^*) = \vec{x} + \vec{e}_1 + \ldots + \vec{e}_{i-1} \in \mathbb{Z}^3.$$

Then, $\pi \circ v$ is a bijection between the faces of a given stepped surface and the 2-dimensional lattice $\mathbb{Z}\pi(\vec{e}_1) + \mathbb{Z}\pi(\vec{e}_2) \subset \Delta$. Thus, the following map codes a stepped surface by a 2-dimensional pointed word over $\{1, 2, 3\}$ (see Fig. 11):

$$\Psi \ : \ \begin{array}{ccc} \mathcal{F} & \to & \mathbb{Z}^2 \times \{1,2,3\} \\ (\vec{x}, i^*) & \to & ((a,b), i), \end{array}$$

where $a\pi(\vec{e}_1) + b\pi(\vec{e}_2) = \pi \circ v(\vec{x}, i^*)$.

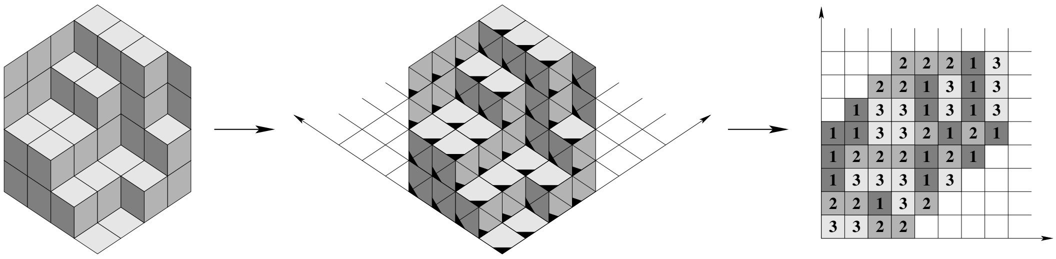

Fig. 11. Consider a stepped surface (left). The vertices $\pi \circ v(\vec{x}, i^*)$, with $(\vec{x}, i^*)$ being a face of the stepped surface, form the lattice $\mathbb{Z}\pi(\vec{e}_1) + \mathbb{Z}\pi(\vec{e}_2)$ (in the middle, with the black corner of a face $(\vec{x}, i^*)$ highlighting the vertex $\pi \circ v(\vec{x}, i^*)$). So that the stepped surface can naturally be coded by a 2-dimensional pointed word (right).

Although a stepped surface can always be coded by a 2-dimensional pointed word, it is not hard to see that the converse does not hold. In fact, 2-dimensional pointed words which are coding of stepped surfaces are characterized in [10]: they are the 2-dimensional pointed words which do not have factors in the finite set of factors depicted on Fig. 12.

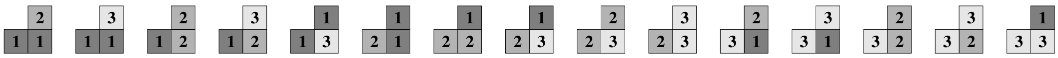

Fig. 12. A 2-dimensional pointed word is the coding of a stepped surface if and only if it avoids these 15 forbidden factors (see, for example, Fig. 11).

Then, it is proven in [2], that if $\mathcal{S}$ and $\mathcal{S}'$ are two stepped surfaces, then $\Psi(\mathcal{S}) = \Psi(\mathcal{S}')$ if and only if $\mathcal{S}$ and $\mathcal{S}'$ are equal *up to translation* by a vector $k(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)$, $k \in \mathbb{Z}$. This yields, in particular, that Ψ is a bijection from the set of stepped surfaces containing the origin $\vec{0}$ onto the set of 2-dimensional pointed words avoiding the forbidden factors of Fig. 12.

4.2 Generalized substitutions

Let σ be a classic substitution on words over the alphabet $\{1, 2, 3\}$. The *incidence matrix* of σ is the 3×3 matrix whose entry at row i and column j is the number of occurrences of the letter i in the word $\sigma(j)$. If $\det M_\sigma = \pm 1$, σ is said to be *unimodular*; in this case, note that M_σ^{-1} has, as well as M_σ, *integer* coefficients.

The *Parikh mapping* $\vec{f} : \{1, 2, 3\}^* \to \mathbb{N}^3$ is defined by $\vec{f}(w) = {}^t(|w|_1, |w|_2, |w|_3)$. In particular, one has $\vec{f}(\sigma(w)) = M_\sigma \vec{f}(w)$ for any word $w \in \{1, 2, 3\}^*$. Then, *generalized substitutions* are defined in [4] as follows:

Definition 10 ([4]) *Let σ be a unimodular substitution over $\{1, 2, 3\}$. The* generalized substitution $\Theta_\sigma^* : \mathcal{F} \to \mathcal{F}$ *is defined by:*

$$\forall \mathcal{E}, \mathcal{E}' \subset \mathcal{F}, \qquad \Theta_\sigma^*(\mathcal{E} \cup \mathcal{E}') = \Theta_\sigma^*(\mathcal{E}) \cup \Theta_\sigma^*(\mathcal{E}'),$$

$$\forall (\vec{x}, i^*) \in \mathcal{F}, \qquad \Theta_\sigma^*(\{(\vec{x}, i^*)\}) = \bigcup_{\substack{j,p,s \\ |\sigma(j) = p \cdot i \cdot s}} \left(M_\sigma^{-1}(\vec{x} + \vec{f}(s)), j^* \right).$$

For example, for $\sigma : 1 \to 13, 2 \to 1, 3 \to 2$, one has:

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This yields (see Fig. 13):

$$\Theta^*_\sigma \ : \ \begin{array}{l} \{(\vec{x}, 1^*)\} \mapsto \{(M_\sigma^{-1}\vec{x} + \vec{e}_1 - \vec{e}_2, 1^*), (M_\sigma^{-1}\vec{x}, 2^*)\} \\ \{(\vec{x}, 2^*)\} \mapsto \{(M_\sigma^{-1}\vec{x}, 3^*)\} \\ \{(\vec{x}, 3^*)\} \mapsto \{(M_\sigma^{-1}\vec{x}, 1^*)\}. \end{array}$$

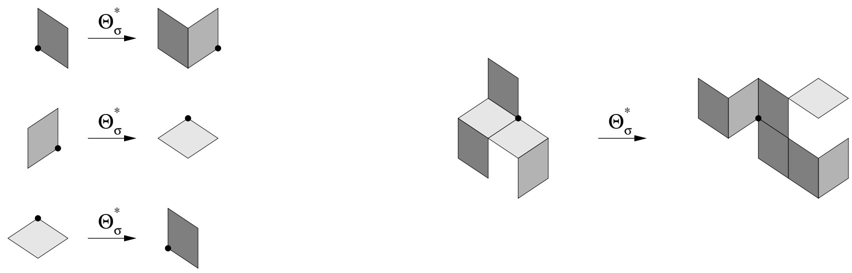

Fig. 13. Action of the generalized substitution Θ^*_σ on some sets of faces.

Although generalized substitutions are defined over all the sets of faces, stepped surfaces and stepped planes are particularly interesting. Indeed, it is proven in [2] that a generalized substitution Θ^*_σ maps a stepped surface, hence a stepped plane, onto a stepped surface, with distinct faces being mapped onto disjoint sets of faces (more exactly, with disjoint interiors). Moreover, stepped planes are mapped onto stepped surfaces which turn out to be also stepped planes. More precisely, one has (see [8]):

$$\Theta^*_\sigma \ : \ \mathcal{P}_{\vec{\alpha},\rho} \ \rightarrow \ \mathcal{P}_{{}^tM_\sigma\vec{\alpha},\rho}.$$

Thus, one can iterate the application of generalized substitutions on both stepped surfaces and stepped planes.

5 Local rule substitutions on stepped surfaces

5.1 Local rules derived from a generalized substitution

Let Θ^*_σ be a generalized substitution and $\mathcal{S}$ be a stepped surface. One defines a map Γ_σ on the pointed letters of the 2-dimensional pointed word $\Psi(\mathcal{S})$ by:

$$\Gamma_\sigma \ : \ \begin{array}{ll} \Psi(\mathcal{S}) \subset \mathcal{P}_2 \rightarrow & \mathcal{P}_2 \\ \Psi(\vec{x}, i^*) & \rightarrow \Psi(\Theta^*_\sigma(\vec{x}, i^*)) \end{array}$$

One also defines a non-pointed substitution $\overline{\sigma^*} : \overline{\mathcal{L}_2} \rightarrow \overline{\mathcal{P}_2}$ by:

$$\overline{\sigma^*} \ : \ \overline{\Psi(\vec{0}, i)} \ \rightarrow \ \overline{\Gamma_\sigma(\vec{0}, i^*)}.$$

Then one has:

Theorem 2 *The map Γ_σ is a global rule on $\Psi(\mathcal{S})$, for $\overline{\sigma^*}$.*

PROOF. Let $\vec{x} \in \mathbb{Z}^3$ and $i \in \{1,2,3\}$. Let $\vec{y} \in \mathbb{Z}^3$ and $(c,d) \in \mathbb{Z}^2$ such that $\pi(\vec{y}) = c\pi(\vec{e}_1) + d\pi(\vec{e}_2)$. One thus has $\Psi(\vec{y} + (\vec{x}, i^*)) = (c,d) + \Psi(\vec{x}, i^*)$. One then computes, for the pointed letter $L = \Psi(\vec{x}, i)$:

$$\begin{aligned}
\Gamma_\sigma((c,d)+L) &= \Gamma_\sigma(\Psi(\vec{y} + (\vec{x}, i^*))) \\
&= \Psi(\Theta_\sigma^*(\vec{y} + (\vec{x}, i^*))) \\
&= \Psi(M_\sigma^{-1}\vec{y} + \Theta_\sigma^*(\vec{x}, i^*)) \\
&= \pi(M_\sigma^{-1}\vec{y}) + \Psi(\Theta_\sigma^*(\vec{x}, i^*)) \\
&= \pi(M_\sigma^{-1}\vec{y}) + \Gamma_\sigma(L).
\end{aligned}$$

Let us show that this yields that Γ_σ satisfies the three properties characterizing a global rule. This will complete the proof.
First, if we take $\vec{y} = -\vec{x}$, then $(c,d) + L$ has support $\vec{0}$, and $\overline{\Gamma_\sigma(L)} = \overline{\sigma^*}(\overline{L})$.
Then, if we denote $\pi(M_\sigma^{-1}\vec{y})$ by $\vec{z}$, one has for a pointed letter L':

$$\vec{v}(\Gamma((c,d)+L), \Gamma((c,d)+L')) = \vec{v}(\vec{z} + \Gamma_\sigma(L), \vec{z} + \Gamma(L')) = \vec{v}(\Gamma(L), \Gamma(L')).$$

Last, since Ψ bijectively associates faces of the stepped surface $\mathcal{S}$ with pointed letters, and since Θ_σ^* maps distinct faces onto disjoint sets of faces, Γ_σ maps distinct pointed letters onto pointed words with disjoint supports. ■

Thus, generalized substitutions provide a wide class of global rules for 2-dimensional pointed words coding stepped surfaces. Moreover, a 2-dimensional pointed word coding a stepped surface has support $\mathbb{Z}^2$, and is thus Σ_1-connected by (recall Def. 7), where:

$$\Sigma_1 = \{(((0,0),i),((0,1),j)),(((0,0),i),((1,0),j)), i,j \in \{1,2,3\}\}.$$

Th. 1 then yields that one can effectively derive from a global rule Γ_σ a finite set of local rules, consistent over the 2-dimensional pointed words coding stepped surfaces. Moreover, since generalized substitutions map stepped surfaces onto stepped surfaces, one can iterate these derived global rules (or the corresponding local rule substitutions) onto 2-dimensional pointed words coding stepped surfaces.

In fact, the local rules depicted on Fig. 6, in Section 3, are derived from the generalized substitution Θ_σ^*, for $\sigma : 1 \to 13, 2 \to 1, 3 \to 2$ (see Fig. 13). For example, Fig. 14 provides a three-dimensional viewpoint, *i.e.*, in terms of stepped surfaces, for Fig. 8. Then, the corresponding local rule substitution can be iterated on the 2-dimensional pointed words coding stepped surfaces.

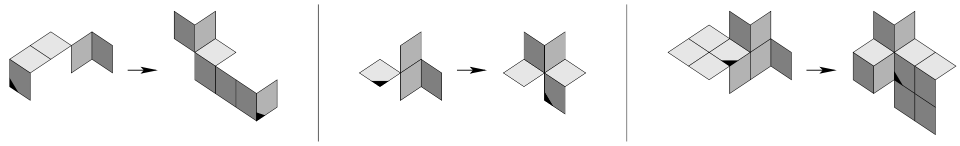

Fig. 14. Action, on some subsets of stepped surfaces, of the generalized substitution from which the local rules of Fig. 6 are derived. This corresponds to a three-dimensional viewpoint of Fig. 7.

5.2 *Local rules not derived from a generalized substitution*

The previous section proved that there are local rule substitutions over stepped surfaces that can be derived from generalized susbtitutions. Here, we provide examples of local rule substitutions over stepped surfaces which cannot be derived from generalized substitutions. We then also discuss the problem of characterizing these local rule substitutions.

Let us first define, as well as for classic substitutions on words, the incidence matrix of a non-pointed substitution $\overline{\sigma}$ defined over $\overline{L}_i = \overline{(\vec{0}, i)}$, $i \in \{1, 2, 3\}$. This is the 3×3 matrix whose coefficient at row i and column j is the number of occurences of $\overline{L}_i$ in the non-pointed word $\overline{\sigma}(\overline{L}_j)$. One easily checks that, if σ is a classic substitution on words and that $\overline{\sigma^*}$ is a non-pointed substitution derived from the generalized substitution Θ^*_σ, then $M_{\overline{\sigma^*}} = {}^tM_\sigma$. Thus, $\det(M_{\overline{\sigma^*}}) = \pm 1$ in such a case.

The uniform-shape substitution defined in Section 2, Fig. 4 (page 7), provides a first example of local rule substitution over stepped surfaces which cannot be derived from a generalized substitution. First, note that the incidence matrix of the associated non-pointed substitution is:

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix},$$

which has determinant zero. Hence, this local rule substitution cannot be derived from a generalized substitution. Then, a case-study, using the characterization by forbidden factors of 2-dimensional pointed words coding stepped surfaces (see Fig. 12), shows that it maps stepped surfaces onto stepped surfaces. This can be easily seen on Fig. 15, which gives a three-dimensional viewpoint of Fig. 4.

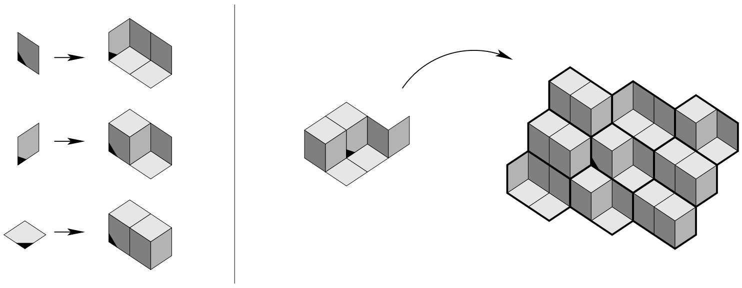

Fig. 15. The three-dimensional viewpoint corresponding to Fig. 4 (page 7).

Fig. 16 and 17 provide a second example of local rule substitution, which is not a uniform-shape substitution, and maps stepped surfaces onto stepped surfaces. The incidence matrix of the associated non-pointed substitution is diagonal, with diagonal $(2,1,2)$, and has thus determinant 4. Hence, this local rule substitution cannot be derived from a generalized substitution.

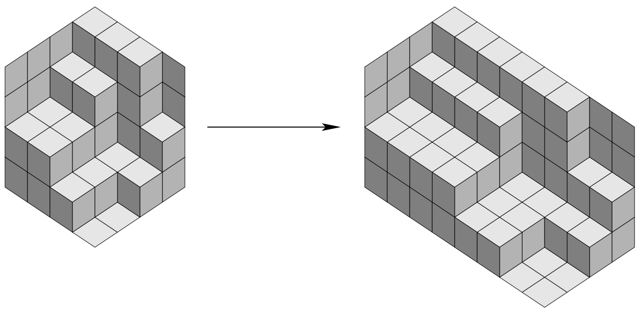

Fig. 16. A global rule which stretches stepped surfaces (three-dim. viewpoint).

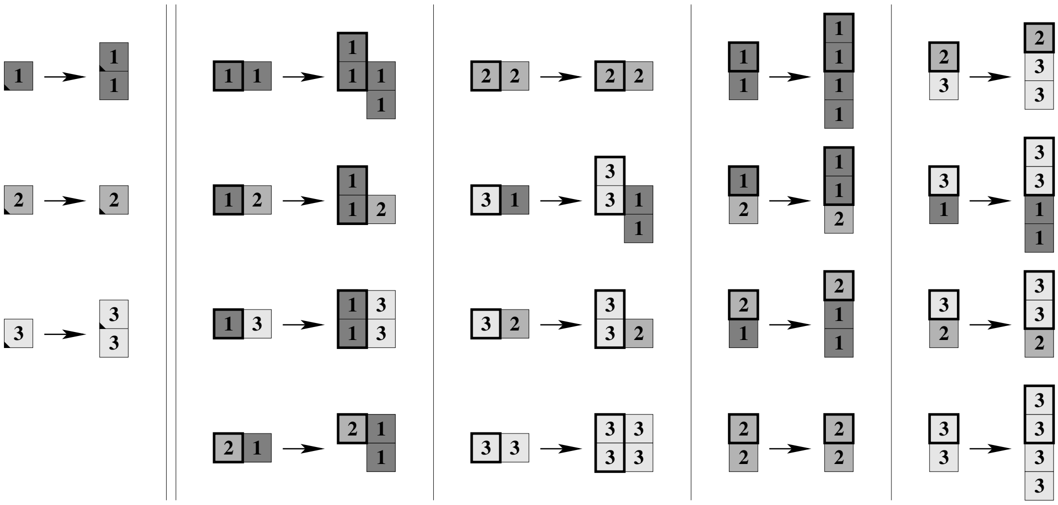

Fig. 17. A set of local rules derived from the global rule of Fig. 16, but which cannot be derived from a generalized substitution.

Thus, among local rule substitutions mapping stepped surfaces onto stepped surfaces, some can be derived from a generalized substitution, but not all. It would be interesting to obtain a characterization of the former.

Acknowledgements. We would like to thank, for suggestions or corrections, P. Arnoux, V. Berthé, J. Cassaigne, J. Sakarovitch, G. Varouchas and anonymous referees of both [9] and this paper.

References

[1] J.-P. Allouche, J. O. Shallit, *Automatic sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.

[2] P. Arnoux, V. Berthé, T. Fernique, D. Jamet, *Functional stepped surfaces, flips and generalized substitutions*, to appear in Theor. Comput. Sci.

[3] P. Arnoux, V. Berthé, A. Siegel, *Two-dimensional iterated morphisms and discrete planes*, Theor. Comput. Sci. **319** (2004), pp. 145–176.

[4] P. Arnoux, S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), pp. 181–207.

[5] P. Arnoux, S. Ito, Y. Sano, *Higher dimensional extensions of substitutions and their dual maps*, J. Anal. Math. **83** (2001), pp. 183–206.

[6] F. M. Dekking, *Recurrent sets*, Adv. in Math. **44** (1982), pp 78–104.

[7] F. M. Dekking, *Replicating superfigures and endomorphisms of free groups*, J. Combin. Theory Ser. A **32** (1982), pp. 315–320.

[8] T. Fernique, *Multidimensional sturmian words and substitutions*, Int. J. Found. of Comput. Sci. **17** (2006) pp. 575–600.

[9] T. Fernique, *Substitutions on multidimensional words*, Publications du LACIM **36** Words 2005 S. Brlek, C. Rentenauer (eds.) (2005) pp. 223–234.

[10] D. Jamet, *On the language of standard discrete planes and surfaces*, Proc. IWCIA, Lect. Notes Comput. Sci. **3322** (2004), pp. 232–247.

[11] N. Priebe Frank, *Detecting combinatorial hierarchy in tilings using derived Voronoï tessellations*, Discrete Comput. Geom. **29** (2003) pp. 459–467.

[12] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Math. **1794** (2002), Springer Verlag.

[13] O. Salon, *Suites automatiques à multi-indices*, Sém. Théorie Nombres Bordeaux (1986-1987) 4.01–4.27, with an appendix of J. Shallit, 4.29A–4.36A.