

Brun expansions, substitutions and discrete geometry

Thomas Fernique, Valerie Berthe

► **To cite this version:**

Thomas Fernique, Valerie Berthe. Brun expansions, substitutions and discrete geometry. Nicolas Bédaride; Julien Cassaigne. WORDS'07: Sixth International Conference on Words, Sep 2007, Marseille, France. pp.7, 2007, <<http://iml.univ-mrs.fr/words2007/index.en.html>>. <lirmm-00182696>

HAL Id: lirmm-00182696

<https://hal-lirmm.ccsd.cnrs.fr/lirmm-00182696>

Submitted on 26 Oct 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Brun expansions, substitutions and discrete geometry

V. Berthé and T. Fernique

LIRMM, Univ. Montpellier 2, CNRS
 161 rue Ada 34392 Montpellier - France,
 {berthe,fernique}@lirmm.fr

Keywords: Brun algorithm, digital planarity, standard discrete plane, flip, free group morphism, multi-dimensional continued fractions, stepped surface.

Abstract. The aim of this lecture is to present a strategy for the problem of discrete plane recognition based on multidimensional continued fractions and S -adic systems. The problem of the discrete plane recognition consists in deciding whether a given set of points with integer coordinates can be described as a plane discretization. The role played respectively by words, substitutions, and classical continued fractions will be played here respectively by stepped surfaces [1], generalized substitutions [2, 4], and Brun's algorithm. We thus give a geometric interpretation of Brun's continued fraction algorithm in terms of the so-called generalized substitutions introduced by Arnoux and Ito.

1 Plane recognition

The discrete plane recognition problem can be stated as follows: given a set of points in \mathbb{Z}^d , does there exist a (standard) arithmetic discrete plane that contains them?

Let us first recall the definition of a standard arithmetic discrete plane, according to [7]. Let (e_1, \dots, e_d) stand for the canonical basis of \mathbb{R}^d . For any $x \in \mathbb{Z}^d$ and $i \in \{1, \dots, d\}$, we denote by (x, i^*) the following translate of a face of the unit hypercube:

$$(x, i^*) = \{e_i + \sum_{j \neq i} \lambda_j e_j \mid 0 \leq \lambda_j \leq 1\}.$$

Then, for any non-negative non-zero vector $\alpha \in \mathbb{R}_+^d \setminus \{0\}$ and $\rho \in \mathbb{R}$, we define

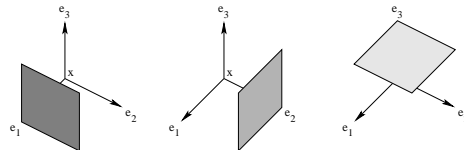


Fig. 1. The faces $(x, 1^*)$, $(x, 2^*)$ and $(x, 3^*)$ (from left to right), in the $d = 3$ case.

the following set of faces:

$$\mathcal{P}_{\alpha, \rho} = \{(\mathbf{x}, i^*) \mid 0 \leq \langle \mathbf{x} | \alpha \rangle - \rho < \langle \mathbf{e}_i | \alpha \rangle\},$$

where $\langle \cdot | \cdot \rangle$ stands for the canonical inner product. One checks that $\mathbf{x} \in \mathbb{Z}^d$ is a *vertex* of $\mathcal{P}_{\alpha, \rho}$ (that is, \mathbf{x} belongs to a face of $\mathcal{P}_{\alpha, \rho}$) if and only if it satisfies:

$$0 \leq \langle \mathbf{x} | \alpha \rangle - \rho < \sum_{i=1}^d \langle \mathbf{e}_i | \alpha \rangle,$$

i.e., the set of vertices of $\mathcal{P}_{\alpha, \rho}$ is a so-called *standard arithmetic discrete hyperplane*.

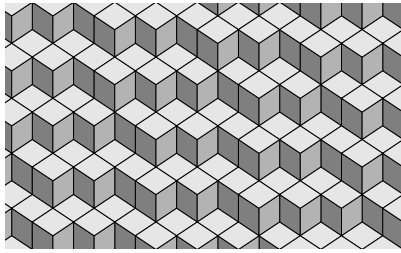


Fig. 2. A piece of a standard arithmetic discrete plane.

There exist various strategies in discrete geometry for the recognition of arithmetic discrete planes, such as described for instance in the survey [3]. The aim of this lecture is to present a strategy based on multidimensional continued fractions and S -adic systems, inspired by the one-dimensional Sturmian case. We will see that the role played respectively by words, substitutions, and classical continued fractions will be played here by stepped surfaces [1], generalized substitutions [2, 4], and Brun's algorithm.

2 Discrete lines and Sturmian words

A standard arithmetic discrete line is made of horizontal and vertical steps. One can code such a standard line by using the *Freeman code* over the two-letter alphabet $\{0, 1\}$ as follows: one codes horizontal steps by a 0, and vertical ones by a 1. One thus gets a Sturmian word $(u_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$.

A natural algorithm for the recognition of finite factors of Sturmian words can be obtained by desubstituting according to the two substitutions

$$\sigma_0(0) = 0, \sigma_0(1) = 10, \sigma_1(0) = 01, \sigma_1(1) = 1,$$

and to the choice of the isolated letter. For more details, see, e.g., [8, 9]. Such a desubstitution/recoding process can be translated in terms of continued fraction algorithm and Ostrowski numeration.

3 Generalized substitutions

We now introduce similar objects in the higher-dimensional case.

The *free group* over $\{1, \dots, d\}$ is denoted by F_d . A *morphism* of F_d is thus a map $\sigma : F_d \rightarrow F_d$ such that, for any u, v in F , $\sigma(uv) = \sigma(u)\sigma(v)$.

A morphism is said to be *non-negative* if it maps each letter of $\{1, \dots, d\}$ to a word over $\{1, \dots, d\}$, and it is said to be *non-erasing* if it does not map any letter to the empty word. Positive non-erasing morphisms are usually called *substitutions*. The *incidence matrix* M_σ of a morphism σ of F_d is the $d \times d$ matrix whose entry at i -th row and j -th column is the number of occurrences of the letter i in $\sigma(j)$. A morphism is said to be *unimodular* if its incidence matrix belongs to the linear group $GL(d, \mathbb{Z})$, that is, has determinant ± 1 . According to the formalism developed in [2, 4], it is possible to associate with any unimodular morphism of the free group σ a so-called *generalized substitution* acting on unions of faces as follows:

$$E_1^*(\sigma)(\mathbf{x}, i^*) = \sum_{j|\sigma(j)=pi^*s} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)), j^*) - \sum_{j|\sigma(j)=pi^{-1}s} (M_\sigma^{-1}(\mathbf{x}-\mathbf{f}(p)+\mathbf{e}_i), j^*).$$

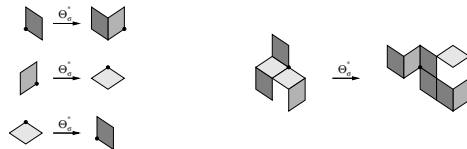


Fig. 3. Generalized substitution associated with a morphism (acting on the faces $(\mathbf{0}, i^*)$)

It is well-known that the image of a discrete plane under the action of $E_1(\sigma)^*$, when σ is a substitution, is again a discrete plane according to [2, 5].

Proposition 1 ([2, 5]). *Let σ be a unimodular substitution. Let $\alpha \in \mathbb{R}_+^d$ be a nonzero vector. The generalized substitution Θ_σ^* maps without overlaps the discrete plane $\mathcal{P}_{\alpha, \rho}$ onto $\mathcal{P}_{M_\sigma \alpha, \rho}$.*

We now extend the domain of definition of generalized substitutions to more general geometric objects.

A *stepped surface* (also called *functional discrete surface*) is defined as a union of pointed faces such that the orthogonal projection π onto the antidiagonal plane $(\mathbf{e}_1 + \dots + \mathbf{e}_d)^\perp$ induces a homeomorphism from the stepped surface onto the antidiagonal plane.

One interest of this notion relies in the fact it is possible to recognize whether a set of points in \mathbb{Z}^d is contained in a stepped surface by considering only a finite neighbour of each point [6]. Furthermore, generalized substitutions act not only on stepped planes but also on stepped surfaces according to [1].

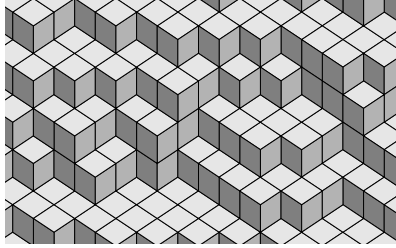


Fig. 4. A piece of a stepped surface

4 Desubstitution

The point now is to try to desubstitute according to generalized substitutions. We will use the following key property

$$E_1^*(\sigma \circ \mu) = E_1^*(\mu) \circ E_1^*(\sigma),$$

that holds for any two morphisms of the free group σ, μ , applied to an invertible substitution σ and to its inverse σ^{-1} . Consequently, desubstitution according to $E_1^*(\sigma)$ consists in the action of $E_1^*(\sigma^{-1})$.

Let σ be a unimodular substitution. A stepped surface is said to be σ -tilable if it is a union of translates of $E_1(\sigma)^*(\mathbf{0}, i^*)$, for $1 \leq i \leq d$. The following question is thus natural: can we desubstitute a σ -tilable stepped surface?

Theorem 1. *Let σ be an invertible substitution. Let \mathcal{S} be a σ -tilable stepped surface. We assume that there exists a non-zero vector α with non-negative entries such that $M_\sigma^{-1}(\alpha)$ has non-negative entries. Then, $E_1^*(\sigma^{-1})(\mathcal{S})$ is a stepped surface.*

Our proof is based on a geometrical approach, using the generation of functional stepped surfaces by *flips*. A flip is a classical notion in the study of dimer tilings and lozenge tilings associated with the triangular lattice. It consists in a local reorganization of tiles that transforms a tiling into another one. Such a reorganization can also be seen in the 3-dimensional space on the functional stepped surface itself. Suppose indeed that a functional stepped surface contains

3 faces that form the lower faces of a unit cube with integer vertices. By replacing these three faces by the upper faces of this cube, one obtains another functional stepped surface. According to [1], any functional stepped surface can be obtained from an arithmetic discrete plane by a sequence of flips, possibly infinite but locally finite, in the sense that, for any bounded neighborhood of the origin in the antidiagonal plane, there is only a finite number of flips whose domain has a projection which intersects this neighborhood.

We thus can come back via the notion of flips to the case of discrete planes. We then use the following:

Proposition 2. *Let σ be a unimodular morphism of the free group. Let $\alpha \in \mathbb{R}_+^d$ be a nonzero vector such that*

$${}^t M_\sigma \alpha \geq 0.$$

Then, $E_1^(\sigma)$ maps without overlaps the discrete plane $\mathcal{P}_{\alpha,\rho}$ onto $\mathcal{P}_{{}^t M_\sigma \alpha,\rho}$, that is, the stepped plane $\mathcal{P}_{\alpha,\rho}$ is σ -tilable, if and only if*

$${}^t M_{\sigma^{-1}} \alpha \geq 0.$$

5 Brun's algorithm

We will see that Proposition 2 can be reformulated in arithmetic terms, when applied to particular substitutions associated with Brun's multidimensional continued fraction algorithm.

Definition 1. *The d -dimensional Brun map T is defined over $[0, 1]^d \setminus \{0\}$ by:*

$$T(\alpha_1, \dots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \dots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \dots, \frac{\alpha_d}{\alpha_i} \right),$$

where i is equal to the smallest index i' such that $\alpha_{i'} = \max_j \alpha_j$.

It is convenient to provide a matrix viewpoint on Brun expansions. For $a \in \mathbb{N}$ and $i \in \{1, \dots, d\}$, let us define the following $(d+1) \times (d+1)$ matrix:

$$B_{a,i} = \begin{pmatrix} a & & & 1 & & \\ & I_{i-1} & & & & \\ & 1 & & 0 & & \\ & & & & I_{d-i} & \end{pmatrix},$$

where I_p stands for the $p \times p$ identity matrix and all the unspecified coefficients are equal to zero. Note that $B_{a,i}$ has integer entries and determinant -1 , and thus belongs to the linear group $GL(d+1, \mathbb{Z})$. Consider now $\alpha = (\alpha_1, \dots, \alpha_d) \in [0, 1]^d \setminus \{0\}$. For $i = \min\{j \mid \alpha_j = \|\alpha\|_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$, an easy computation shows that

$$(1, \alpha) = \|\alpha\|_\infty B_{a,i}(1, T(\alpha)),$$

where $(1, \mathbf{u}) = (1, u_1, \dots, u_n)$ for $\mathbf{u} = (u_1, \dots, u_n)$. In particular, if α has Brun expansion $(a_n, i_n)_n$, this yields, for every suitable n :

$$(1, \alpha) = \mu_n M_n(1, T^{n+1}(\alpha)),$$

where $\mu_n = \|T^0(\boldsymbol{\alpha})\|_\infty \times \dots \times \|T^n(\boldsymbol{\alpha})\|_\infty$ and $M_n = B_{a_0, i_0} \dots B_{a_n, i_n}$.

With each step of the algorithm, we can associate an invertible substitution $\beta_{a,i}$.

Definition 2 (Brun morphism). Let $\beta_{a,i}$ be the morphism of free group over $\{1, \dots, d+1\}$ defined, for $a \in \mathbb{N}$ and $i \in \{1, \dots, d\}$, by:

$$\beta_{a,i} : \begin{cases} 1 & \mapsto 1^a(i+1) \\ (i+1) & \mapsto 1 \\ j & \mapsto j \text{ for } j \neq 1, i+1. \end{cases}$$

It is easily checked that $\beta_{a,i}$ has incidence matrix $B_{a,i}$. In particular, $\beta_{a,i}$ is unimodular since $B_{a,i} \in GL(d+1, \mathbb{Z})$. Note that $\beta_{a,i}$ is a substitution, that is, a positive and non-erasing morphism. Let us note furthermore that the $B_{a,i}$'s are symmetric matrices. The substitution $\beta_{a,i}$ is moreover an automorphism:

$$\beta_{a,i}^{-1} : \begin{cases} 1 & \mapsto (i+1) \\ (i+1) & \mapsto (i+1)^{-a} 1 \\ j & \mapsto j \end{cases} \quad \text{and} \quad M_{\beta_{a,i}^{-1}} = B_{a,i}^{-1} = \begin{pmatrix} 0 & 1 & & \\ & I_{i-1} & & \\ 1 & & -a & \\ & & & I_{d-i} \end{pmatrix}.$$

One gets

Proposition 3. For any $\boldsymbol{\alpha} \in [0, 1]^d \setminus \{\mathbf{0}\}$ and $\rho \in \mathbb{R}$,

$$\mathcal{P}_{(1, \boldsymbol{\alpha}), \rho} = E_1^*(\beta_{a,i})(\mathcal{P}_{\|\boldsymbol{\alpha}\|_\infty(1, T(\boldsymbol{\alpha})), \rho}),$$

or, equivalently:

$$E_1^*(\beta_{a,i}^{-1})(\mathcal{P}_{(1, \boldsymbol{\alpha}), \rho}) = \mathcal{P}_{\|\boldsymbol{\alpha}\|_\infty(1, T(\boldsymbol{\alpha})), \rho},$$

where $i = \min\{j \mid \alpha_j = \|\boldsymbol{\alpha}\|_\infty\}$ and $a = \lfloor \alpha_i^{-1} \rfloor$.

Thus, we can relate the action of the Brun map T on a vector $\boldsymbol{\alpha}$ to the action of a dual map $E_1^*(\beta_{a,i}^{-1})$ on the stepped plane $\mathcal{P}_{(1, \boldsymbol{\alpha}), \rho}$. Hence, by identifying the vector $\boldsymbol{\alpha}$ with the hyperplane with normal vector $(1, \boldsymbol{\alpha})$, it is possible to define the Brun's expansion of a discrete plane, and even of a stepped surface.

We will use here the unimodularity and the weak convergence of Brun's algorithm. The main result of our lecture is the following:

Theorem 2. If a stepped surface S has the same Brun expansion as a totally irrational vector $\boldsymbol{\alpha} \in [0, 1]^d \setminus \{0\}$, then it is a stepped plane of normal vector $\boldsymbol{\alpha}$.

We conclude this lecture by discussing the application of this theorem to the recognition problem for discrete planes.

References

1. P. Arnoux, V. Berthé, T. Fernique, D. Jamet, *Generalized substitutions, functional stepped surfaces and flips*, Theor. Comput. Sci. **380** (2007), 251–267.

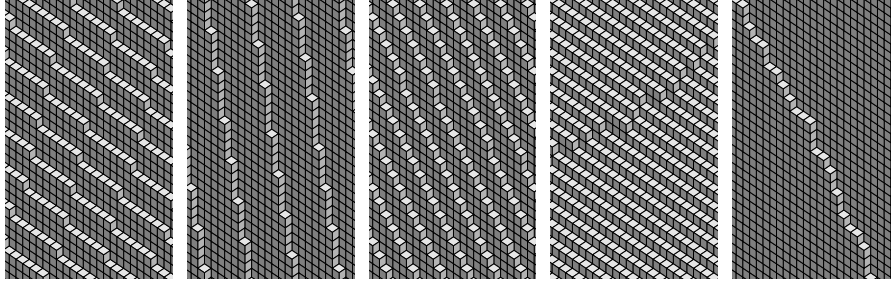


Fig. 5. An examples of Brun's expansion for a nonplane stepped surface

2. P. Arnoux, S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Bel. Math. Soc. Simon Stevin **8** (2001), 181–207.
3. V. Brimkov, D. Coeurjolly, R. Klette, *Digital Planarity - A Review*, Discrete Applied Mathematics **155** (2007), 468–495.
4. H. Ei, *Some properties of invertible substitutions of rank d , and higher dimensional substitutions*, Osaka Journal of Mathematics **40** (2003), pp. 543–562.
5. T. Fernique, *Multidimensional sturmian words and substitutions*, Int. J. of Found. Comput. Sci. **17** (2006), 575–600.
6. D. Jamet, *On the Language of Discrete Planes and Surfaces*, In Proceedings of the Tenth International Workshop on Combinatorial Image Analysis, pages 227-241. Springer-Verlag, 2004.
7. J.-P. Reveillès, "Calcul en nombres entiers et algorithmique," Ph. D Thesis, Univ. Louis Pasteur, Strasbourg, France, 1991.
8. A. Troesch, *Interprétation géométrique de l'algorithme d'Euclide et reconnaissance de segments*, Theor. Comput. Sci. **115** (1993), 291–320.
9. L.-D. Wu, *On the chain code of a line*, IEEE Transactions on Pattern Analysis and Machine Intelligence **4** (1982), 359–368.