Functional Stepped Surfaces, Flips and Generalized Substitutions

Pierre Arnoux, Valerie Berthe, Thomas Fernique, Damien Jamet

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Pierre Arnoux,

IML – CNRS UMR 6206– 163, avenue de Luminy, Case 907, 13288 Marseille Cedex 09, France

Valérie Berthé, Thomas Fernique,

LIRMM – UMR 5506 – Univ. Montpellier II, 161 rue Ada, 34392 Montpellier Cedex 05, France

Damien Jamet,

LORIA - Campus Scientifique - BP 239 - 54506 Vandoeuvre-lès-Nancy Cedex, France

Abstract

A substitution is a non-erasing morphism of the free monoid. The notion of multidimensional substitution of non-constant length acting on multidimensional words is proved to be well-defined on the set of two-dimensional words related to discrete approximations of irrational planes. Such a multidimensional substitution can be associated with any usual unimodular substitution. The aim of this paper is to extend the domain of definition of such multidimensional substitutions to functional stepped surfaces. One central tool for this extension is the notion of flips acting on tilings by lozenges of the plane.

Key words: Generalized substitution, discrete geometry, arithmetical discrete plane, discrete surface, word combinatorics, flip, lozenge tiling, dimer tiling, Sturmian word.

Email addresses: `arnoux@iml.univ-mrs.fr` (Pierre Arnoux), `berthe@lirmm.fr` (Valérie Berthé), `fernique@lirmm.fr` (Thomas Fernique), `jamet@loria.fr` (Damien Jamet).

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1 Introduction

Sturmian words are known to be codings of digitizations of an irrational straight line [KR04,Loth02]. One could expect a higher-dimensional extension of Sturmian words to correspond to a digitization of a hyperplane with irrational normal vector. It is thus natural to consider the digitization scheme corresponding to the notion of standard arithmetical plane introduced in [Rev91]: this notion consists in approximating a plane in $\mathbb{R}^3$ by selecting points with integral coordinates above and within a bounded distance of the plane; more precisely, given $\mathbf{v} \in \mathbb{R}^3$, and $(\mu, \omega) \in \mathbb{R}^2$, the *lower* (resp. *upper*) *arithmetical hyperplane* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is the set of points $\mathbf{x} \in \mathbb{Z}^3$ satisfying $0 \leq \langle \mathbf{x}, \mathbf{v} \rangle + \mu < \omega$ (resp. $0 < \langle \mathbf{x}, \mathbf{v} \rangle + \mu \leq \omega$). If $\omega = \sum |v_i| = \|\mathbf{v}\|_1$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said to be *standard*.

In this latter case, one approximates a plane with normal vector $\mathbf{v} \in \mathbb{R}^3$ by square faces oriented along the three coordinate planes. The union of all these faces is called a *stepped plane* (see Figure 2); the standard discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \|\mathbf{v}\|_1)$ is then equal to the set of points with integer coordinates that belong to the stepped plane; after orthogonal projection onto the plane $x_1 + x_2 + x_3 = 0$, one obtains a tiling of the plane with three kinds of lozenges, namely the projections of the three possible unit faces. One can code this projection over $\mathbb{Z}^2$ by associating with each lozenge the name of the projected face it corresponds to. These words are in fact three-letter two-dimensional Sturmian words (see, e.g., [BV00]).

It is natural to try to endow arithmetic discrete planes with a relevant notion of discrete surface. There is a vast literature devoted to discrete surface developed during the last 25 years with various approaches. For instance, in [MR81], Morgenthaler and Rosenfeld introduce a notion of discrete surface based on a graph theoretical approach using adjacency relations. Nevertheless, this definition is not relevant for arithmetic discrete planes. In [Fra96,KI00,KI03], the authors show that an appropriate way to provide arithmetic discrete planes with a discrete surface structure is to consider two-dimensional combinatorial manifolds. For instance, Françon shows in [Fra96] that the 2-adjacency graph of a rational standard arithmetic discrete plane has a natural underlying structure of two-dimensional combinatorial manifold.

As a particular case of this latter approach, functional discrete surfaces are introduced in [Jam04,JP05]. A functional discrete surface is defined as a union of pointed faces such that the orthogonal projection onto the diagonal plane $x_1 + x_2 + x_3 = 0$ induces an homeomorphism from the functional discrete surface onto the diagonal plane. As done for stepped planes, one provides any functional discrete surface with a two-dimensional coding over a three-letter alphabet. In the present paper, we refer functional discrete surfaces to *functional* *stepped surfaces*, since such objects are not discrete, in the sense that they are not subsets of $\mathbb{Z}^3$. Note that one could define more general stepped surfaces, for instance approximations of spheres. Nevertheless, we restrict ourselves here to *functional* surfaces, that is, surfaces that project homeomorphically onto the diagonal plane and that can be described as graphs of piecewise affine maps defined on the diagonal plane.

Let us recall that a substitution is a non-erasing morphism of the free monoid. It acts naturally on all finite and infinite words. In particular, it maps a two-sided word to a two-sided word. We are interested here in higher dimensional analogues of substitutions. It is easy to define a two-dimensional substitution which replaces each letter by a rectangle of fixed size. This is the analogue of substitutions of constant length, and such a substitution acts on the set of all two-dimensional words. For such examples, see for instance [AS03]. In the present paper, we deal with substitutions of non-constant length; one easily sees that such a substitution can never be defined on the set of *all* two-dimensional words: if two letters are replaced by patterns of different shapes, and if we consider two two-dimensional words that differ in exactly one place by the corresponding letters, it is not possible that both two-dimensional words are sent by the substitution to complete two-dimensional words. In fact, it is not even clear that a two-dimensional substitution can act on at least one two-dimensional word.

A notion of multidimensional substitution of non-constant length acting on multidimensional words is studied in [AI01,AIS01,ABI02,ABS04,Fer05b,Fer05c], inspired by the geometrical formalism of [IO93,IO94]. According to [AI01], these multidimensional substitutions are proved to be well defined on multidimensional Sturmian words. Given any usual unimodular substitution, then such a multidimensional substitution can be associated with it (a substitution is said *unimodular* if the determinant of its incidence matrix equals ± 1). The aim of the present paper is to explore the domain of definition of such multidimensional substitutions. Our main result is the following: the image of a functional stepped surface under the action of a two-dimensional substitution is still a functional stepped surface.

Our proofs are based on a geometrical approach, using the generation of functional stepped surfaces by *flips*. A flip is a classical notion in the study of dimer tilings and lozenge tilings associated with the triangular lattice; e.g., see [Thu89]. It consists in a local reorganization of tiles that transforms a tiling into another one. Such a reorganization can also be seen in the three-dimensional space on the functional stepped surface itself. Suppose indeed that a functional stepped surface contains three faces that form the lower faces of a unit cube with integer vertices. By replacing these three faces by the upper faces of this cube, one obtains another functional stepped surface (see Figure 7). We prove that any functional stepped surface can be obtained from a stepped plane by a sequence of flips, possibly infinite but locally finite, in the sense that, for any bounded neighborhood of the origin in the diagonal plane, there is only a finite number of flips whose domain has a projection which intersects this neighborhood (see Theorem 12).

This paper is organized as follows. In Section 2 and 3, we give definitions for stepped planes, functional stepped surfaces, their codings and review their basic properties. Section 4 is devoted to the generation of a functional stepped surface by a locally finite sequence of flips performed on a given stepped plane. Generalized substitutions associated with a unimodular substitution are introduced in Section 5.1; we prove that the image of a stepped plane by such a susbtitution is still a stepped plane, whose parameters can be explicitly computed. Finally, in Section 5.2, we prove that generalized substitutions act on the set of functional stepped surfaces. Furthermore the main result of the present paper is proved, namely, the image of a functional stepped surface is still a functional stepped surface.

We remark that we deal here with three types of objects: functional stepped surfaces, lozenge tilings of the plane and two-dimensional words. There is a straightforward relation between these objects: there is a one-to-one correspondence between lozenge tilings and functional stepped surfaces containing the origin, or functional stepped surfaces up to a translation by a multiple of the diagonal vector $(1, 1, 1)$ (of course, the translate of a stepped surface by this vector gives the same lozenge tiling by projection): any tiling can be lifted in a unique way, up to translation, to a functional stepped surface, as it is intuitively clear by looking at a tiling (see for instance Figure 3 and Theorem 9). The map which associates with a lozenge tiling the corresponding symbolic coding is obviously one-to-one, but not onto; the set of words obtained in this way can be completely described by a local condition (see [Jam04,JP05]). Hence the multidimensional substitutions we deal with here can be equivalently defined as acting either on functional stepped surfaces, or on their codings as a two-dimensional word over a three-letter alphabet, or lastly, on the corresponding tiling of the plane by lozenges. For the sake of clarity, we choose here to focus on the first point of view, that is, on multidimensional substitutions acting on faces of functional stepped surfaces.

2 Stepped planes

There are several ways to approximate planes by integer points such as illustrated in the survey [BCK04]. All these methods boil down to selecting integer points within a bounded distance from the considered plane. Such objects are called *discrete planes*. In the present paper, we deal with an approach inspired by the formalism of [AI01], see also [IO93,IO94,BV00,ABI02].

Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ stand for the canonical basis of $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$. The *face* $(\mathbf{x}, i^\star)$ is the subset of $\mathbb{R}^3$ defined as follows (see Figure 1):

$$(\mathbf{x}, i^*) = \left\{ \mathbf{x} + \sum_{j \neq i} \lambda_j \mathbf{e_j},\ \lambda_j \in [0, 1] \right\}.$$

The integer $i \in \{1, 2, 3\}$ is called the *type* of the face $(\mathbf{x}, i^*)$. We denote by $\mathfrak{F} = \{(\mathbf{x}, i^*),\ \mathbf{x} \in \mathbb{Z}^3,\ i \in \{1, 2, 3\}\}$ the set of faces, and by $\mathcal{G}$, the set of (finite or infinite) unions of faces of $\mathcal{F}$. Endowed with the union operation, $\mathcal{G}$ is a monoid. We provide $\mathcal{G}$ with a distance as follows:

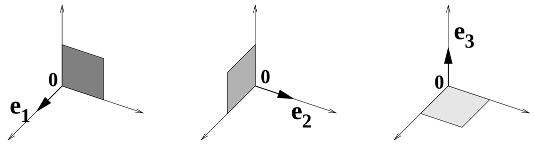

Figure 1. An example of faces in $\mathbb{R}^3$.

Definition 1 (Distance between two sets of faces) *Given $\mathcal{E}$ and $\mathcal{E}'$ in $\mathcal{G}$, we set $d(\mathcal{E}, \mathcal{E}') = 0$ if $\mathcal{E} = \mathcal{E}'$. Otherwise:*

$$d(\mathcal{E}, \mathcal{E}') = 2^{-\min\{\|v\|_\infty,\, (v, i*) \subseteq (\mathcal{E} \setminus \mathcal{E}') \cup (\mathcal{E}' \setminus \mathcal{E})\}},$$

with $\|\mathbf{v}\|_\infty = \max\{|\mathbf{v}_1|, |\mathbf{v}_2|, |\mathbf{v}_3|\}$.

One easily checks that $d : \mathcal{G} \times \mathcal{G} \longrightarrow [0, 1]$ defines a distance on the set $\mathcal{G}$. Roughly speaking, the larger the balls $B(0, r) = \{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\|_\infty < r\}$ the sets $\mathcal{E}$ and $\mathcal{E}'$ coincide on, the closer the sets $\mathcal{E}$ and $\mathcal{E}'$ are. In all that follows, $\mathcal{G}$ stands for the union $\mathcal{G}$ provided with the topology induced by the distance d.

From now on, we denote by $\mathbb{R}^3_+$ the set of vectors in $\mathbb{R}^3$ with positive coordinates. We then define *stepped planes* as a particular set of faces as follows:

Definition 2 (Stepped plane) *Let $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$. The* stepped plane *with* normal vector $\mathbf{v}$ *and* translation parameter μ *is the subset $\mathfrak{P}(\mathbf{v}, \mu)$ of $\mathcal{G}$ defined as follows (see Figure 2):*

$$\mathfrak{P}(\mathbf{v}, \mu) = \bigcup_{i=1}^{3} \bigcup_{\substack{\mathbf{x} \in \mathbb{Z}^3 \\ 0 \leq \langle \mathbf{x}, \mathbf{v} \rangle + \mu < v_i}} (\mathbf{x}, i^*).$$

In other words, one has:

Proposition 1 ([IO93,IO94]) *Let $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$. The* stepped plane *$\mathfrak{P}(\mathbf{v}, \mu)$ is the boundary of the union of the unit cubes intersecting the open*

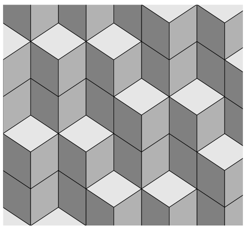

Figure 2. A piece of a stepped plane in $\mathbb{R}^3$.

half-space $\{\mathbf{x} \in \mathbb{R}^3,\ \langle \mathbf{v}, \mathbf{x}\rangle + \mu < 0\}$. *The set* $\mathfrak{P}(\mathbf{v}, \mu) \cap \mathbb{Z}^3$ *is called the* set of vertices *of* $\mathfrak{P}(\mathbf{v}, \mu)$.

Let Δ be the diagonal plane of equation $x_1 + x_2 + x_3 = 0$ and let π be the orthogonal projection onto Δ. Note that $\pi(\mathbb{Z}^3)$ is a lattice in Δ with basis $\pi(e_1)$, $\pi(e_2)$, and that $\pi(e_3) = -\pi(e_1) - \pi(e_2)$. If we use this basis for $\pi(\mathbb{Z}^3)$, then the restriction of π to $\mathbb{Z}^3$ becomes the following map, also denoted by π by abuse of notation:

$$\pi \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2,\ \mathbf{x} \longmapsto (x_1 - x_3, x_2 - x_3).$$

By construction, a stepped plane is a union of (closed) faces of type 1, 2 or 3. Let us introduce the map $v : \mathcal{F} \longrightarrow \mathbb{Z}^3$ defined by $v(\mathbf{x}, i^*) = \mathbf{x} + \mathbf{e_1} + \cdots + \mathbf{e_{i-1}}$, for $i \in \{1, 2, 3\}$, which associates with each face (x, i^*) a *distinguished vertex* $v(x, i^*)$. One proves that the set of vertices of a stepped plane is in one-to-one correspondence with the set of distinguished vertices of the faces of this stepped plane. More precisely, one gets:

Proposition 2 ([BV00,ABI02]) *Let* $\mathbf{v} \in \mathbb{R}^3_+$, $\mu \in \mathbb{R}$. *One has*

$$\mathfrak{P}(\mathbf{v}, \mu) \cap \mathbb{Z}^3 = \{v(\mathbf{x}, i^*),\ (\mathbf{x}, i^*) \subset \mathfrak{P}(\mathbf{v}, \mu)\},$$

and thus

$$\forall (m_1, m_2) \in \mathbb{Z}^2,\ \exists! (\mathbf{x}, i^*) \in \mathfrak{P}(\mathbf{v}, \mu),\ \pi \circ v(\mathbf{x}, i^*) = (m_1, m_2).$$

Furthermore, the restriction of the projection map π *to* $\mathfrak{P}(\mathbf{v}, \mu)$ *is one-to-one and onto* Δ*; the projections of the faces of the stepped plane* $\mathfrak{P}(\mathbf{v}, \mu)$ *tile the diagonal plane* Δ *with three kinds of lozenges (see Figure 3).*

Note that we recover here some classical notions of discrete geometry. According to J.-P. Reveillès' terminology [Rev91], given $\mathbf{v} \in \mathbb{R}^3$ and $(\mu, \omega) \in \mathbb{R}^2$, the *lower* (resp. *upper*) *arithmetical hyperplane* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is defined as the set of points $\mathbf{x} \in \mathbb{Z}^3$ satisfying $0 \leq \langle \mathbf{x}, \mathbf{v}\rangle + \mu < \omega$ (resp. $0 < \langle \mathbf{x}, \mathbf{v}\rangle + \mu \leq \omega$). Moreover, if $\omega = \sum |v_i| = \|\mathbf{v}\|_1$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said to be *standard*, while it is said to be *naive* if $\omega = \max |v_i| = \|\mathbf{v}\|_\infty$. One checks that the set $\{\mathbf{x} \in \mathbb{Z}^3,\ \exists i \in \{1, 2, 3\},\ (\mathbf{x}, i^*) \subset \mathfrak{P}(\mathbf{v}, \mu)\}$ is the lower naive arithmetical

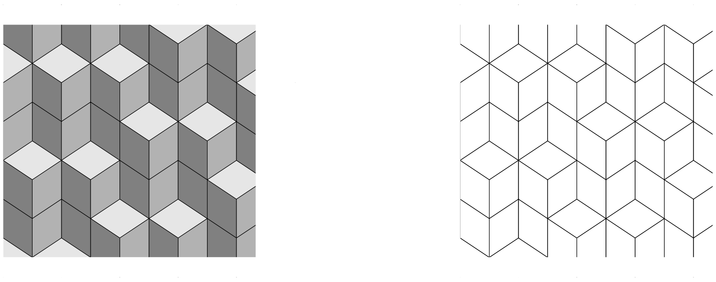

Figure 3. From a stepped plane to a tiling of the plane Δ by three kinds of lozenges.

plane $\mathfrak{P}(\mathbf{v}, \mu, \|\mathbf{v}\|_\infty)$, whereas $\mathfrak{P}(\mathbf{v}, \mu) \cap \mathbb{Z}^3$ is the lower standard arithmetical plane $\mathfrak{P}(\mathbf{v}, \mu, \|\mathbf{v}\|_1)$ (see [CDS04]).

The bijection between the faces of $\mathfrak{P}(\mathbf{v}, \mu)$ and the lattice $\mathbb{Z}^2$ ensures us, that, given a point $(m_1, m_2) \in \mathbb{Z}^2$, there exists one and only one face $(\mathbf{x}, i^*)$ of $\mathfrak{P}(\mathbf{v}, \mu)$ such that $\pi \circ v(\mathbf{x}, i^*) = (m_1, m_2)$ (see Proposition 2). We thus provide each stepped plane with a two-dimensional coding as follows:

Definition 3 (Two-dimensional coding of a stepped plane) *Let $\mathfrak{P}(\mathbf{v}, \mu)$ be a stepped plane with $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$. The* two-dimensional coding *of the stepped plane $\mathfrak{P}(\mathbf{v}, \mu)$ is the two-dimensional word $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$ defined by: for all $(m_1, m_2) \in \mathbb{Z}^2$ and all $i \in \{1, 2, 3\}$,*

$$u_{m_1,m_2} = i \iff \exists (\mathbf{x}, i^*) \subset \mathfrak{P}(\mathbf{v}, \mu) \text{ such that } (m_1, m_2) = \pi \circ v(\mathbf{x}, i^*).$$

From Definition 3 and Proposition 2, an easy computation gives:

Proposition 3 ([BV00]) *Let $\mathbf{v} \in \mathbb{R}^3_+$, $\mu \in \mathbb{R}$ and $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$ be the two-dimensional coding of the stepped plane $\mathfrak{P}(\mathbf{v}, \mu)$. Let $(m_1, m_2) \in \mathbb{Z}^2$ and $i \in \{1, 2, 3\}$. Then $u_{m_1,m_2} = i$ if and only if:*

$$m_1 v_1 + m_2 v_2 + \mu \mod v_1 + v_2 + v_3 \in [v_1 + \cdots + v_{i-1}, v_1 + \cdots + v_i[.$$

Of course not all the two-dimensional words over the three-letter alphabet $\{1, 2, 3\}$ code a stepped plane. For instance, a word containing two consecutive 1's and two consecutive 2's in the same row cannot be the two-dimensional coding of a stepped plane.

In order to generalize the notion of stepped plane to the one of functional stepped surface (see Section 3), we use a slightly more precise property of the restriction of the projection map π to $\mathfrak{P}(\mathbf{v}, \mu)$.

Proposition 4 *The restriction of the map π to $\mathfrak{P}(\mathbf{v}, \mu)$ is a homeomorphism onto the plane Δ.*

PROOF. We already know from Proposition 2 that the restriction of π to

$\mathfrak{P}(\mathbf{v}, \mu)$ is a bijection. The restriction of the map π to $\mathfrak{P}(\mathbf{v}, \mu)$ is closed since each compact subset of $\mathfrak{P}(\mathbf{v}, \mu)$ is contained in a finite number of faces. This implies that $\pi^{-1} : \Delta \longrightarrow \mathfrak{P}(\mathbf{v}, \mu)$ is continuous. It follows that the map $\pi : \mathfrak{P}(\mathbf{v}, \mu) \longrightarrow \Delta$ is a homeomorphism. ■

3 Functional stepped surface

It is natural to try to extend the previous definitions and results to more general objects:

Definition 4 (Stepped surface [Jam04]) *A union $\mathfrak{S}$ of faces $(\mathbf{x}, i^*)$, where $\mathbf{x} \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$, is called a* functional stepped surface *if the restriction of the projection map π to $\mathfrak{S}$ is a homeomorphism. The set of integer points included in $\mathfrak{S}$ is called the set of* vertices *of $\mathfrak{S}$.*

In particular, a stepped plane is a functional stepped surface, according to Proposition 4. Furthermore, let us note that a functional stepped surface $\mathfrak{S}$ is a connected subset of $\mathbb{R}^3$; indeed, it is the image of the connected set Δ by a continuous map.

Proposition 5 ([Jam04,JP05]) *Let $\mathfrak{S}$ be a functional stepped surface. One has $\pi(\mathbb{Z}^3) = \pi \circ v(\{(\mathbf{x}, i^*),\ (\mathbf{x}, i^*) \subset \mathfrak{S}\})$. Furthermore, given $(m_1, m_2) \in \mathbb{Z}^2$, there exists a unique face $(\mathbf{x}, i^*) \subset \mathfrak{S}$ such that $(m_1, m_2) = \pi \circ v(\mathbf{x}, i^*)$.*

PROOF. The proof is deduced from a simple case study. ■

The following coding is thus well defined:

Definition 5 ($2D$-coding of a stepped surface) *A two-dimensional word $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$ is said to be the* coding *of the functional stepped surface $\mathfrak{S}$ if for all $(m_1, m_2) \in \mathbb{Z}^2$ and for every $i \in \{1, 2, 3\}$: $u_{m_1,m_2} = i \iff \exists (\mathbf{x}, i^*) \subset \mathfrak{S}$ such that $(m_1, m_2) = \pi \circ v(\mathbf{x}, i^*)$.*

Definition 6 (Lozenge tiling) *A* lozenge tiling *of Δ is defined as a subset $\mathcal{T}$ of $\{\pi(\mathbf{x}, i^*),\ (\mathbf{x}, i^\star) \in \mathcal{F}\}$ such that the union of the lozenges contained in $\mathcal{T}$ covers entirely Δ and furthermore, the interiors of two distinct lozenges do not intersect.*

An example of a piece of a lozenge tiling is depicted in Figure 4. Let $\mathfrak{S}$ be a functional stepped surface. By definition, let us note that $\{\pi(\mathbf{x}, i^*),\ (\mathbf{x}, i^*) \subset \mathfrak{S}\}$ is a lozenge tiling.

Proposition 6 *A union of faces $\mathfrak{S} \subset \mathcal{G}$ is a functional stepped surface if and*

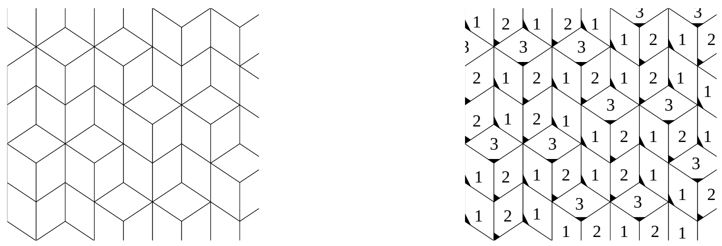

Figure 4. From a lozenge tiling of Δ to a 2D-word.

only if the restriction of π to $\mathfrak{S}$ is a bijection onto Δ.

Proof. Let $\mathfrak{S}$ be a union of faces such that the restriction of π to $\mathfrak{S}$ is a (continuous) bijection onto Δ. Every compact subset of Δ is included in a finite union L of lozenges of the form $\pi(\mathbf{x}, i^*)$, for $(\mathbf{x}, i^*) \subset \mathfrak{S}$. Furthermore, the preimage of L in $\mathfrak{S}$ by π is a finite union of faces, by injectivity of π. We deduce that the restriction of π to $\mathfrak{P}(\mathbf{v}, \mu)$ is closed, and similarly as in the proof of Proposition 4 that $\pi : \mathfrak{S} \longrightarrow \Delta$ is a homeomorphism. ■

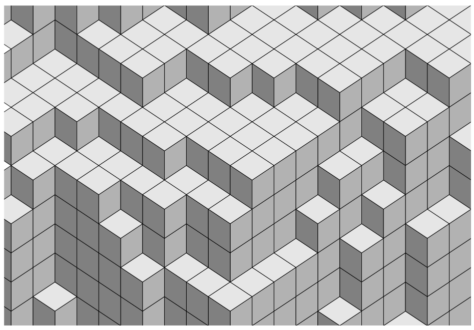

Figure 5. A stepped surface.

Proposition 7 *Let $\mathfrak{S}$ and $\mathfrak{S}'$ be two functional stepped surfaces. Then one has:*

$$\mathfrak{S} = \mathfrak{S}' \quad \Leftrightarrow \quad \mathfrak{S} \cap \mathbb{Z}^3 = \mathfrak{S}' \cap \mathbb{Z}^3.$$

In other words, a functional stepped surface is entirely characterized by the set of its vertices.

Proof. Let $\mathfrak{S}$ be a functional stepped surface and let u be the coding of $\mathfrak{S}$ (see Definition 5). It is sufficient to prove that, if the four vertices of a face $(\mathbf{x}, i^*)$ belong to $\mathfrak{S}$, then the whole face $(\mathbf{x}, i^*)$ is included in $\mathfrak{S}$. On the contrary and with no loss of generality, let us suppose that the four vertices $\mathbf{0}$, $\mathbf{e_2}$, $\mathbf{e_3}$ and $\mathbf{e_2} + \mathbf{e_3}$ of the face $(\mathbf{0}, 1^\star)$ belong to $\mathfrak{S}$, and that the face $(0, 1^\star)$ is not included in $\mathfrak{S}$. One thus has $u_{0,0} \neq 1$. If $u_{0,0} = 2$, then $-\mathbf{e_1} \in \mathfrak{S}$ and $\pi(-\mathbf{e_1}) = \pi(\mathbf{e_2} + \mathbf{e_3})$. Hence we obtain a contradiction with the bijectivity of $\pi : \mathfrak{S} \longrightarrow \Delta$. A similar investigation holds for $u_{0,0} = 3$, and for the general case of a face $(\mathbf{x}, i^*)$. ■

Definition 7 *(i) Let* $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ *and* $\mathbf{x}' = (x'_1, x'_2, x'_3) \in \mathbb{Z}^3$ *such that* $\pi(\mathbf{x}) = \pi(\mathbf{x}')$. *We say that* $\mathbf{x}$ *is* above $\mathbf{x}'$ *if* $x_1 + x_2 + x_3 \geq x'_1 + x'_2 + x'_3$, *otherwise we say that* $\mathbf{x}$ *is* below $\mathbf{x}'$.
(ii) We then say that a functional stepped surface $\mathfrak{S}$ *is* above *(resp.* below*) a stepped surface* $\mathfrak{S}'$ *if, for any* $\mathbf{x} \in \mathfrak{S} \cap \mathbb{Z}^3$ *and* $\mathbf{x}' \in \mathfrak{S}' \cap \mathbb{Z}^3$ *such that* $\pi(\mathbf{v}) = \pi(\mathbf{v}')$, $\mathbf{x}$ *is above (resp. below)* $\mathbf{x}'$.

Notation 1 *Given* $\mathbf{s} \in \mathbb{Z}^3$, *according to Proposition 7, one defines two particular functional stepped surfaces* $\hat{\mathcal{C}}_{\mathbf{s}}$ *and* $\check{\mathcal{C}}_{\mathbf{s}}$ *by their set of vertices as follows:*

$$\hat{\mathcal{C}}_{\mathbf{s}} \cap \mathbb{Z}^3 = \{\mathbf{s}' \in \mathbb{Z}^3,\, (s_1 - s'_1)(s_2 - s'_2)(s_3 - s'_3) = 0 \text{ and } s'_i \leq s_i,\, i \in \{1,2,3\}\},$$

$$\check{\mathcal{C}}_{\mathbf{s}} \cap \mathbb{Z}^3 = \{\mathbf{s}' \in \mathbb{Z}^3,\, (s_1 - s'_1)(s_2 - s'_2)(s_3 - s'_3) = 0 \text{ and } s'_i \geq s_i,\, i \in \{1,2,3\}\}.$$

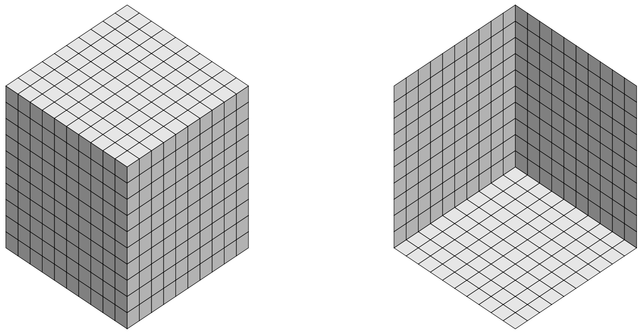

Figure 6. The stepped surfaces $\hat{\mathcal{C}}_{\mathbf{s}}$ (left) and $\check{\mathcal{C}}_{\mathbf{s}}$ (right).

Proposition 8 *Let* $\mathfrak{S}$ *be a stepped surface and let* $\mathbf{s} \in \mathfrak{S} \cap \mathbb{Z}^3$. *Then* $\hat{\mathcal{C}}_{\mathbf{s}}$ *(resp.* $\check{\mathcal{C}}_{\mathbf{s}}$*) is below (resp. above)* $\mathfrak{S}$.

Proof. Let $\mathfrak{S}$ be a stepped surface, $\mathbf{s} \in \mathfrak{S} \cap \mathbb{Z}^3$ and $\mathbf{s}' \in \check{\mathcal{C}}_{\mathbf{s}}$ such that $s_3 = s'_3$. The other cases can be similarly handled. We first introduce the finite sequence of integer points $(\mathbf{w}'_k)_{0 \leq k \leq s'_1 - s_1 + s'_2 - s_2}$ with values in $\check{\mathcal{C}}_{\mathbf{s}}$ defined as follows:

$$\mathbf{w}'_k = \begin{cases} \mathbf{s} & \text{if } k = 0, \\ \mathbf{s} + k\mathbf{e_1} & \text{if } k \in \{1, \ldots, s'_1 - s_1\}, \\ \mathbf{s} + (s'_1 - s_1)\mathbf{e_1} + (k - (s'_1 - s_1))\mathbf{e_2} & \text{if } k \in \{s'_1 - s_1 + 1, \cdots, \\ & (s'_1 - s_1) + (s'_2 - s_2)\}. \end{cases}$$

In particular, one notes that $\mathbf{w}'_{s'_1 - s_1 + s'_2 - s_2} = \mathbf{s}'$.

Let us now introduce the finite sequence of integer points $(\mathbf{w}_k)_{0 \leq k \leq s'_1 - s_1 + s'_2 - s_2}$ with values in $\mathfrak{S}$ defined as follows: for all $k \in \{0, \cdots, s'_1 - s_1 + s'_2 - s_2\}$, $\pi(\mathbf{w}_k) = \pi(\mathbf{w}'_k)$. In other words, $\mathbf{w}_k$ is the unique preimage in $\mathfrak{S}$ of $\pi(\mathbf{w}'_k)$

by π. One has, for $0 \leq k \leq s'_1 - s_1 - 1$, $\mathbf{w}_{k+1} - \mathbf{w}_k = \mathbf{e}_1$. Recall that the functional stepped surface $\mathfrak{S}$ is a connected subset of $\mathbb{R}^3$. Hence, $\mathbf{w'}_{k+1} - \mathbf{w}_{k+1} \in \{\mathbf{e}_1 + \mathbf{e_2} + \mathbf{e}_3, 0\}$. Similarly, for $s'_1 - s_1 \leq k \leq s'_1 - s_1 + s'_2 - s_2 - 1$, one has $\mathbf{w'}_{k+1} - \mathbf{w}_k = \mathbf{e}_2$, which also yields $\mathbf{w'}_{k+1} - \mathbf{w}_{k+1} \in \{\mathbf{e}_1 + \mathbf{e_2} + \mathbf{e}_3, 0\}$. One thus gets that $\mathbf{s'} = \mathbf{w'}_{s'_1 - s_1 + s'_2 - s_2}$ is above $\mathbf{w}_{s'_1 - s_1 + s'_2 - s_2}$. We similarly prove that $\hat{\mathcal{C}}_{\mathbf{s}}$ is below $\mathfrak{S}$. ∎

According to [Thu89], it is well-known that for any lozenge tiling of a region R of Δ bounded by a polygonon, there exists a three-dimensional interpretation, i.e., R can be lifted as a 2-skeleton of a cubical tiling of $\mathbb{R}^3$. This result naturally can be reformulated in terms of stepped surfaces.

Theorem 9 ([Thu89]) *Let $\mathcal{T}$ be a lozenge tiling of Δ. Then there exists a unique functional stepped surface $\mathfrak{S}$, up to translation by the vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, of the form $\bigcup_{\pi(\mathbf{x},i^*) \in \mathcal{T}} \pi(\mathbf{y}(\mathbf{x}), i^*)$ with $(\mathbf{y}(\mathbf{x}), i^*) \in \mathcal{F}$, and $\pi(\mathbf{x}, i^*) = \pi(\mathbf{y}(\mathbf{x}), i^*)$, for all $(\mathbf{x}, i^*)$ such that $\pi(\mathbf{x}, i^*) \in \mathcal{T}$. Such a functional stepped surface is said to* project *onto $\mathcal{T}$.*

Proof. We follow here the proof of [Thu89]. Let $\mathcal{T}$ be a lozenge tiling of Δ. Let us note that there is no reason for the union of faces $\bigcup_{\pi(\mathbf{x},i^*) \in \mathcal{T}} (\mathbf{x}, i^*)$ to be a functional stepped surface.

Let Γ be the lattice of Δ generated by the vectors $\pi(\mathbf{e}_1)$, $\pi(\mathbf{e_2})$, and $\pi(\mathbf{e_3})$. Similarly as in the proof of Proposition 5 (e.g., see [Jam04,JP05]), one proves by a finite case study that Γ is equal to the set of vertices of the lozenges $\pi(\mathbf{x}, i^*)$, as well as to the set the points $\pi \circ v(\mathbf{x}, i^*)$, for $\pi(\mathbf{x}, i^*) \in \mathcal{T}$. In other words, we have chosen a distinguished vertex for each lozenge $\pi(\mathbf{x}, i^*)$: for any $\gamma \in \Gamma$, there exists a unique $i^*_\gamma \in \{1, 2, 3\}$ such that i^*_γ is the type of the lozenge whose distinguished vertex is γ. One thus gets $\mathcal{T} = \{\pi(\gamma, i^*_\gamma),\ \gamma \in \Gamma\}$. Furthermore, there is a one-to-one correspondence between the lozenges $\pi(\mathbf{x}, i^*)$ of $\mathcal{T}$, and the faces (γ, i^*_γ), for $\gamma \in \Gamma$. Hence a functional stepped surface projects onto $\mathcal{T}$ if and only if it is of the form $\bigcup_{\gamma \in \Gamma} (\mathbf{x}_\gamma, i^*_\gamma)$, with $\pi(\mathbf{x}_\gamma) = \gamma$, for every $\gamma \in \Gamma$.

Let us first exhibit a functional stepped surface of the form $\bigcup_{\gamma \in \Gamma} (\mathbf{x}_\gamma, i^*_\gamma)$ with $\pi(\mathbf{x}_\gamma) = \gamma$, for every $\gamma \in \Gamma$. For that purpose, we introduce the oriented graph $G = (V, E)$ whose set of vertices is $V = \Gamma$, and whose set of edges E is equal to the set of edges of the lozenges $\pi(\mathbf{x}, i^*)$, for $\pi(\mathbf{x}, i^*) \in \mathcal{T}$, endowed with both orientations. We first define a *weight* function on the edges of G as follows: for any $\gamma, \gamma' \in \Gamma$ such that the oriented edge $e(\gamma, \gamma')$ from γ to γ' belongs to E, then one sets $w(\gamma, \gamma') = 1$, if $\gamma' = \gamma + \pi(\mathbf{e}_3)$, $w(\gamma, \gamma') = -1$, if $\gamma' = \gamma - \pi(\mathbf{e}_3)$, and 0, otherwise. One checks by induction on the lengths of the cycles of G that the sum of the weights of a cycle is equal to zero. We thus can define a *height* function on the vertices of G as follows: one sets $h_0 = 0$, and for any $\gamma, \gamma' \in \Gamma$ such that the edge with vertices γ and γ' belongs to E, then $h_{\gamma'} = h_\gamma + 1$, if

$\gamma' = \gamma + \pi(\mathbf{e}_3)$, $h_{\gamma'} = h_\gamma - 1$, if $\gamma' = \gamma - \pi(\mathbf{e}_3)$, and $h_{\gamma'} = h_\gamma$, otherwise. One checks that this function is well defined for any vertex of G since the graph G is connected, and according to the properties of the weight function. We then define for $\gamma \in \Gamma$, $\mathbf{x}_\gamma$ as the point of $\mathbb{R}^3$ equal to $\gamma + h_\gamma(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, and i^*_γ as the type of the unique lozenge whose distinguished vertex has coordinates γ. We now consider $\mathfrak{S} = \bigcup_{\gamma\in\Gamma}(\mathbf{x}_\gamma, i^*_\gamma)$. It remains to prove that $\mathfrak{S}$ is a functional stepped surface. According to Proposition 6, this is a direct consequence of the fact that the restriction of π to $\mathfrak{S}$ is a bijection: one first notes that the restriction of π to $\mathfrak{S} \cap \mathbb{Z}^3$ is one-to-one and onto Γ by construction; we conclude similarly as in the proof of Proposition 7.

Let us consider now a functional stepped surface that contains the origin $\mathbf{0}$ of $\mathbb{R}^3$ and that projects onto $\mathcal{T}$; it is of the form $\bigcup_{\gamma\in\Gamma}(\mathbf{y}_\gamma, i^*_\gamma)$. with $\mathbf{y}_\gamma - \gamma \in \mathbb{Z}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, for all $\gamma \in \Gamma$. A functional stepped surface is connected, hence one checks that, necessarily, $\mathbf{y}_\gamma = \gamma + h_\gamma$. ■

4 Flips acting on stepped surfaces

Let us define, for $\mathbf{s} \in \mathbb{Z}^3$, two specific unions of faces (see Figure 7):

$$\check{c}_\mathbf{s} = \bigcup_{i=1}^{3}(\mathbf{s}, i^*) \qquad \text{and} \qquad \hat{c}_\mathbf{s} = \bigcup_{i=1}^{3}(\mathbf{s} + \mathbf{e}_i, i^*).$$

Let us note that a functional stepped surface cannot contain simultaneously $\hat{c}_\mathbf{s}$ and $\check{c}_\mathbf{s}$. Furthermore, these two unions have the same boundary after projection by π. Hence, thanks to Theorem 9, if a functional stepped surface contains one of them, then by exchanging both unions, we obtain a functional stepped surface. This leads us to define a simple operation on functional stepped surfaces, the so-called *flip*, such as depicted in Figure 7:

Definition 8 (Flip) *Let* $\mathbf{s} \in \mathbb{Z}^3$*. The* flip *map* $\varphi_\mathbf{s} : \mathcal{G} \to \mathcal{G}$ *is defined as follows: if a union of faces* $\mathcal{E} \in \mathcal{G}$ *contains* $\hat{c}_\mathbf{s}$ *(resp.* $\check{c}_\mathbf{s}$*), then* $\varphi_\mathbf{s}(\mathcal{E})$ *is obtained by replacing* $\hat{c}_\mathbf{s}$ *by* $\check{c}_\mathbf{s}$ *(resp.* $\hat{c}_\mathbf{s}$ *by* $\check{c}_\mathbf{s}$*); otherwise,* $\varphi_\mathbf{s}(\mathcal{E}) = \mathcal{E}$*.*

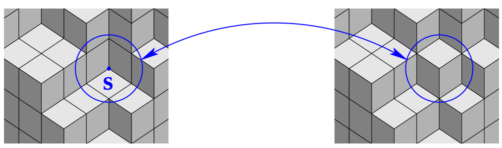

Figure 7. The action of flip $\varphi_\mathbf{s}$, for $\mathbf{x} \in \mathbb{Z}^3$: $\check{c}_\mathbf{s}$ (left) is exchanged with $\hat{c}_\mathbf{s}$ (right).

According to Theorem 9, we can perform a flip on a functional stepped surface if and only if one can perform a classic flip in the sense, e.g., of [Thu89], on the lozenge tiling of the plane which corresponds to this functional stepped surface.

We are now interested in performing on a functional stepped surface, not only one flip, but a sequence of flips. We first need to introduce the following notion:

Definition 9 (Locally finiteness) *A sequence of flips $(\varphi_{\mathbf{s}_n})_{n\in\mathbb{N}^*}$ is said to be* locally finite *if, for any $n_0 \in \mathbb{N}^*$, the set $\{\mathbf{s}_n \in \mathbb{Z}^3,\ \pi(\mathbf{s}_n) = \pi(\mathbf{s}_{n_0})\}$ is finite.*

Let us recall that the set $\mathcal{G}$ of unions of faces is provided with the topology induced by the distance d defined in Definition 1. Then one has:

Proposition 10 *Let $\mathfrak{S}$ be a functional stepped surface and $(\varphi_{\mathbf{s}_n})_{n\in\mathbb{N}^\star}$ be a locally finite sequence of flips such that the following limit exists:*

$$\mathfrak{S}' = \lim_{n\to\infty} \varphi_{\mathbf{s}_n} \circ \ldots \circ \varphi_{\mathbf{s}_1}(\mathfrak{S}).$$

Then, $\mathfrak{S}'$ is a stepped surface.

Proof. By performing a single flip on a stepped surface, one easily checks that one obtains a union of faces still homeomorphic by π to Δ, that is, a functional stepped surface. The case of the action of a finite number of flips is straightforward. Suppose now that we perform a locally finite sequence of flips $(\varphi_{\mathbf{s}_n})_{n\in\mathbb{N}^*}$ on the functional stepped surface $\mathfrak{S}$ such that $(\varphi_{\mathbf{s}_n}\circ\ldots\circ\varphi_{\mathbf{s}_1}(\mathfrak{S}))_{n\in\mathbb{N}^*}$ is convergent in the set $\mathcal{G}$ of unions of faces. According to Proposition 6, it is sufficient to prove that the restriction of π to $\mathfrak{S}'$ is a bijection onto Δ. Let $\mathbf{x}$ and $\mathbf{y}$ be two points of $\mathfrak{S}'$ such that $\pi(\mathbf{x}) = \pi(\mathbf{y})$. There exists $n \in \mathbb{N}$ such that $\mathbf{x}, \mathbf{y} \in \varphi_{\mathbf{s}_n} \circ \ldots \circ \varphi_{\mathbf{s}_1}(\mathfrak{S})$. Since $\varphi_{\mathbf{s}_n} \circ \ldots \circ \varphi_{\mathbf{s}_1}(\mathfrak{S})$ is a functional stepped surface, it follows that $\mathbf{x} = \mathbf{y}$. We thus have proved that the restriction of π is one-to-one.
Let $\mathbf{z} \in \Delta$. Let A be a bounded subset of Δ containing $\mathbf{z}$. By the local finiteness of the sequence $(\varphi_{\mathbf{s}_n})_{n\in\mathbb{N}^\star}$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$, then $\pi(\mathbf{s}_n) \notin A$. Take $n_1 \geq n_0$; we also assume n_1 large enough for $\mathfrak{S}'$ and $\varphi_{\mathbf{s}_{n_1}} \circ \ldots \circ \varphi_{\mathbf{s}_1}(\mathfrak{S})$ to coincide on their intersection with $\pi^{-1}(A)$. Let $\mathbf{y} \in \varphi_{\mathbf{s}_{n_1}} \circ\ldots\circ \varphi_{\mathbf{s}_1}(\mathfrak{S})$ such that $\pi(\mathbf{y}) = \mathbf{z}$. Then one has $\mathbf{y} \in \varphi_{\mathbf{s}_n} \circ\ldots\circ \varphi_{\mathbf{s}_1}(\mathfrak{S})$ for all $n \geq n_1$, and thus $\mathbf{y} \in \mathfrak{S}'$. We have proved that the restriction of π is onto, which concludes the proof. ■

Thus, flips allow to transform functional stepped surfaces into functional stepped surfaces. However, one cannot necessarily transform a *given* functional stepped surface into another *given* one by a locally finite sequence of flips. See Figure 8 for some examples of (un)accessibility by flips.

In order to characterize the (un)accessibility by flips between stepped surfaces, we introduce the notion of *shadows*, illustrated in Figure 9:

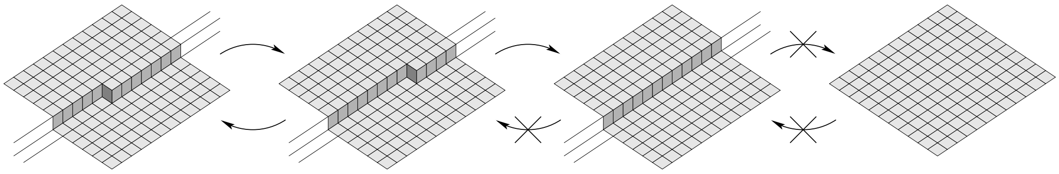

Figure 8. One transform the first stepped surface into the second one, and conversely, by performing a finite number of flips. A locally finite sequence of flips allows one to transform the second stepped surface into the third one (we perform an infinite and locally finite sequence of flips which rejects to the infinity the only face of type 1*), but the converse transformation is impossible (no flip can be performed). Lastly, we can neither transform by flips the fourth stepped surface into the third one, nor conversely.

Definition 10 (Shadows) *Let $\mathfrak{S}$ be a functional stepped surface. We define three projection maps from $\mathbb{R}^3$ to $\mathbb{R}^2$ by:*

and

$$\pi_3 : (x_1, x_2, x_3) \longmapsto (x_1, x_2).$$

The shadows of $\mathfrak{S}$ are respectively defined as the three images of the stepped surface $\mathfrak{S}$ by these maps.

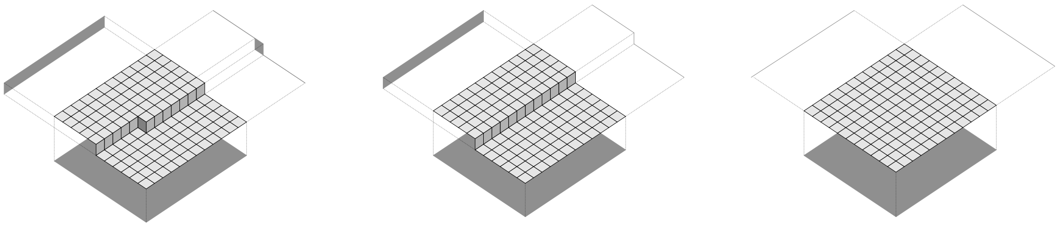

Figure 9. The shadows of the stepped surfaces of Figure 8. The central stepped surface has all its shadows included in the corresponding ones of the leftmost stepped surface. The shadows of the rightmost stepped surface are neither included in the shadow of the other stepped surfaces, nor contain them.

Considering the functional stepped surfaces of Figure 8, it is worth remarking that one can transform one functional stepped surface into another one if and only if the shadows of the first one contain the respective shadows of the second one (see Figure 9). This turns out to be a general fact:

Proposition 11 *Let $\mathfrak{S}$ and $\mathfrak{S}'$ be two functional stepped surfaces. The following assertions are equivalent:*
(i) There exists a locally finite sequence $(\varphi_{\mathbf{s}_n})_{n\in\mathbb{N}^*}$ of flips such that

$$\mathfrak{S}' = \lim_{n\to\infty} \varphi_{\mathbf{s}_n} \circ \ldots \circ \varphi_{\mathbf{s}_1}(\mathfrak{S});$$

(ii) the three shadows of $\mathfrak{S}'$ are included in the corresponding shadows of $\mathfrak{S}$.

PROOF. Since $\hat{c}_{\mathbf{s}}$ and $\check{c}_{\mathbf{s}}$ have the same shadows, performing a flip does not modify the shadows of a functional stepped surface. By performing a *sequence* of flips, the shadows cannot be extended. However, note that they can be reduced (recall the example of Figure 9). Thus, if the stepped surface $\mathfrak{S}'$ can be obtained by performing a locally finite sequence of flips on the stepped surface $\mathfrak{S}$, then the three shadows of $\mathfrak{S}'$ are included in the corresponding shadows of $\mathfrak{S}$.
Conversely, let $\mathfrak{S}'$ and $\mathfrak{S}$ be two functional stepped surfaces such that the three shadows of $\mathfrak{S}'$ are included in the corresponding shadows of $\mathfrak{S}$. Let us consider a vertex $\mathbf{x} \in \mathfrak{S}' \cap \mathbb{Z}^3$ of the functional stepped surface $\mathfrak{S}'$. With no loss of generality, we suppose that $\mathbf{x}$ is above the stepped surface $\mathfrak{S}$, according to Definition 7. We associate with this vertex $\mathbf{x} \in \mathfrak{S}'$ the following union of faces of $\mathfrak{S}$ (see Figure 10):

$$\mathcal{T}_{\mathbf{x}} = \bigcup_{\substack{(\mathbf{x}',i^*)\subset\mathfrak{S} \\ x'_j \leq x_j,\ j=1,2,3}} (\mathbf{x}', i^*) \subset \mathfrak{S}.$$

Let us prove that $\mathcal{T}_{\mathbf{x}}$ is a *finite* union of faces. By assumption, the shadow

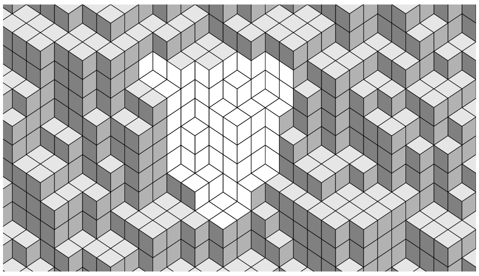

Figure 10. Given a stepped surface $\mathfrak{S}$, a vertex $\mathbf{x} \in \mathfrak{S}' \cap \mathbb{Z}^3$ defines a subset $\mathcal{T}_{\mathbf{x}}$ of $\mathfrak{S}$ (in white). To $\mathcal{T}_{\mathbf{x}}$ corresponds a lozenge tiling of a bounded and simply connected domain of $\mathbb{R}^2$.

$\pi_1(\mathfrak{S}')$ is included in the shadow $\pi_1(\mathfrak{S})$. In particular, $\pi_1(\mathbf{x}) \in \pi_1(\mathfrak{S})$: there exists $x'_1 \in \mathbb{Z}$ such that $(x'_1, x_2, x_3) \in \mathfrak{S}$. Then, according to Proposition 8, $\mathcal{T}_{\mathbf{x}} \subset \mathfrak{S}$ is above $\hat{\mathcal{C}}_{(x'_1,x_2,x_3)}$. Consequently, for any $\mathbf{x}'' \in \mathcal{T}_{\mathbf{x}}$, one has $x''_2 \leq x_2$ and $x''_3 \leq x_3$; this yields that $x'_1 \leq x''_1 \leq x_1$. Similarly, there exist $x'_2 \in \mathbb{Z}$ and $x'_3 \in \mathbb{Z}$ such $(x_1, x'_2, x_3) \in \mathfrak{S}$ and $(x_1, x_2, x'_3) \in \mathfrak{S}$, and for any $\mathbf{x}'' \in \mathcal{T}_{\mathbf{x}}$, then $x'_2 \leq x''_2 \leq x_2$ and $x'_3 \leq x''_3 \leq x_3$. Thus, $\mathcal{T}_{\mathbf{x}}$ is bounded, that is, it is a finite union of faces.
Let us now consider the union of faces $\hat{\mathcal{T}}_{\mathbf{x}}$ which is included in $\hat{\mathcal{C}}_{\mathbf{x}}$ and satisfies $\pi(\hat{\mathcal{T}}_{\mathbf{x}}) = \pi(\mathcal{T}_{\mathbf{x}})$ (see Figure 11, left). Similarly as $\mathcal{T}_{\mathbf{x}}$, $\hat{\mathcal{T}}_{\mathbf{x}}$ is a finite union of faces. A classic result of the theory of lozenge tilings (see, e.g., [Thu89]) yields that the tiling corresponding to $\mathcal{T}_{\mathbf{x}}$ can be transformed by performing a finite number of flips into the tiling corresponding to $\hat{\mathcal{T}}_{\mathbf{x}}$. In terms of stepped surfaces, this means that a finite number of flips transforms $\mathcal{S}$ (which contains $\mathcal{T}_{\mathbf{x}}$) into a stepped surface which contains $\hat{\mathcal{T}}_{\mathbf{x}}$, hence the vertex $\mathbf{x}$ of $\mathcal{S}'$ (since $\mathbf{x} \in \hat{\mathcal{T}}_{\mathbf{x}}$)

too.

Now, we would like to perform such a finite number of flips for each $\mathbf{x} \in \mathfrak{S}' \cap$

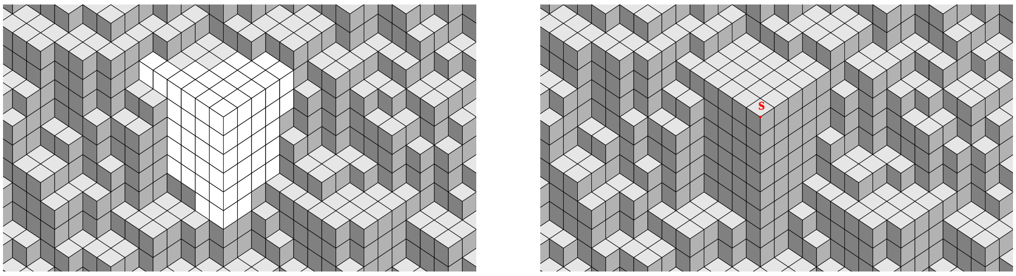

Figure 11. By performing a finite number of flips, one transforms $\mathcal{T}_{\mathbf{x}}$ (Figure 10, right) into the union of faces $\hat{\mathcal{T}}_{\mathbf{x}}$ (left, with white faces). We obtain a stepped surface which contains the vertex $\mathbf{x}$ of $\mathfrak{S}'$, similarly as $\hat{\mathcal{T}}_{\mathbf{x}}$ does (right). By performing such a finite number of flips for each vertex of $\mathcal{S}'$, this transforms the stepped surface $\mathfrak{S}$ into the stepped surface $\mathfrak{S}'$.

$\mathbb{Z}^3$, in order to transform by an infinite sequence of flips the functional stepped surface $\mathfrak{S}$ into a functional stepped surface which would contain all the vertices $\mathfrak{S}' \cap \mathbb{Z}^3$, that is, into $\mathfrak{S}'$, by Proposition 7. The only problem could be the following one: by performing the flips to obtain a stepped surface containing a given $\mathbf{x}$ in $\mathfrak{S}' \cap \mathbb{Z}^3$, we could lose a vertex $\mathbf{x}'$ of $\mathfrak{S}' \cap \mathbb{Z}^3$ previously obtained by performing flips. However, the flips performed to obtain $\mathbf{x} \in \mathfrak{S}' \cap \mathbb{Z}^3$ are performed *below* $\hat{\mathcal{T}}_{\mathbf{x}}$, in particular *below* $\mathfrak{S}'$ since $\hat{\mathcal{T}}_{\mathbf{x}} \subset \hat{\mathcal{C}}_{\mathbf{x}}$ and $\hat{\mathcal{C}}_{\mathbf{x}}$ is below $\mathfrak{S}'$ by Proposition 8. Hence, we do not lose the previously obtained vertices of $\mathfrak{S}' \cap \mathbb{Z}^3$, and the whole (infinite) sequence of flips thus transforms $\mathfrak{S}$ into $\mathfrak{S}'$.

To conclude, we note that the finite number of flips performed to obtain a stepped surface containing a vertex $\mathbf{x}$ of $\mathfrak{S}' \cap \mathbb{Z}^3$ are performed at a *bounded* distance from $\mathbf{x}$. This yields that the previous sequence of flips (that is, the one used to obtain the stepped surface containing *all* the vertices of $\mathfrak{S}' \cap \mathbb{Z}^3$) contains, for each $\pi(\mathbf{x}) \in \pi(\mathfrak{S}' \cap \mathbb{Z}^3) = \pi(\mathbb{Z}^3)$, a *finite* number of flips $\varphi_{\mathbf{x}'}$ such that $\pi(\mathbf{x}') = \pi(\mathbf{x})$. Thus, this is a *locally finite* sequence of flips. This completes the proof. ■

Hence, flips transform functional stepped surfaces into functional stepped surfaces, and we have obtained a necessary and sufficient condition - in terms of shadows - under which a given functional stepped surface can be transformed by flips into another one. In particular, we can use these results to give an equivalent definition of functional stepped surfaces:

Theorem 12 *A union of faces $\mathfrak{U} \in \mathcal{G}$ is a functional stepped surface if and only if there exist a stepped plane $\mathfrak{P}$ and a locally finite sequence of flips $(\varphi_{\mathbf{s}_n})_{n \in \mathbb{N}}$ such that*

$$\mathfrak{U} = \lim_{n \to \infty} \varphi_{\mathbf{s}_n} \circ \ldots \circ \varphi_{\mathbf{s}_1}(\mathfrak{P}).$$

Proof. Since a stepped plane is a functional stepped surface, Proposition 10 yields that the limit of a sequence of functional stepped surfaces obtained by performing a locally finite sequence of flips over a stepped plane is a functional stepped surface. Conversely, it is easy to check that the three shadows of a stepped plane with normal vector $\mathbf{v} \in \mathbb{R}^3_+$ such that $v_1 v_2 v_3 \neq 0$ are equal to the whole plane $\mathbb{R}^2$. Therefore, according to Proposition 11, one can transform by flips any stepped plane $\mathfrak{P}$ into a given stepped surface $\mathfrak{S}$. ■

5 Generalized substitutions

We first review in Section 5.1 the notion of *generalized substitutions* [AI01]; we then discuss in Section 5.2 the way they act on stepped planes and more generally on functional stepped surfaces.

5.1 First definitions

Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{A}^\star$ be the set of finite words over $\mathcal{A}$. The empty word is denoted by ε. A *substitution* is an endomorphism of the free-monoid $\mathcal{A}^\star$ such that the image of every letter of $\mathcal{A}$ is non-empty. Such a definition naturally extends to infinite or biinfinite words in $\mathcal{A}^\mathbb{N}$ and $\mathcal{A}^\mathbb{Z}$.

Assume $\mathcal{A} = \{1, 2, 3\}$ and let σ be a substitution over $\mathcal{A}$. The *incidence matrix* $\mathbf{M}_\sigma$ of σ is the 3×3 matrix defined by:

$$\mathbf{M}_\sigma = (|\sigma(j)|_i)_{(i,j)\in\{1,2,3\}^2},$$

where $|\sigma(j)|_i$ is the number of occurrences of the letter i in $\sigma(j)$.

A substitution σ is then said to be *unimodular* if $\det \mathbf{M}_\sigma = \pm 1$. In particular, $\mathbf{M}_\sigma^{-1}$ has *integer* coefficients. Let $\mathbf{f} : \{1, 2, 3\}^\star \longrightarrow \mathbb{N}^3$ be the map defined by $\mathbf{f}(w) = {}^t(|w|_1, |w|_2, |w|_3)$. The map $\mathbf{f}$ is usually called the *Parikh mapping* and is the homomorphism obtained by abelianization of the free monoid $\mathcal{A}^\star$. One has for every $w \in \{1, 2, 3\}^\star$, $\mathbf{f}(\sigma(w)) = \mathbf{M}_\sigma \cdot \mathbf{f}(w)$.

Definition 11 (Generalized substitution [AI01]) *Let σ be a unimodular substitution over $\{1, 2, 3\}$. The* generalized substitution $\Theta^*_\sigma : \mathcal{G} \longrightarrow \mathcal{G}$ *is defined by:*

$$\forall (\mathbf{v}, i^*) \in \mathfrak{F}, \qquad \Theta^*_\sigma(\mathbf{v}, i^*) = \bigcup_{\substack{j,p,s \\ \sigma(j)=p\cdot i\cdot s}} \left(\mathbf{M}_\sigma^{-1}(\mathbf{v} + \mathbf{f}(s)), j^*\right)$$

and

$$\forall \mathcal{E} \in \mathcal{G}, \qquad \Theta^*_\sigma(\mathcal{E}) = \bigcup_{(\mathbf{v},i^*)\subseteq\mathcal{E}} \Theta^*_\sigma\left((\mathbf{v}, i^*)\right).$$

Example 1 *Let* $\sigma : \{1,2,3\}^\star \longrightarrow \{1,2,3\}^\star$ *be the substitution defined by* $\sigma : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$. *Then,*

$$\mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{M}_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This yields (see Figure 12):

$$\Theta^*_\sigma \ : \ \begin{array}{l} (\mathbf{v}, 1^*) \mapsto (\mathbf{M}_\sigma^{-1}\mathbf{v} + \mathbf{e_1} - \mathbf{e_2}, 1^*) \cup (\mathbf{M}_\sigma^{-1}\mathbf{v}, 2^*) \\ (\mathbf{v}, 2^*) \mapsto (\mathbf{M}_\sigma^{-1}\mathbf{v}, 3^*) \\ (\mathbf{v}, 3^*) \mapsto (\mathbf{M}_\sigma^{-1}\mathbf{v}, 1^*). \end{array}$$

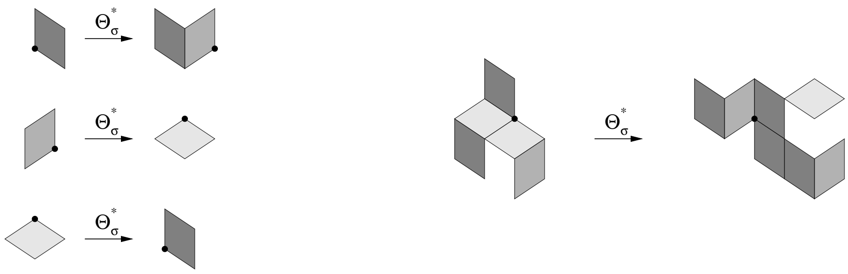

Figure 12. Action of Θ^*_σ on single faces and on a given union of faces.

There is a natural measure μ defined on the elements of $\mathcal{G}$, obtained by extension of the two-dimensional Lebesgue measure. Two elements $\mathcal{E}$ and $\mathcal{E}'$ of $\mathcal{G}$ are then said to be *μ-disjoint* if $\mu(\mathcal{E} \cap \mathcal{E}') = 0$. In other words, this means that both sets do not intersect, except possibly on edges. A generalized substitution does not necessarily map μ-disjoint faces to μ-disjoint unions of faces. Consider in Example 1, $\Theta^*_\sigma(\mathbf{0}, 1^*) \cap \Theta^*_\sigma(\mathbf{e_3}, 3^*)$. One has $\Theta^*_\sigma(\mathbf{0}, 1^*) = (\mathbf{e_1} - \mathbf{e_2}, 1^*) \cup (\mathbf{0}, 2^*)$ and $\Theta^*_\sigma(\mathbf{e_3}, 3^*) = (\mathbf{e_1} - \mathbf{e_2}, 1^*)$, whence $\mu(\Theta^*_\sigma(\mathbf{0}, 1^*) \cap \Theta^*_\sigma(\mathbf{e_3}, 3^*)) \neq 0$.

Definition 12 *A generalized substitution* Θ^*_σ *is said to* act properly *on a union of faces* $\mathcal{E} \subset \mathcal{G}$ *if μ-disjoint faces of* $\mathcal{E}$ *are mapped onto μ-disjoint unions of faces.*

Stepped planes are particularly interesting with respect to this property as shown by Theorem 13 below. Let us assume that the substitution σ is primitive, that is, $\mathbf{M}_\sigma$ admits a power with positive entries. Let $\mathbf{v} \in \mathbb{R}^3_+$ be a Perron-Frobenius left eigenvector of $\mathbf{M}_\sigma$ having only positive entries. Then, the generalized substitution Θ_σ^* is proved in [AI01] to act properly on the stepped plane $\mathfrak{P}(\mathbf{v}, 0)$, and to map it onto itself. More generally, one has the following:

Theorem 13 ([Fer05b]) *Let σ be a unimodular substitution over $\{1, 2, 3\}$, $\mathbf{v} \in \mathbb{R}_+^3$ and $\mu \in \mathbb{R}$. The generalized substitution Θ_σ^* acts properly on the stepped plane $\mathfrak{P}(\mathbf{v}, \mu)$; furthermore Θ_σ^* maps $\mathfrak{P}(\mathbf{v}, \mu)$ onto the stepped plane $\mathfrak{P}({}^t\mathbf{M}_\sigma \mathbf{v}, \mu)$.*

5.2 Generalized substitutions and functional stepped surfaces

The aim of this section is to extend the previous results to functional stepped surfaces, by proving the main theorem of this paper:

Theorem 14 *Let σ be a unimodular substitution over $\{1, 2, 3\}$. The generalized substitution Θ_σ^* acts properly on every functional stepped surface. Furthermore, the image by Θ_σ^* of a functional stepped surface is a functional stepped surface.*

Let us note that a partial version of Theorem 14 has been stated in [ABJ05]. An illustration of Theorem 14 is depicted in Figure 13.

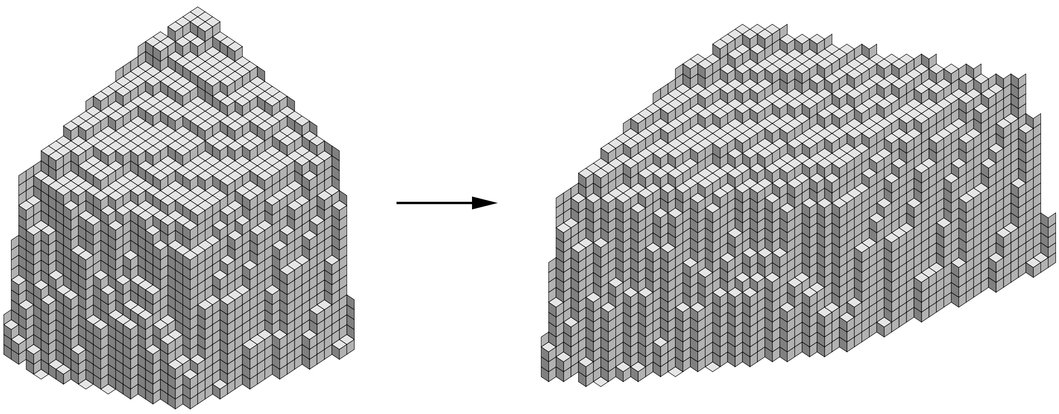

Figure 13. A generalized substitution maps a stepped surface onto a stepped surface.

Several lemmas are required to prove Theorem 14. Let us first prove the continuity of any generalized substitution as a map from $\mathcal{G}$ to $\mathcal{G}$ provided with the distance d (see Definition 1):

Lemma 15 *Let $(\mathcal{E}_n)_{n\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ be a convergent sequence in $\mathcal{G}$. Then the sequence $(\Theta_\sigma^*(\mathcal{E}_n))_{n\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ is a convergent sequence in $\mathcal{G}$. One thus gets:*

$$\lim_{n\to\infty} \Theta_\sigma^*(\mathcal{E}_n) = \Theta_\sigma^*(\lim_{n\to\infty} \mathcal{E}_n).$$

PROOF. Let $\mathcal{E}$ stand for the limit of the sequence $(\mathcal{E}_n)_{n\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$. Let us prove that the sequence $(\Theta_\sigma^*(\mathcal{E}_n))_{n\in\mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ converges towards $\Theta_\sigma^*(\mathcal{E})$. For $n \in \mathbb{N}$, let r_n be such that for all $m \geq r_n$, then $\mathcal{E}_m$ and $\mathcal{E}$ contain the same faces in a ball of radius n centered on $\mathbf{0}$. Let $M = \max\{\|\mathbf{M}_\sigma \mathbf{f}(s)\|_\infty,\ s$ suffix of $\sigma(i),\ i \in \{1,2,3\}\}$. Let α be the modulus of the smallest eigenvalue of $\mathbf{M}_\sigma^{-1}$. Let us recall that, for all $\mathbf{x} \in \mathbb{R}^3$, $\|\mathbf{M}_\sigma^{-1}\mathbf{x}\|_\infty \geq \alpha\|\mathbf{x}\|_\infty$. Let $n > M$, $m \geq r_n$, and let $(\mathbf{y}, j^*) \subseteq \Theta_\sigma^*(\mathcal{E}_m)$ such that $\|\mathbf{y}\|_\infty \leq \alpha(n-M)$. Let $(\mathbf{x}, i^*) \subseteq \mathcal{E}_m$ such that $(\mathbf{y}, j^*) \subseteq \Theta_\sigma^*(\mathbf{x}, i^*)$; one has $\mathbf{y} = \mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{f}(s)$, with $\sigma(j) = p \cdot i \cdot s$. One deduces $\|\mathbf{x} + \mathbf{M}_\sigma \mathbf{f}(s)\|_\infty \leq n - M$. Hence $\|\mathbf{x}\|_\infty \leq n$ and $(\mathbf{y}, j^*) \subseteq \Theta_\sigma^*(\mathcal{E})$. We show in a similar way that any face $(\mathbf{y}, j^*)$ included in $\Theta_\sigma^*(\mathcal{E})$ and satisfying $\|\mathbf{y}\|_\infty \leq \alpha(n-M)$ is included in $\Theta_\sigma^*(\mathcal{E}_m)$. In other words, $d(\Theta^*(\mathcal{E}_m), \Theta^*(\mathcal{E})) \leq 2^{-\alpha(n-M)}$, for every $m \geq r_n$, which concludes the proof. ∎

The following lemma plays a key role by relating the action of generalized susbtitutions to the action of flips, such as depicted in Figure 14:

Lemma 16 *Let Θ_σ^* be a generalized substitution that acts properly on $\mathcal{E} \subset \mathcal{G}$. Then, for any $\mathbf{x} \in \mathbb{Z}^3$, Θ_σ^* acts properly on $\varphi_{\mathbf{s}}(\mathcal{E})$, and furthermore, Θ_σ^* maps $\varphi_{\mathbf{s}}(\mathcal{E})$ onto $\varphi_{\mathbf{M}_\sigma^{-1}\mathbf{x}}(\Theta_\sigma^*(\mathcal{E}))$.*

PROOF. Let us first compute $\Theta_\sigma^*(\check{c}_{\mathbf{x}})$. One has:

$$\begin{aligned}
\Theta_\sigma^*(\check{c}_{\mathbf{x}}) &= \bigcup_{i=1,2,3} \Theta_\sigma^*(\mathbf{x}, i^*) = \bigcup_{\substack{j,p,i,s\\ \sigma(j)=p\cdot i\cdot s}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(s)), j^*)\\
&= \bigcup_{\substack{j,p'\neq\varepsilon,s\neq\varepsilon\\ \sigma(j)=p'\cdot s}} (M_\sigma^{-1}(\mathbf{x} + \mathbf{f}(s)), j^*) \cup \underbrace{\bigcup_{j=1,2,3} (\mathbf{M}_\sigma^{-1}\mathbf{x}, j^*)}_{\check{c}_{\mathbf{M}_\sigma^{-1}\mathbf{x}}}.
\end{aligned}$$

and

$$\begin{aligned}
\Theta_\sigma^*(\hat{c}_{\mathbf{x}}) &= \bigcup_{i=1,2,3} \Theta_\sigma^*(\mathbf{x} + \mathbf{e_i}, i^*) = \bigcup_{\substack{j,p,i,s\\ \sigma(j)=p\cdot i\cdot s}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{e_i} + \mathbf{f}(s)), j^*)\\
&= \bigcup_{\substack{j,p,i,s\\ \sigma(j)=p\cdot i\cdot s}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(i\cdot s)), j^*) = \bigcup_{\substack{j,p,s'\neq\varepsilon\\ \sigma(j)=p\cdot s'}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(s')), j^*)\\
&= \bigcup_{\substack{j,p\neq\varepsilon,s'\neq\varepsilon\\ \sigma(j)=p\cdot s'}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(s')), j^*) \cup \bigcup_{j=1,2,3} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(\sigma(j))), j^*)\\
&= \bigcup_{\substack{j,p\neq\varepsilon,s'\neq\varepsilon\\ \sigma(j)=p\cdot s'}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(s')), j^*) \cup \bigcup_{j=1,2,3} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{M}_\sigma \mathbf{e_j}), j^*)\\
&= \bigcup_{\substack{j,p\neq\varepsilon,s'\neq\varepsilon\\ \sigma(j)=p\cdot s'}} (\mathbf{M}_\sigma^{-1}(\mathbf{x} + \mathbf{f}(s')), j^*) \cup \underbrace{\bigcup_{j=1,2,3} (\mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{e_j}, j^*)}_{\hat{c}_{\mathbf{M}_\sigma^{-1}\mathbf{x}}},
\end{aligned}$$

since $\mathbf{e_i} = \mathbf{f}(i)$ and $\mathbf{f}(\sigma(j)) = \mathbf{M}_\sigma \mathbf{e_j}$. The desired result easily follows. ■

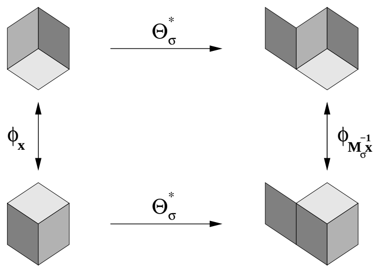

Figure 14. If two unions of faces differ by the flip $\varphi_{\mathbf{s}}$, then their images by Θ_σ^* differ by the flip $\varphi_{M_\sigma^{-1}\mathbf{x}}$ (one has here $\sigma \ : \ 1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2$).

Lemma 17 *Let $\mathfrak{S}$ be a stepped surface and $(\varphi_{\mathbf{x}_n})_{n\in\mathbb{N}^*}$ be a locally finite sequence of flips such that the sequence $(\varphi_{\mathbf{x_n}} \circ \ldots \circ \varphi_{\mathbf{x_1}}(\mathfrak{S}))_{n\in\mathbb{N}^*}$ is convergent in $\mathcal{G}$. Then, the sequence of flips $(\varphi_{M_\sigma^{-1}\mathbf{x}_n})_{n\in\mathbb{N}^*}$ is* locally finite.

Proof. We set $\mathfrak{S}' = \lim_{n\to\infty} \varphi_{\mathbf{x_n}} \circ \ldots \circ \varphi_{\mathbf{x_1}}(\mathfrak{S})$. According to Proposition 10, $\mathfrak{S}'$ is a stepped surface. Suppose that $(\varphi_{M_\sigma^{-1}\mathbf{x_n}})_{n\in\mathbb{N}^*}$ is not locally finite. Let us prove that this implies $\mathfrak{S}$ is not a stepped surface, which yields a contradiction. We first assume w.l.o.g. that for all $n \in \mathbb{N}^*$, either $\check{c}_{\mathbf{x_n}}$ or $\hat{c}_{\mathbf{x_n}}$ is a subset of $\varphi_{\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{x}_1}(\mathfrak{S})$. Since $(\varphi_{M_\sigma^{-1}\mathbf{x}_n})_{n\in\mathbb{N}^*}$ is not locally finite, there exists a subsequence $(\mathbf{y}_n)_{n\in\mathbb{N}^*}$ of $(\mathbf{x}_n)_{n\in\mathbb{N}^*}$, with $\sup_n \|\mathbf{y}_n\| = \infty$, such that:

$$\forall (m,n) \in (\mathbb{N}^*)^2, \qquad \pi(\mathbf{M}_\sigma^{-1}\mathbf{y}_m) = \pi(\mathbf{M}_\sigma^{-1}\mathbf{y}_n).$$

If we denote by $\mathbf{u}$ the vector $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, this is equivalent to say that there exists a sequence $(\lambda_n) \in \mathbb{Z}^{\mathbb{N}^*}$, with $\sup_n |\lambda_n| = \infty$, such that:

$$\forall n \in \mathbb{N}^*, \qquad \mathbf{M}_\sigma^{-1}(\mathbf{y}_n - \mathbf{y}_1) = \lambda_n \mathbf{u}.$$

The matrix $\mathbf{M}_\sigma$ admits nonnegative entries, and at least one positive entry in each row, since $\det(\mathbf{M}_\sigma) \neq 0$. Hence the vector $\mathbf{M}_\sigma\mathbf{u}$ has positive entries. Moreover, one can assume $\sup_n \lambda_n = \infty$ (the case $\inf_n \lambda_n = -\infty$ can be similarly handled). In addition with $\mathbf{y}_n = \mathbf{y}_1 + \lambda_n \mathbf{M}_\sigma \mathbf{u}$, where $(y_{n,1}, y_{n,2}, y_{n,3})$ stands for the entries of $\mathbf{y}_n$, this yields:

$$\lim_{n\to\infty} y_{n,1} = \lim_{n\to\infty} y_{n,2} = \lim_{n\to\infty} y_{n,3} = \infty.$$

For all n, $\mathbf{y_n}$ belongs to the stepped surface $\varphi_{\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{x_1}}(\mathfrak{S})$, which is hence above $\hat{\mathcal{C}}_{\mathbf{y}_n}$, according to Proposition 8. Let us consider the vertex $\mathbf{a}_n$ of this stepped surface whose image by π is $\mathbf{0}$. This vertex has three identical entries, say, $\mathbf{a}_n = (a_n, a_n, a_n)$ and is above $\hat{\mathcal{C}}_{\mathbf{y}_n}$. Hence, $a_n \geq \min(y_{n,1}, y_{n,2}, y_{n,3})$, and therefore, $\lim_n a_n = \infty$. Consider now the vertex $\mathbf{a}_\infty = (a_\infty, a_\infty, a_\infty)$ of $\mathfrak{S}'$ whose image by π is $\mathbf{0}$. For n large enough, $\mathbf{a}_\infty$ belongs to $\varphi_{\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{x}_1}(\mathfrak{S})$

and $\mathbf{a}_\infty \neq \mathbf{a}_n$, which yields a contradiction. ■

We are now in a position to prove Theorem 14:

PROOF. Let us consider a stepped surface $\mathfrak{S}$. According to Theorem 12, there exist a locally finite sequence of flips $(\varphi_{\mathbf{x}_n})_{n\in\mathbb{N}^*}$ and a stepped plane $\mathfrak{P}$ such that $\mathfrak{S}$ can be obtained by performing on $\mathfrak{P}$ the sequence of flips $(\varphi_{\mathbf{x}_n})$:

$$\mathfrak{S} = \lim_{n\to\infty} \varphi_{\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{x}_1}(\mathfrak{P}).$$

Then, Lemma 15 yields:

$$\Theta_\sigma^*(\mathfrak{S}) = \Theta_\sigma^*\left(\lim_{n\to\infty} \varphi_{\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{x}_1}(\mathfrak{P})\right) = \lim_{n\to\infty} \Theta_\sigma^*(\varphi_{\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{x}_1}(\mathfrak{P})),$$

and by Lemma 16 one has:

$$\Theta_\sigma^*(\mathfrak{S}) = \lim_{n\to\infty} \varphi_{\mathbf{M}_\sigma^{-1}\mathbf{x}_n} \circ \ldots \circ \varphi_{\mathbf{M}_\sigma^{-1}\mathbf{x}_1}(\Theta_\sigma^*(\mathfrak{P})).$$

By Theorem 13, Θ_σ^* maps properly $\mathfrak{P}$ onto the stepped plane $\Theta_\sigma^*(\mathfrak{P})$. The sequence of flips $(\varphi_{\mathbf{M}_\sigma^{-1}\mathbf{x}_n})_n$ is locally finite by Lemma 17, hence Theorem 12 yields that $\Theta_\sigma^*(\mathfrak{S})$ is a stepped surface. ■

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