

# The Hoàng-Reed Conjecture holds for tournaments

Frédéric Havet, Stéphan Thomassé, Anders Yeo

► **To cite this version:**

Frédéric Havet, Stéphan Thomassé, Anders Yeo. The Hoàng-Reed Conjecture holds for tournaments. Discrete Mathematics, Elsevier, 2008, 308, pp.3412-3415. <10.1016/j.disc.2007.06.033>. <lirmm-00292710>

**HAL Id: lirmm-00292710**

**<https://hal-lirmm.ccsd.cnrs.fr/lirmm-00292710>**

Submitted on 2 Jul 2008

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Hoàng-Reed conjecture holds for tournaments

Frédéric Havet <sup>\*</sup>      Stéphan Thomassé <sup>†</sup>      Anders Yeo <sup>‡</sup>

## Abstract

Hoàng-Reed conjecture asserts that every digraph  $D$  has a collection  $\mathcal{C}$  of circuits  $C_1, \dots, C_{\delta^+}$ , where  $\delta^+$  is the minimum outdegree of  $D$ , such that the circuits of  $\mathcal{C}$  have a forest-like structure. Formally,  $|V(C_i) \cap (V(C_1) \cup \dots \cup V(C_{i-1}))| \leq 1$ , for all  $i = 2, \dots, \delta^+$ . We verify this conjecture for the class of tournaments.

## 1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph  $D$  on  $n$  vertices and with minimum outdegree  $n/k$  has a circuit of length at most  $k$ . Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A *circuit-tree* is either a singleton or consists of a set of circuits  $C_1, \dots, C_k$  such that  $|V(C_i) \cap (V(C_1) \cup \dots \cup V(C_{i-1}))| = 1$  for all  $i = 2, \dots, k$ , where  $V(C_j)$  is the set of vertices of  $C_j$ . A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique  $xy$ -directed path for every distinct vertices  $x$  and  $y$ . A vertex-disjoint union of circuit-trees is a *circuit-forest*. When all circuits have length three, we speak of a *triangle-tree*. For short, a  $k$ -circuit-forest is a circuit-forest consisting of  $k$  circuits.

**Conjecture 1** (Hoàng and Reed [3]) *Every digraph has a  $\delta^+$ -circuit-forest.*

This conjecture is not even known to be true for  $\delta^+ = 3$ . In the case  $\delta^+ = 2$ , C. Thomassen proved in [6] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament  $T$ , that is the 3-uniform hypergraph on vertex set  $V$  which edges are the 3-circuits of  $T$ .

Indeed, if a tournament  $T$  has a  $\delta^+$ -circuit-forest, by the fact that every circuit contains a directed triangle,  $T$  also has a  $\delta^+$ -triangle-forest. Observe that a  $\delta^+$ -triangle-forest spans exactly  $2\delta^+ + c$  vertices, where  $c$  is the number of components of the triangle-forest. When  $T$  is a regular tournament with outdegree  $\delta^+$ , hence with  $2\delta^+ + 1$  vertices, a  $\delta^+$ -triangle-forest of  $T$  is necessarily a spanning  $\delta^+$ -triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

**Theorem 1** *Every tournament has a  $\delta^+$ -triangle-tree.*

---

<sup>\*</sup>Projet Mascotte, CNRS/INRIA/UNSA, INRIA Sophia-Antipolis, 2004 route des Lucioles BP 93, 06902 Sophia-Antipolis Cedex, France [fhavet@sophia.inria.fr](mailto:fhavet@sophia.inria.fr)

<sup>†</sup>LIRMM, 161 rue Ada, 34392 Montpellier Cedex 5, France, [thomasse@lirmm.fr](mailto:thomasse@lirmm.fr)

<sup>‡</sup>Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, [anders@cs.rhul.ac.uk](mailto:anders@cs.rhul.ac.uk)

## 2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

**Lemma 1** *Let  $k \geq 1$  and let  $a_1, a_2, \dots, a_k$  and  $b_1, b_2, \dots, b_k$  be two sequences of positive reals. Let  $A = \sum_{i=1}^k a_i$  and  $B = \sum_{j=1}^k b_j$ . If  $\sum_{i=1}^k a_i b_i = \frac{AB}{2} + q$ , where  $q \geq 0$ , then there is an  $i$  such that  $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$ .*

**Proof.** If  $k = 1$ , then the lemma follows immediately as  $q = \frac{AB}{2}$  and  $A + B \geq \frac{A+B}{2} + \sqrt{AB}$ . So assume that  $k > 1$ . Without loss of generality, we may assume that  $(a_1, b_1) \geq (a_2, b_2) \geq \dots \geq (a_k, b_k)$  in the lexicographical order. Let  $r$  be the minimum value such that  $b_r \geq b_i$  for all  $i = 1, 2, \dots, k$ . Note that  $a_1 \geq |A|/2$ , since otherwise  $\sum_{i=1}^k a_i b_i < \sum_{i=1}^k A b_i / 2 = AB/2$ . Analogously  $b_r \geq |B|/2$ . Define  $a'$  and  $b'$  so that  $a_1 = A/2 + a'$  and  $b_r = B/2 + b'$ .

If  $r \neq 1$ , then the following holds:

$$\begin{aligned} \sum_{i=1}^k a_i b_i &\leq a_1 b_1 + \sum_{i=2}^k a_i b_r \\ &\leq a_1 (B - b_r) + (A - a_1) b_r \\ &= \left(\frac{A}{2} + a'\right) \left(\frac{B}{2} - b'\right) + \left(\frac{A}{2} - a'\right) \left(\frac{B}{2} + b'\right) \\ &= \frac{AB}{2} - 2a'b' \\ &\leq \frac{AB}{2} \end{aligned}$$

As  $q \geq 0$ , this implies we have equality everywhere above, which means that  $b_1 = B - b_r$ . As  $B = b_1 + b_r$ , we must have  $k = 2$ . As there was equality everywhere above we have  $b' = 0$  or  $a' = 0$  which implies that  $a_1 = a_2 = A/2$  or  $b_1 = b_2 = B/2$ . In both cases we would have  $r = 1$ , a contradiction.

Suppose now that  $r = 1$ . Then

$$\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = \left(\frac{A}{2} + a'\right) \left(\frac{B}{2} + b'\right) + \left(\frac{A}{2} - a'\right) \left(\frac{B}{2} - b'\right)$$

This implies that  $q \leq 2a'b'$ . The minimum value of  $a' + b'$  is obtained when  $a' = b' = \sqrt{q/2}$ . Therefore the minimum value of  $a_1 + b_1$  is  $A/2 + B/2 + 2\sqrt{q/2}$ . This completes the proof of the lemma.  $\blacksquare$

**Corollary 1** *Let  $G$  be a bipartite graph with partite sets  $A$  and  $B$ . If  $|E(G)| = \frac{|A||B|}{2} + q$ , where  $q \geq 0$ , then there is a component in  $G$  of size at least  $|V(G)|/2 + \sqrt{2q}$ .*

**Proof.** Let  $Q_1, Q_2, \dots, Q_k$  be the components of  $G$ . Let  $a_i = |A \cap Q_i|$  and  $b_i = |B \cap Q_i|$  for all  $i = 1, 2, \dots, k$ . We note that  $\sum_{i=1}^k a_i b_i \geq \frac{|A||B|}{2} + q$ . By Lemma 1, we have  $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$  for some  $i$ . This completes the proof.  $\blacksquare$

**Lemma 2** *Let  $T$  be a triangle-tree in a digraph  $D$ , and let  $X \subseteq V(T)$  and  $Y \subseteq V(T)$  be such that  $|X| + |Y| \geq |V(T)| + 2$ . Then there exists a triangle  $C$  in  $T$  such that the three disjoint triangle-trees in  $T - E(C)$  can be named  $T_1, T_2, T_3$  such that  $Y$  intersects both  $T_1$  and  $T_2$  and  $X$  intersects both  $T_2$  and  $T_3$ .*

**Proof.** We show this by induction. As  $|X| + |Y| \geq |V(T)| + 2$ , we note that  $T$  contains at least one triangle. If  $T$  only contains one triangle then the lemma holds as either  $X$  or  $Y$  equals  $V(T)$ , and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that  $T$  contains at least two triangles. Let  $T = T_1 \cup C$ , where  $C$  is a triangle and  $T_1$  is a triangle-tree. If  $|X \cap V(T_1)| + |Y \cap V(T_1)| \geq |V(T_1)| + 2$ , then we are done by induction. So assume that this is not the case. As  $|V(T_1)| = |V(T)| - 2$  this implies that  $|X \setminus V(T_1)| + |Y \setminus V(T_1)| \geq 3$ .

Without loss of generality assume that  $|X \setminus V(T_1)| \geq 2$  and  $|Y \setminus V(T_1)| \geq 1$ . Let  $T_2$  be the singleton-tree consisting of a vertex in  $Y \setminus V(T_1)$  and let  $T_3$  be the singleton-tree  $X \setminus (V(T_1) \cup V(T_2))$ . Note that

$T - E(C)$  consists of the triangle-trees  $T_1, T_2$  and  $T_3$ . By definition,  $X$  intersects both  $T_2$  and  $T_3$  and  $Y$  intersects  $T_2$ . If  $Y$  also intersects  $T_1$ , we have our conclusion. If not, since  $|X| + |Y| \geq |V(T)| + 2$ , we have  $Y = T_2 \cup T_3$  and  $X = V(T)$ , and free to rename  $T_1, T_2, T_3$ , we have our conclusion.  $\blacksquare$

### 3 Proof of Theorem 1.

We will need the following results:

**Theorem 2** (Tewes and Volkmann [5]) *Let  $D$  be a  $p$ -partite tournament with partite sets  $V_1, V_2, \dots, V_p$ . Then there exists a partition  $Q_1, Q_2, \dots, Q_k$  of  $D$  such that*

- each  $Q_i$  induces an independent set or a strong component,
- there are no arcs from  $Q_j$  to  $Q_i$  for all  $j > i$ , and there is an arc from  $Q_i$  to  $Q_{i+1}$  for all  $i = 1, 2, \dots, k-1$ .

**Theorem 3** (Guo and Volkmann [2]) *Let  $D$  be a strong  $p$ -partite tournament with partite sets  $V_1, V_2, \dots, V_p$ . For every  $1 \leq i \leq p$ , there exists a vertex  $x \in V_i$  which belongs to a  $k$ -circuit for all  $3 \leq k \leq p$ .*

Now, we assume that  $D$  is a strong tournament as otherwise we just consider the terminal strong component. Let  $T$  be a maximum size triangle-tree in  $D$ , and assume for the sake of contradiction that  $|V(T)| < 2\delta^+(D) + 1$ . Let  $D^{MT}$  be the multipartite tournament obtained from  $D$  by deleting all the arcs with both endpoints in  $V(T)$ . Let  $V_1, V_2, \dots, V_i$  be the partite sets in  $D^{MT}$  such that  $V_1 = V(T)$  and  $|V_i| = 1$  for all  $i > 1$ .

Let  $Q_1, Q_2, \dots, Q_k$  be a partition of  $V(D^{MT})$  given by Theorem 2.

If there is a  $Q_i$  with  $Q_i \cap V_1 \neq \emptyset$  and  $Q_i \not\subseteq V_1$  then we obtain the following contradiction. Since  $Q_i \not\subseteq V_1$ , we observe that  $Q_i$  contains at least two partite set. In addition, note that at least three partite sets intersect  $Q_i$  as  $D^{MT} \langle Q_i \rangle$  would not be strong if there were only two partite sets since  $|V_i| = 1$  for all  $i > 1$ . By Theorem 3, in the subgraph of  $D^{MT}$  induced by  $Q_i$ , there is a 3-circuit containing exactly one vertex from  $V_1$ . This contradicts the maximality of  $T$ . So every set  $Q_i$  is either a subset of  $V_1$  or is disjoint from  $V_1$ .

Note that  $Q_1 \cap V_1 \neq \emptyset$  and  $Q_k \cap V_1 \neq \emptyset$ , as otherwise  $D$  would not be strong. Applying the observation above, we obtain  $Q_1 \cup Q_k \subset V_1$ . Let  $D' = D \langle V_1 \rangle$ . If there is a vertex  $x \in Q_k$  with  $d_{D'}^+(x) \leq \frac{|V_1|-1}{2}$ , then  $d_D^+(x) \leq \frac{|V_1|-1}{2}$ , which implies that  $|V(T)| \geq 2\delta^+(D) + 1$ , a contradiction. So  $d_{D'}^+(x) \geq \frac{|V_1|+1}{2}$  for all  $x \in Q_k$ , as  $|V_1|$  is odd.

Let  $G_1$  denote the bipartite graph with partite sets  $Q_k$  and  $V_1 - Q_k$ , and with  $E(G_1) = \{uv \mid u \in Q_k, v \in V_1 - Q_k, uv \in E(D)\}$ . Note that the following now holds by the above.

$$|Q_k| \frac{|V_1|+1}{2} \leq \sum_{u \in Q_k} d_{D'}^+(u) = \binom{|Q_k|}{2} + |E(G_1)| \quad (1)$$

This implies that  $|E(G_1)| \geq \frac{|Q_k|(|V_1|-|Q_k|)}{2} + |Q_k|$ , which by Corollary 1 implies that there is a component in  $G_1$  of size at least  $|V_1|/2 + \sqrt{2|Q_k|} \geq |V_1|/2 + \sqrt{2}$ . As the size of the maximum component in  $G_1$  is an integer it is at least  $|V_1|/2 + 3/2$ . Two cases can now occur:

- If  $|Q_{k-1}| > 1$  or  $Q_{k-2} \not\subseteq V_1$  (or both). If  $|Q_{k-1}| > 1$  then let  $Z = \{z_1, z_2\}$  be any two distinct vertices in  $Q_{k-1}$  otherwise let  $Z$  be any two distinct vertices in  $Q_{k-1} \cup Q_{k-2}$ . By the definition of the  $Q_i$ 's we note that  $Z \cap V_1 = \emptyset$  and there are all arcs from  $(V_1 - Q_k)$  to  $Z$  and from  $Z$  to  $Q_k$ . We let  $X = Y$  be the vertices of a component in  $G_1$  of size at least  $(|V_1|+3)/2$  and use Lemma 2 to find a triangle  $C$  in  $T$ , such that the three disjoint triangle-trees,  $T_1, T_2$  and  $T_3$ , of  $T - E(C)$  all intersect

$X$  (as  $X = Y$ ). As  $X$  are the vertices of a component in  $G_1$  there are edges,  $u_1v_1$  and  $u_2v_2$ , from  $G_1$  such that the following holds. The edge  $u_1v_1$  connects  $T_3$  and  $T_j$ , where  $u_2v_2$  connects  $T_{3-j}$  and  $T_j \cup T_3$ . generality assume that  $u_1, u_2 \in Q_k$  and  $v_1, v_2 \in V_1 - Q_k$ . Now  $T - E(C)$  together with the vertices  $z_1$  and  $z_2$  as well as the 3-circuits  $v_1z_1u_1v_1$  and  $v_2z_2u_2v_2$  is a triangle-tree in  $D$  with more triangles than  $T$ , a contradiction.

- If  $|Q_{k-1}| = 1$  and  $Q_{k-2} \subseteq V_1$ . Note that  $k > 3$ , as otherwise  $|V(D) \setminus V(T)| = 1$  and we have a contradiction to our assumption. This implies that  $k > 4$  as  $Q_1 \subseteq V_1$ , which implies that  $Q_2 \not\subseteq V_1$ . Now let  $Q_{k-1} = \{z_1\}$  and let  $z_2 \in Q_{k-3}$  be arbitrary. Let  $G_2$  denote the bipartite graph with partite sets  $A = Q_k \cup Q_{k-2}$  and  $B = V_1 - A$ , and with  $E(G_2) = \{uv \mid u \in A, v \in B, uv \in E(D)\}$ . Recall that  $d_{D'}^+(x) \geq \frac{|V_1|+1}{2}$  for all  $x \in Q_k$ . Analogously we get that  $d_{D'}^+(y) \geq \frac{|V_1|+1}{2} - 1$  for all  $y \in Q_{k-2}$  (as  $|Q_{k-1}| = 1$ ). This implies the following.

$$|A| \frac{|V_1|+1}{2} - |Q_{k-2}| \leq \sum_{u \in A} d_{D'}^+(u) = \binom{|A|}{2} + |E(G_2)| \quad (2)$$

This implies that  $|E(G_2)| \geq \frac{|A|(|V_1|-|A|)}{2} + |A| - |Q_{k-2}|$ , which by Corollary 1 implies that there is a component in  $G_2$  of size at least  $|V_1|/2 + \sqrt{2|Q_k|}$ , as  $|A| - |Q_{k-2}| = |Q_k|$ . Note that  $|Q_k| > 1$ , as otherwise the vertex in  $Q_{k-1}$  only has out-degree one, a contradiction. Therefore there is a component in  $G_2$  of size at least  $|V_1|/2 + 2$  and so at least  $|V_1|/2 + 5/2$  as  $V_1$  is odd.

Let  $X$  be the vertices of a component in  $G_1$  of size at least  $|V_1|/2 + 3/2$  and let  $Y$  be the vertices in a connected component of  $G_2$  of size at least  $|V_1|/2 + 5/2$ . Now use Lemma 2 to find a triangle  $C$  in  $T$ , such that the three disjoint triangle-trees,  $T_1, T_2$  and  $T_3$ , of  $T - E(C)$  have the following property. The set  $Y$  intersects  $T_1$  and  $T_2$  and the set  $X$  intersects  $T_2$  and  $T_3$ . Due to the definition of  $X$  and  $Y$  there exists edges,  $u_1v_1 \in E(G_1)$  and  $u_2v_2 \in E(G_2)$ , such that the following holds. The edge  $u_1v_1$  connects  $T_3$  and  $T_j$ , where  $j \in \{1, 2\}$  and  $u_2v_2$  connects  $T_{3-j}$  and  $T_j \cup T_3$ . Without loss of generality assume that  $u_1, u_2 \in Q_k$  and  $v_1, v_2 \in V_1 - Q_k$ . Now  $T - E(C)$  together with the vertices  $z_1$  and  $z_2$  as well as the 3-circuits  $v_1z_1u_1v_1$  and  $v_2z_2u_2v_2$  is a triangle-tree in  $D$  with more triangles than  $T$ , a contradiction. This completes the proof. ■

## References

- [1] L. Caccetta and R. Häggkvist, On minimal digraphs with given girth, Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congress. Numer.*, **XXI** (1978), 181–187.
- [2] Y. Guo and L. Volkmann, Cycles in multipartite tournaments. *Journal of Combinatorial Theory, Series B*, **62** (1994), 363–366.
- [3] C.T. Hoàng and B. Reed, A note on short cycles in digraphs, *Discrete Math.*, **66** (1987), 103–107.
- [4] B.D. Sullivan, A summary of results and problems related to the Caccetta-Häggkvist conjecture, preprint.
- [5] M. Tewes and L. Volkmann, Vertex deletion and cycles in multipartite tournaments, *Discrete Math.*, **197/198** (1999), 769–779.
- [6] C. Thomassen, The 2-linkage problem for acyclic digraphs, *Discrete Math.*, **55** (1985), 73–87.