Proof of NP-completeness for a scheduling problem with coupled-tasks and compatibility graph

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In this technical rapport, we investigate a scheduling problem with coupled-tasks (denoted acquisition tasks) and a compatibility graph. We will show that the problem $1|prec, coupled-task, (p_{a_i} = p_{b_i} = 1, L_i = \alpha) \cup (p_{T_i}, pmtn), G_c|C_{max}$ is $\mathcal{NP}$-complete, with $\alpha \geq 3$. This problem is denoted by Π.

Theorem 1 *Let n be the acquisition tasks number, the problem to decide if an instance of the problem Π has a scheduling length $C_{max} = 2n + \sum_{T_i \in \mathcal{T}} p_{T_i}$, is $\mathcal{NP}$-complete.*

Proof

Our approach is similar to the proof of Lenstra and Rinnoy Kan [1] for the problem $P|prec; p_j = 1|C_{max}$. This demonstration is based on the *Clique* decision problem (see Garey and Johnson GT19 [2]):
INSTANCE: A graph $G = (V, E)$ where $|V| = n$, and an integer K.
QUESTION: Can we find a clique of size K in G?

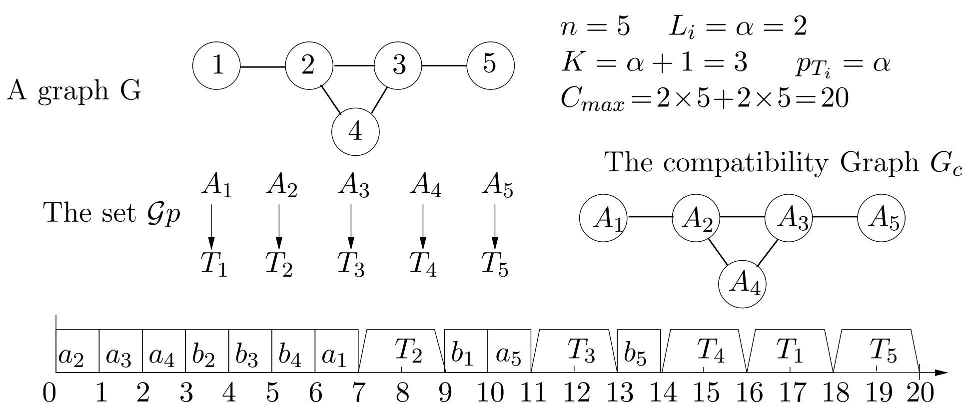

Figure 1: Illustration of polynomial-time transformation Clique $\propto$ Π

Our proof is based on the polynomial-time transformation Clique $\propto$ Π. It is easy to see that the problem Π is in $\mathcal{NP}$.

Let I^* an instance of Clique, we will construct an instance I of Π with $C_{max} = 2n + \sum_{T_i \in \mathcal{T}} p_{T_i}$ in the following way:
Let $G = (V, E)$ a graph in the instance I, with $|V| = n$:

- $\forall v \in V$, an acquisition task A_v is introduced, composed of two sub-tasks a_v and a'_v with processing time $p_{a_v} = p_{a'_v} = 1$ and with a latency time, between these two sub-tasks, of length $\alpha = (K - 1)$, called *slot*.

- For each edge $e = (v, w) \in E$, there is a compatibility relation between the two acquisition tasks A_v and A_w.

- For each task A_v, we introduce a treatment task T_v which is its successor.

- Each T_v has a processing time noted $p_{T_v} = \alpha$. Thus, the treatment tasks will replace all the inactivity slot of all the A_v after the clique.

- We suppose that there is a clique of length $K = (\alpha + 1)$ in the graph G. Let us show that there is a scheduling in $C_{max} = (2n + \sum_{T_i \in \mathcal{T}} p_{T_i})$ units of time. For that, consider the following scheduling:
  - From $t = 0$ to $t = \alpha$, we schedule the $K = (\alpha + 1)$ tasks which represent the vertices of the clique of size K.
  - From $t = (2\alpha + 2)$, we schedule the $(n - K)$ remaining tasks A_v.
  - In each slot from these $(n - K)$ tasks A_v, we schedule the tasks T_v. Since each T_v has as a value $p_{T_v} = \alpha$, by scheduling $(n - K)$ tasks T_v, we will fill each slot of length α of the $(n - K)$ tasks A_v.
  - Remeaning treatment tasks are scheduled at the end of the schedule.

  With this allocation, we fill all the slots and we give a valid scheduling in $(2n + \sum_{T_i \in \mathcal{T}} p_{T_i})$ units of time.

- Reciprocally, let us suppose that there is a scheduling in $(2n + \sum_{T_i \in \mathcal{T}} p_{T_i})$ units of time without inactivity time, then let us show that the graph G contains a clique of size $K = (\alpha + 1)$.
  From these suppositions, we make essential comments:
  - With the precedence constraints between the tasks A_v and T_v, it is easy to see that we can schedule only tasks A_v at $t = 0$, $\forall v \in V$. Thus, the first treatment task could be scheduled only starting from $t = (\alpha + 2)$.
  - Let a_{p_1} be the first sub-task of acquisition scheduled at $t = 0$, with a slot of length α. We need a clique of size $(\alpha + 1)$ to obtain a scheduling without inactivity slot.

Thus we have $(\alpha + 1)$ acquisition tasks which are compatibles. And in the compatibility graph G_c, we will have an edge between each couple of these tasks A_v. Consequently, the tasks $A_{p_1}, A_{p_2}, \ldots, A_{p_\alpha}$, associated to the vertices of the graph G, form a clique of size $K = (\alpha + 1)$.

This concludes our proof of Theorem 1.

$\square$

Bibliographie

1. R.L. Graham, E.L. Lawler, J.K. Lenstra, and A.H.G. Rinnooy Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. *Annals of Discrete Mathematics*, 5:287–326, 1979.
2. M.R. Garey and D.S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness.* W. H. Freeman & Co., New York, NY, USA, 1990.