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Arithmetic Charaterization of Polynomial Based Discrete Curves

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ABSTRACT

In the present paper, we investigate discretization of curves based on polynomials in the 2-dimensional space. Under some assumptions, we propose an arithmetic characterization of thin and connected discrete approximations of such curves. In fact, we reach usual discretization models, that is, GIQ, OBQ and BBQ but with a generic arithmetic definition.

Keywords: discrete geometry, polynomial curve, arithmetic characterization

1. INTRODUCTION

Discrete geometry attempts to provide an analogue of the Euclidean geometry for the discrete space $\mathbb{Z}^d$. Such an investigation is not only theoretical, but has also practical applications since digital images can be seen as arrays of pixels, in other words, as subsets of $\mathbb{Z}^2$ or $\mathbb{Z}^3$.

Discrete lines are currently the better known discrete objects. At first, J. Bresenham,[1] H. Freeman[2] and A. Rosenfeld[3] have followed algorithmic approaches and have defined them as digitizations of Euclidean lines. They have provided tools for drawing and recognition.[1–3] Later, J.-P. Reveillès has initiated the arithmetic discrete geometry[4] and has introduced the notion of arithmetic discrete line as the solution of a system of diophantine inequalities, that is, as the subset of $\mathbb{Z}^2$ contained in a band. Its definition is based on a single parameter, the arithmetic thickness and includes all the previously defined lines, that is, 0-connected or 1-connected ones, centered (GIQ) or not (BBQ and OBQ) (see Section 2 for explanations about GIQ, BBQ and OBQ) on the underlying Euclidean line. Such an approach enhances the knowledge of discrete lines. In addition to give new drawing[4] and recognition[5] algorithms, it directly links topological and geometrical properties of an arithmetic discrete line with its definition. For instance, its connectedness is entirely characterized by its arithmetic thickness.

Similarly, first investigations into discrete circles have been algorithmic ones.[6–8] Discrete circles were only considered as digitizations of Euclidean circles. It is thus natural to ask whether or not J.-P. Reveillès' arithmetic approach is extendable to discrete circles and can supply an arithmetic discrete definition of circles independent of Euclidean circles. Such an extension has been proposed by É. Andres.[9] He has defined the discrete analytical circles as the solutions of a system of diophantine inequalities, the subsets of $\mathbb{Z}^2$ contained in a ring of width its arithmetic thickness. Discrete analytical circles tile the plane but do not have topological properties contrary to circles algorithmically defined. Recently, another arithmetic definition characterized GIQ circles,[10] BBQ and OBQ circles.[11] The key-point is the use of a thickness function instead of the constant thickness found in the previous approaches. This last definition also applies to discrete lines and is in fact a generalization of the one introduced by J.-P. Reveillès.

In the present paper, we go further by investigating polynomial based discrete curves. The main idea remains the use of a non constant thickness in order to have more accurate discretizations of curves. Starting from the

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strict definition of discretizations, based on global distance criteria and not suitable for applicative purpose, we do some approximation and finally characterize GIQ, BBQ and OBQ of curves defined by polynomials.

First, we will recall some basic notions of discrete geometry useful to understand the present matter. Then, we will define what are, for us, strictly good curve discretizations, that is, sets of discrete points, with topological properties, close to a curve according to global distance criteria. Finally, since we are not able to exactly and easily define those sets, we approximate them and reach usual discretization models as GIQ, BBQ and OBQ. Moreover, we verified that this approach is in agreement with arithmetic definitions of already known discrete objects.

2. BASIC NOTIONS

The aim of this section is to introduce the basic notions of discrete geometry used throughout the present paper. Let $\mathbf{v} \in \mathbb{Z}^2$, $\mathbf{v}$ is then called a discrete point or pixel.

Definition 1 (0-Adjacency and 1-Adjacency) Let $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ be both pixels.

$\mathbf{v}$ and $\mathbf{w}$ are 0-*adjacent* if and only if $\|\mathbf{v} - \mathbf{w}\|_\infty = \max\{|v_1 - w_1|, |v_2 - w_2|\} = 1$.

$\mathbf{v}$ and $\mathbf{w}$ are 1-*adjacent* if and only if $\|\mathbf{v} - \mathbf{w}\|_1 = |v_1 - w_1| + |v_2 - w_2| = 1$.

Let $k \in \{0, 1\}$. A discrete set E is said to be k-*connected* if for each pair of voxels $(\mathbf{v}, \mathbf{w}) \in \mathrm{E}^2$, there exists a finite sequence of voxels $(\mathbf{s_1}, \dots, \mathbf{s_p}) \in \mathrm{E}^p$ such that $\mathbf{v} = \mathbf{s_1}$, $\mathbf{w} = \mathbf{s_p}$ and the voxels $\mathbf{s_j}$ and $\mathbf{s_{j+1}}$ are k-adjacent, for each $j \in \{1, \dots, p-1\}$.

For the sake of clarity, a 0-connected discrete set is simply said to be *connected* in the sequel.

Let E be a discrete set, $\mathbf{v}$ be a voxel of E and $k \in \{0, 1\}$. The k-*connected component* of $\mathbf{v}$ in E is the maximal k-connected subset of E (with respect to set inclusion) containing $\mathbf{v}$.

Definition 2 (k-Separability) A discrete set E is k-*separating* in a discrete set F if its complement in F, $\overline{\mathrm{E}} = \mathrm{F} \setminus \mathrm{E}$, has two distinct k-connected components. E is called a *separator* of F.

Definition 3 (k-Simple Points, k-Minimality) Let F and E be two discrete sets such that E is k-separating in F. A voxel $\mathbf{v} \in \mathrm{E}$ is said to be k-*simple* if $\mathrm{E} \setminus \{\mathbf{v}\}$ remains k-separating in F. Moreover, a k-separating discrete set in F without k-simple points is said to be k-*minimal* in F.

In general, a discrete object is a set of points of $\mathbb{Z}^2$ obtained from the digitization of a real object. There are three classical methods of discretization which allows to obtain connected discrete sets.[12,13] We will recall them here, using the example of a line. Let $\mathcal{L}_{\mathbf{n},\mu}$ be the line directed by $\mathbf{n} = (a, b) \in \mathbb{N}^2$ ($0 \leq a \leq b$ and $b \neq 0$) and with shift $\mu \in \mathbb{Z}$. We can define it analytically in two equivalent way:

$$\begin{aligned} \mathcal{L}_{\mathbf{n},\mu} &= \left\{\mathbf{x} = (x, y) \in \mathbb{R}^2;\ y = -\frac{a}{b}x - \frac{\mu}{b}\right\}, \\ \mathcal{L}_{\mathbf{n},\mu} &= \left\{\mathbf{x} = (x, y) \in \mathbb{R}^2;\ ax + by + \mu = 0\right\}. \end{aligned}$$

The BBQ[12,14] (*Background Boundary Quantization*) keeps a pixel outside an object each time its border crosses an edge of the square grid of $\mathbb{Z}^2$. Since we need an object to define a BBQ, we consider that the BBQ of the line $\mathcal{L}_{\mathbf{n},\mu}$ is the BBQ of the half-space defined by the set of points where $ax + by + \mu \leq 0$ and we have:

$$BBQ(\mathcal{L}_{\mathbf{n},\mu}) = \{\mathbf{v} = (i, j) \in \mathbb{Z}^2;\ 0 \leq j + \frac{a}{b}i + \frac{\mu}{b} < 1\},$$

or with the second analytic form of $\mathcal{L}_{\mathbf{n},\mu}$:

$$BBQ(\mathcal{L}_{\mathbf{n},\mu}) = \{\mathbf{v} = (i, j) \in \mathbb{Z}^2;\ 0 \leq ai + bj + \mu \leq b\}.$$

This last defintion is the one introduced by J.-P. Reveills.[4] Since the arithmetic thickness ω is here equal to $\|\mathbf{n}\|_\infty = b$, the discrete line is the thinnest 0-connected one and is called to be *naive*.

The OBQ[12,14] (*Object Boundary Quantization*) keeps a pixel inside an object each time its border crosses an edge of the square grid of $\mathbb{Z}^2$. With the previous considerations, the OBQ of the line $\mathcal{L}_{\mathbf{n},\mu}$ is the following set:

$$OBQ(\mathcal{L}_{\mathbf{n},\mu}) = \{\mathbf{v} = (i,j) \in \mathbb{Z}^2;\ -\|\mathbf{n}\|_\infty \leq ai + bj + \mu \leq 0\}.$$

The GIQ (*Grid Intersect Quantization*) is the one used by Bresenham[1] and Freeman.[15] It keeps pixel which are the closest to the curve, each time the curve crosses an edge of the square grid of $\mathbb{Z}^2$. The GIQ of the line $\mathcal{L}_{\mathbf{n},\mu}$ is the set $GIQ(\mathcal{L}_{\mathbf{n},\mu})$ defined by:

$$GIQ(\mathcal{L}_{\mathbf{n},\mu}) = \left\{\mathbf{v} = (i,j) \in \mathbb{Z}^2;\ -\frac{\|\mathbf{n}\|_\infty}{2} \leq ai + bj + \mu < \frac{\|\mathbf{n}\|_\infty}{2}\right\}.$$

3. DISCRETIZATION OF POLYNOMIAL BASED CURVES

Let f be a polynomial of degree n in two variables. We are interested in a discrete curve close to the Euclidean curve defined by the set $\{\mathbf{x} \in \mathbb{R}^2; f(\mathbf{x}) = 0\}$. Strictly, we are looking for the closest discrete points to this set according to a distance criterion. Basically, we defined discretization as:

$$\mathfrak{C}_{\|\cdot\|}(f,\lambda) = \left\{\mathbf{v} \in \mathbb{Z}^2; \exists \mathbf{x} \in \mathbb{R}^2, f(\mathbf{x}) = 0 \text{ and } \|\mathbf{v} - \mathbf{x}\| \leq \lambda\right\}, \tag{1}$$

that is the set of discrete points $\mathbf{v}$ for which a ball centered on them, based on the norm $\|\cdot\|$ and of radius λ contains at least a point $\mathbf{x}$ of the curve $f(\mathbf{x}) = 0$.

Such a vision of discretization allows to reach closed naive model[16] when using $\mathfrak{C}_1\left(f, \frac{1}{2}\right)$ or supercover model[16,17] when using $\mathfrak{C}_\infty\left(f, \frac{1}{2}\right)$ of polynomial based curves. Nevertheless, most popular discretization models are naive[18] and standard[19] ones. They are not symmetric and required the definition of other discrete sets to be reached:

$$\begin{aligned}
\mathfrak{C}^{+*}_{\|\cdot\|}(f,\lambda) &= \left\{\mathbf{v} \in \mathbb{Z}^2; \exists \mathbf{x} \in \mathbb{R}^2, f(\mathbf{x}) = 0 \text{ and } f(\mathbf{v}) \in \mathbb{Z}^+ \text{ and } \|\mathbf{v} - \mathbf{x}\| < \lambda\right\}, \\
\mathfrak{C}^{-}_{\|\cdot\|}(f,\lambda) &= \left\{\mathbf{v} \in \mathbb{Z}^2; \exists \mathbf{x} \in \mathbb{R}^2, f(\mathbf{x}) = 0 \text{ and } f(\mathbf{v}) \in \mathbb{Z}^- \text{ and } \|\mathbf{v} - \mathbf{x}\| \leq \lambda\right\}.
\end{aligned}$$

In fact, we select the discrete points $\mathbf{v}$ such that $f(\mathbf{v})$ is positive (respectively negative) and such that the distance between $\mathbf{x}$ and $\mathbf{v}$ with regards to the norm $\|\cdot\|$ is lower (respectively lower or equal) than a fixed value λ.

Of course, such discretizations are meaninful only if the function never vanishes without changing sign. We restrict thus our study to polynomial f such that it does not exist polynomials g and h such that $f = g^2 h$.

We can now define centered, but not symmetric discretizations:

$$\mathfrak{C}^{*}_{\|\cdot\|}(f,\lambda) = \mathfrak{C}^{-}_{\|\cdot\|}(f,\lambda) \bigcup \mathfrak{C}^{+*}_{\|\cdot\|}(f,\lambda). \tag{2}$$

Here, we select discrete points $\mathbf{v}$ such that $f(\mathbf{v})$ is positive or negative, that is, the thickness is centered on the curve defined by f. Depending on the choice of $\|\cdot\|$, we reach the standard representation[19] with $\mathfrak{C}^*_\infty(f, \frac{1}{2})$ or the naive one with $\mathfrak{C}^*_1(f, \frac{1}{2})$.

In the sequel of this paper we will only focus on the naive and GIQ representations, keeping in mind that other mentioned representations are also reachable with our definition.

4. ARITHMETIC CHARACTERIZATION OF POLYNOMIAL BASED CURVES

The previous definition is mainly theoretical. It is hard to directly deduce the set of discrete points $\mathfrak{C}_1^*(f, \frac{1}{2})$, since we do not know the point $\mathbf{x}$ of the curve the closest to $\mathbf{v}$. So we are rather interested in an arithmetic representation of the form:

$$\{\mathbf{v} \in \mathrm{E}; \omega_{min}(\mathbf{v}) \le f(\mathbf{v}) < \omega_{max}(\mathbf{v})\}\,, \tag{3}$$

with $\mathrm{E} \subseteq \mathbb{Z}^2$.

Such a representation is a way to simulate the previous geometric definition of the discretization we gave, i.e., the set of discrete points inside a ball running along the curve. It implies that in order to determine the membership of a discrete point $\mathbf{v} \in \mathbb{Z}^2$ to the discretization, we have to determine the maximum ($\omega_{max}(\mathbf{v})$) and the minimum ($\omega_{min}(\mathbf{v})$) of f in $\{\mathbf{x} \in \mathbb{R}^2; \|\mathbf{v} - \mathbf{x}\| < \lambda\}$ and verify that it is enough close to $f(\mathbf{v})$ ($\{\mathbf{v} \in \mathrm{E}; \omega_{min}(\mathbf{v}) \le f(\mathbf{v}) < \omega_{max}(\mathbf{v})\}$). Doing this way leads to two problems:

1. an extremum of f, different from 0, but between the computed bounds, can lead to select discrete points where f does not vanish.
2. evaluation of the bounds requires an accurate analyze of f in the neighborhood of $\mathbf{v}$. It can be hard and computing time consuming.

As we want to have a discrete approximation of $f\{\mathbf{x}\} = 0$, the two following conditions on the polynomial f allow to ensure that we avoid these two problems, that is, no point out of the discretization will be considered and the evaluation of the maximum and minimum of f stays simple:

1. f does not have any extremum $f(\mathbf{m})$ such that $f(\mathbf{m}) \in [\Omega_{min}, 0[\cup]0, \Omega_{max}]$ with $\Omega_{min} = \min_{\mathbf{v} \in \mathrm{E}}\{\omega_{min}(\mathbf{v})\}$ and $\Omega_{max} = \max_{\mathbf{v} \in \mathrm{E}}\{\omega_{max}(\mathbf{v})\}$,
2. f does not have any extremum $f(\mathbf{m})$ such that $0 < \|\mathbf{m} - \mathbf{x}\| \le \lambda$.

Note that the second condition is not a required condition. It is there to ensure that the evaluation of the local maximum and minimum of f will be on the border of the ball, so, easy to evaluate. It can also be viewed as a condition which ensures that the discrete curve will show all the variations of the polynomial curve. If it is not fulfilled, it means that the resolution (the quantization step) is not sufficiently small to have an accurate discretization of the curve.

Now, we can try to go from the geometric definition to an analytic characterization. In the geometric definition, evaluation by the function f and distance are mixed up. So, we have to express $f(\mathbf{v})$ according to the differences $(x - i)$ and $(y - j)$. More precisely, It can be done with the Taylor series of the polynomial f (of degree n) in $\mathbf{v}$.

$$f(\mathbf{x}) = f(\mathbf{v}) + \sum_{k=1}^{n} \left(\frac{\partial^k f}{\partial x^k}(\mathbf{v}) \frac{(x - i)^k}{k!} + \frac{\partial^k f}{\partial y^k}(\mathbf{v}) \frac{(y - j)^k}{k!} \right). \tag{4}$$

Since $f(\mathbf{x}) = 0$, we have:

$$f(\mathbf{v}) = -\sum_{k=1}^{n} \left(\frac{\partial^k f}{\partial x^k}(\mathbf{v}) \frac{(x - i)^k}{k!} + \frac{\partial^k f}{\partial y^k}(\mathbf{v}) \frac{(y - j)^k}{k!} \right). \tag{5}$$

According to the two conditions above, we just have to take care of the maximum of f on the border of the ball $\{\mathbf{x} \in \mathbb{R}^2; \|\mathbf{v} - \mathbf{x}\|_1 = \lambda\}$ to determine if a discrete point belongs to the discrete curve. Unfortunately, it stay hard to determine.

To overcome this limitation, we have to find good approximation. This is possible, in particular, if we take $\lambda = \frac{1}{2}$. In such a case, we can only consider the furthest points from the center of the ball, that is its vertices $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $(0, -\frac{1}{2})$, $(0, \frac{1}{2})$ and we obtain:

$$\begin{cases} \omega_{min}(\mathbf{v}) = \min\left\{-\sum_{k=1}^{n} \frac{\partial^k f}{\partial x^k}(\mathbf{v})\frac{(-1/2)^k}{k!}, -\sum_{k=1}^{n} \frac{\partial^k f}{\partial x^k}(\mathbf{v})\frac{(1/2)^k}{k!}, -\sum_{k=1}^{n} \frac{\partial^k f}{\partial y^k}(\mathbf{v})\frac{(1/2)^k}{k!}, -\sum_{k=1}^{n} \frac{\partial^k f}{\partial y^k}(\mathbf{v})\frac{(-1/2)^k}{k!}\right\} \\ \omega_{max}(\mathbf{v}) = \max\left\{-\sum_{k=1}^{n} \frac{\partial^k f}{\partial x^k}(\mathbf{v})\frac{(-1/2)^k}{k!}, -\sum_{k=1}^{n} \frac{\partial^k f}{\partial x^k}(\mathbf{v})\frac{(1/2)^k}{k!}, -\sum_{k=1}^{n} \frac{\partial^k f}{\partial y^k}(\mathbf{v})\frac{(1/2)^k}{k!}, -\sum_{k=1}^{n} \frac{\partial^k f}{\partial y^k}(\mathbf{v})\frac{(-1/2)^k}{k!}\right\} \end{cases}$$

As mentioned above, we make an approximation. We consider less space than in $\mathfrak{C}_1^*\left(f, \frac{1}{2}\right)$. Nevertheless each connected component of the Euclidean curve gives a 0-connected component in the resulting discrete set.

Proposition 1 Let f be a polynomial of degree n in two variables. Let $\{\mathbf{x} \in \mathbb{R}^2;\ f(\mathbf{v}) = 0\}$ be a curve and let $\left\{\mathbf{v} \in \mathbb{Z}^2; \omega_{min}(\mathbf{v}) \leq f(\mathbf{v}) < \omega_{max}(\mathbf{v})\right\}$, with $\omega_{min}(\mathbf{v})$ and $\omega_{max}(\mathbf{v})$ defined as above, be its discretization. Then, each connected component of the curve gives a 0-connected component in its discretization.

In fact, we analytically characterize the *GIQ* of polynomial curves. And if we have focused on $\mathfrak{C}_1^{+*}(f, 1)$ or $\mathfrak{C}_1^{-*}(f, 1)$, with a similar approximation, we would have reached *BBQ* and *OBQ* of the curve define by f.

Figure 1 presents some examples of discrete curves we are able to determine with our arithmetic definition. Example (a) is the *BBQ* curve with an intersection point. We can see that it is not 1-minimal. But no simple point can be removed without distorting the curve. Example (b) is the *GIQ* of a simple curve which has in the Euclidean space a point of symmetry. Example (c) presents the *OBQ* of a curve with several connected components.

4.1. Discrete lines

Let us consider the polynomial $l_{\mathbf{n},\mu}(\mathbf{x}) = ax + by + \mu$. We then obtain the following bounds:

$$\begin{cases} \omega_{min}(\mathbf{v}) &= \min\left\{\frac{-a}{2}, \frac{a}{2}, \frac{-b}{2}, \frac{b}{2}\right\} &= -\frac{1}{2}\max\{|a|, |b|\} &= -\frac{\|\mathbf{n}\|_\infty}{2} \\ \omega_{max}(\mathbf{v}) &= \max\left\{\frac{-a}{2}, \frac{a}{2}, \frac{-b}{2}, \frac{b}{2}\right\} &= \frac{1}{2}\max\{|a|, |b|\} &= \frac{\|\mathbf{n}\|_\infty}{2} \end{cases}$$

and the arithmetic characterization of a discrete line:

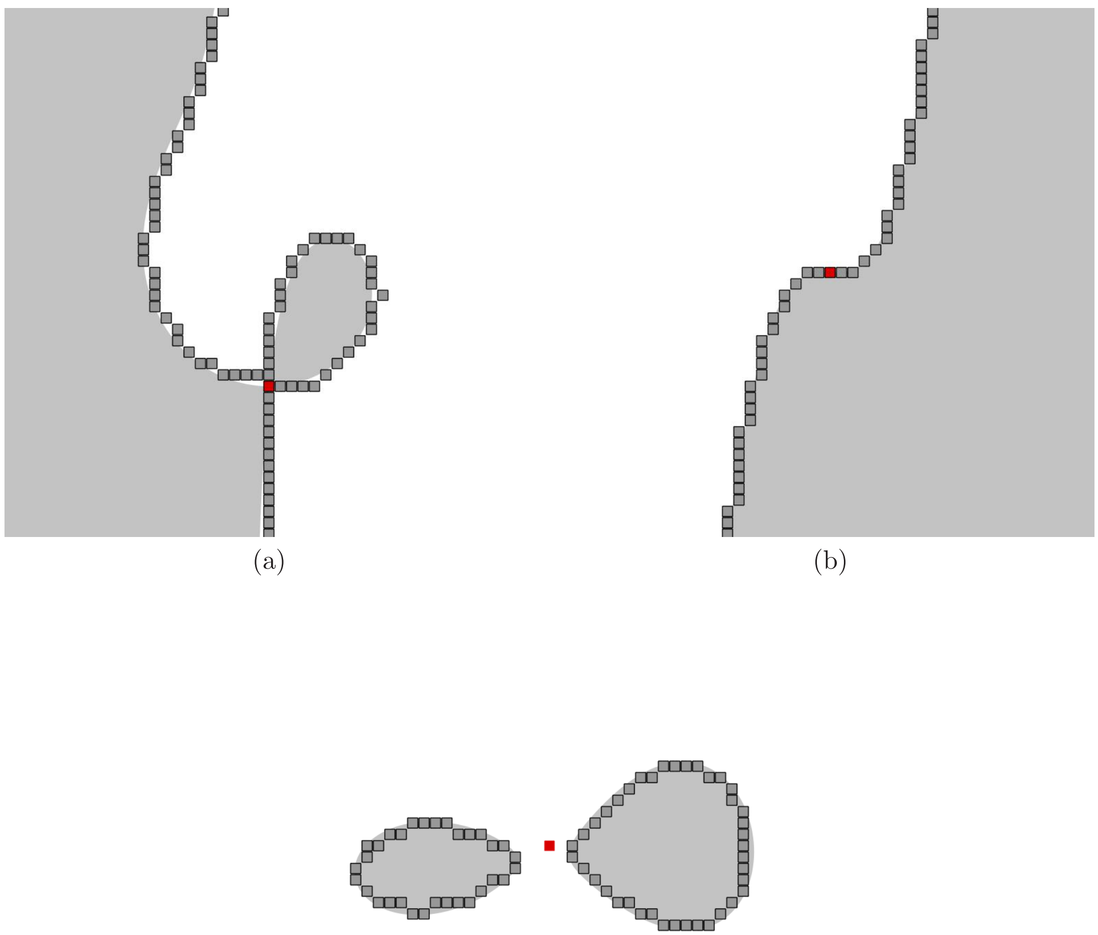
$$\left\{\mathbf{v}(i,j) \in \mathbb{Z}^2; -\frac{\|\mathbf{n}\|_\infty}{2} \leq ai + bj + \mu < \frac{\|\mathbf{n}\|_\infty}{2}\right\} \tag{6}$$

This discrete set is equivalent to the J.-P. Reveills' discrete line. And finally for linear polynomials, $GIQ(\mathcal{L}_{\mathbf{n},\mu})$ is equal to $\mathfrak{C}_1^*\left(l_{\mathbf{n},\mu}, \frac{1}{2}\right)$.

4.2. Discrete circles

Let us consider the polynomial $c_{\mathbf{o},r}(\mathbf{x}) = (x - x_0)^2 + (y - y_0)^2 - r^2$. We then obtain the following bounds:

$$\begin{cases} \omega_{min}(\mathbf{v}) &= \min\left\{(x - x_0) - \frac{1}{4}, -(x - x_0) - \frac{1}{4}, (y - y_0) - \frac{1}{4}, -(y - y_0) - \frac{1}{4}\right\} &= -\|(x - x_0, y - y_0)\|_\infty - \frac{1}{4} \\ \omega_{max}(\mathbf{v}) &= \max\left\{(x - x_0) - \frac{1}{4}, -(x - x_0) - \frac{1}{4}, (y - y_0) - \frac{1}{4}, -(y - y_0) - \frac{1}{4}\right\} &= \|(x - x_0, y - y_0)\|_\infty - \frac{1}{4} \end{cases}$$



(a) (b)

(c)

Figure 1. Some discrete curves. (a) Background Boundary Quantization of an ophiuride, (b) Grid Intersect Quantization of a cubic curve $ay - bx^3 = 0$, (c) Object Boundary Quantization of a curve of degree 4.

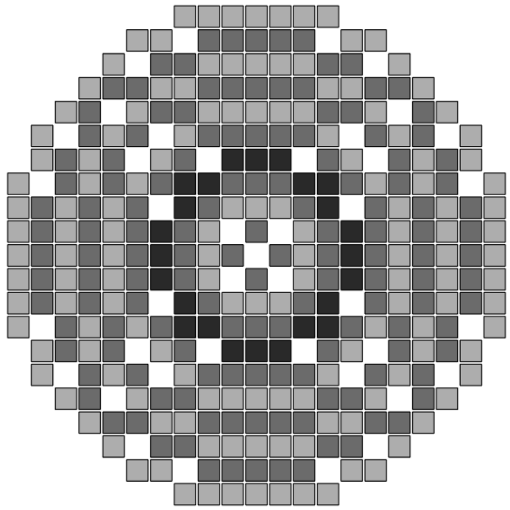


Figure 2. *GIQ* of circles

and the arithmetic characterization of a discrete circle:

$$\left\{\mathbf{v}(i,j) \in \mathbb{Z}^2; -\|(i-x_0, j-y_0)\|_\infty - \frac{1}{4} \leq (i-x_0)^2 + (j-y_0)^2 - r^2 < \|(i-x_0, j-y_0)\|_\infty - \frac{1}{4}\right\} \tag{7}$$

This discrete set is equivalent to the discrete arithmetic circle.[11] This set is not minimal in $\mathbb{Z}^2$ but is minimal in $\{(i,j) \in \mathbb{Z}^2; j \leq i\}$. Indeed, on the diagonal, a simple point can appear[20] (Figure 2) even if we are thinnner than expected in $\mathfrak{C}_1^*(c_{\mathbf{o},r}, \frac{1}{2})$. In such a case the diagonal point is closer to the Euclidean circle than the both other pixels playing a part in the thickening and we can not reach a minimal set.

5. CONCLUSION AND FURTHER WORKS

In the present paper, we have proposed an arithmetic characterization of the thinnest disrete approximations of polynomial based curves. Whatever the degree of the polynomial may be, we are able to determine arithmetically a connected set of close integer points to its associated curve. For some polynomials and in some locations, simple points can be found. For example, such simple points are present on the diagonals of arithmetic discrete circles with particular radii when thickness is centered[11] or in Figure 1(a). Indeed, our approach has the default of all transformation from the continuous to the discrete space: when the step of discretization is too great compared to the signal variation, results could be different from the expectations.

Our study highlights another point which seems important to us. Discrete symmetries are not already defined concepts. Nevertheless, in a natural way, we can simply define them as the restriction of Euclidean symmetries to discrete points. In this case, our discretizations do not maintain the points of symmetry of original Euclidean curves as you can see Figure 1(b). It could be avoid by thickening the curve, but more simple point will appear.

In further works, we will focus on the *d*-dimensional case and try to fix the symmetry problem.

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