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Hoàng-Reed conjecture holds for tournaments

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Abstract

Hoàng-Reed conjecture asserts that every digraph $D$ has a collection $C$ of circuits $C_1, \ldots, C_{\delta^+}$, where $\delta^+$ is the minimum outdegree of $D$, such that the circuits of $C$ have a forest-like structure. Formally, $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| \leq 1$, for all $i = 2, \ldots, \delta^+$. We verify this conjecture for the class of tournaments.

1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph $D$ on $n$ vertices and with minimum outdegree $n/k$ has a circuit of length at most $k$. Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A circuit-tree is either a singleton or consists of a set of circuits $C_1, \ldots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| = 1$ for all $i = 2, \ldots, k$, where $V(C_j)$ is the set of vertices of $C_j$. A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique $xy$-directed path for every distinct vertices $x$ and $y$. A vertex-disjoint union of circuit-trees is a circuit-forest. When all circuits have length three, we speak of a triangle-tree. For short, a $k$-circuit-forest is a circuit-forest consisting of $k$ circuits.

Conjecture 1 (Hoàng and Reed [3]) Every digraph has a $\delta^+$-circuit-forest.

This conjecture is not even known to be true for $\delta^+ = 3$. In the case $\delta^+ = 2$, C. Thomassen proved in [6] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament $T$, that is the 3-uniform hypergraph on vertex set $V$ which edges are the 3-circuits of $T$.

Indeed, if a tournament $T$ has a $\delta^+$-circuit-forest, by the fact that every circuit contains a directed triangle, $T$ also has a $\delta^+$-triangle-forest. Observe that a $\delta^+$-triangle-forest spans exactly $2\delta^+ + c$ vertices, where $c$ is the number of components of the triangle-forest. When $T$ is a regular tournament with outdegree $\delta^+$, hence with $2\delta^+ + 1$ vertices, a $\delta^+$-triangle-forest of $T$ is necessarily a spanning $\delta^+$-triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

Theorem 1 Every tournament has a $\delta^+$-triangle-tree.
2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

**Lemma 1** Let $k \geq 1$ and let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be two sequences of positive reals. Let $A = \sum_{i=1}^k a_i$ and $B = \sum_{j=1}^k b_j$. If $\sum_{i=1}^k a_i b_i = \frac{AB}{2} + q$, where $q \geq 0$, then there is an $i$ such that $a_i + b_i \geq \frac{A+B}{2} + \sqrt{q}$. 

**Proof.** If $k = 1$, then the lemma follows immediately as $q = \frac{AB}{2}$ and $A + B \geq \frac{A+B}{2} + \sqrt{AB}$. So assume that $k > 1$. Without loss of generality, we may assume that $(a_1, b_1) \geq (a_2, b_2) \geq \ldots \geq (a_k, b_k)$ in the lexicographical order. Let $r$ be the minimum value such that $b_r \geq b_i$ for all $i = 1, 2, \ldots, k$. Note that $a_1 \geq |A|/2$, since otherwise $\sum_{i=1}^k a_i b_i < \sum_{i=1}^k A b_i/2 = AB/2$. Analogously $b_r \geq |B|/2$. Define $a'$ and $b'$ so that $a_1 = A/2 + a'$ and $b_r = B/2 + b'$.

If $r \neq 1$, then the following holds:

$$\sum_{i=1}^k a_i b_i \leq a_1 b_1 + \sum_{i=2}^k a_i b_r = a_1 (B - b_r) + (A - a_1) b_r = (\frac{A}{2} + a') (\frac{B}{2} - b') + (\frac{A}{2} - a') (\frac{B}{2} + b') = \frac{AB}{2} - 2a'b' \leq \frac{AB}{2}$$

As $q \geq 0$, this implies we have equality everywhere above, which means that $b_1 = B - b_r$. As $B = b_1 + b_r$, we must have $k = 2$. As there was equality everywhere above we have $b' = 0$ or $a' = 0$ which implies that $a_1 = a_2 = A/2$ or $b_1 = b_2 = B/2$. In both cases we would have $r = 1$, a contradiction.

Suppose now that $r = 1$. Then

$$\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = (\frac{A}{2} + a')(\frac{B}{2} + b') + (\frac{A}{2} - a')(\frac{B}{2} - b')$$

This implies that $q \leq 2a'b'$. The minimum value of $a'+b'$ is obtained when $a' = b' = \sqrt{q/2}$. Therefore the minimum value of $a_1 + b_1$ is $A/2 + B/2 + 2\sqrt{q/2}$. This completes the proof of the lemma.

**Corollary 1** Let $G$ be a bipartite graph with partite sets $A$ and $B$. If $|E(G)| = \frac{|A||B|}{2} + q$, where $q \geq 0$, then there is a component in $G$ of size at least $|V(G)|/2 + \sqrt{2q}$.

**Proof.** Let $Q_1, Q_2, \ldots, Q_k$ be the components of $G$. Let $a_i = |A \cap Q_i|$ and $b_i = |B \cap Q_i|$ for all $i = 1, 2, \ldots, k$. We note that $\sum_{i=1}^k a_i b_i \geq \frac{|A||B|}{2} + q$. By Lemma 1, we have $a_i + b_i \geq \frac{A+B}{2} + \sqrt{q}$ for some $i$. This completes the proof.

**Lemma 2** Let $T$ be a triangle-tree in a digraph $D$, and let $X \subseteq V(T)$ and $Y \subseteq V(T)$ be such that $|X| + |Y| \geq |V(T)| + 2$. Then there exists a triangle $C$ in $T$ such that the three disjoint triangle-trees in $T - E(C)$ can be named $T_1, T_2, T_3$ such that $Y$ intersects both $T_1$ and $T_2$ and $X$ intersects both $T_2$ and $T_3$.

**Proof.** We show this by induction. As $|X| + |Y| \geq |V(T)| + 2$, we note that $T$ contains at least one triangle. If $T$ only contains one triangle then the lemma holds as either $X$ or $Y$ equals $V(T)$, and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that $T$ contains at least two triangles. Let $T = T_1 \cup C$, where $C$ is a triangle and $T_1$ is a triangle-tree. If $|X \cap V(T_1)| + |Y \cap V(T_1)| \geq |V(T_1)| + 2$, then we are done by induction. So assume that this is not the case. As $|V(T_1)| = |V(T)| - 2$ this implies that $|X \setminus V(T_1)| + |Y \setminus V(T_1)| \geq 3$.

Without loss of generality assume that $|X \setminus V(T_1)| \geq 2$ and $|Y \setminus V(T_1)| \geq 1$. Let $T_2$ be the singleton-tree consisting of a vertex in $Y \setminus V(T_1)$ and let $T_3$ be the singleton-tree $X \setminus (V(T_1) \cup V(T_2))$. Note that
$T - E(C)$ consists of the triangle-trees $T_1$, $T_2$ and $T_3$. By definition, $X$ intersects both $T_2$ and $T_3$ and $Y$ intersects $T_2$. If $Y$ also intersects $T_1$, we have our conclusion. If not, since $|X| + |Y| \geq |V(T)| + 2$, we have $Y = T_2 \cup T_3$ and $X = V(T)$, and free to rename $T_1, T_2, T_3$, we have our conclusion.  

3 Proof of Theorem 1.

We will need the following results:

**Theorem 2** (Tewes and Volkmann [5]) Let $D$ be a $p$-partite tournament with partite sets $V_1, V_2, \ldots, V_p$. Then there exists a partition $Q_1, Q_2, \ldots, Q_k$ of $D$ such that

- each $Q_i$ induces an independent set or a strong component,
- there are no arcs from $Q_j$ to $Q_i$ for all $j > i$, and there is an arc from $Q_i$ to $Q_{i+1}$ for all $i = 1, 2, \ldots, k - 1$.

**Theorem 3** (Guo and Volkmann [2]) Let $D$ be a strong $p$-partite tournament with partite sets $V_1, V_2, \ldots, V_p$. For every $1 \leq i \leq p$, there exists a vertex $x \in V_i$ which belongs to a $k$-circuit for all $3 \leq k \leq p$.

Now, we assume that $D$ is a strong tournament as otherwise we just consider the terminal strong component. Let $T$ be a maximum size triangle-tree in $D$, and assume for the sake of contradiction that $|V(T)| < 2\delta^+(D) + 1$. Let $D^{MT}$ be the multipartite tournament obtained from $D$ by deleting all the arcs with both endpoints in $V(T)$. Let $V_1, V_2, \ldots, V_i$ be the partite sets in $D^{MT}$ such that $V_i = V(T)$ and $|V_i| = 1$ for all $i \geq 1$.

Let $Q_1, Q_2, \ldots, Q_k$ be a partition of $V(D^{MT})$ given by Theorem 2. If there is a $Q_i$ with $Q_i \cap V_1 \neq \emptyset$ and $Q_i \subseteq V_1$ then we obtain the following contradiction. Since $Q_i \subseteq V_1$, we observe that $Q_i$ contains at least two partite set. In addition, note that at least three partite sets intersect $Q_i$ as $D^{MT}(Q_i)$ would not be strong if there were only two partite sets since $|V_i| = 1$ for all $i > 1$. By Theorem 3, in the subgraph of $D^{MT}$ induced by $Q_1$, there is a 3-circuit containing exactly one vertex from $V_1$. This contradicts the maximality of $T$. So every set $Q_i$ is either a subset of $V_1$ or is disjoint from $V_1$.

Note that $Q_1 \cap V_1 \neq \emptyset$ and $Q_k \cap V_1 \neq \emptyset$, as otherwise $D$ would not be strong. Applying the observation above, we obtain $Q_1 \cup Q_k \subset V_1$. Let $T' = D(V_1)$. If there is a vertex $x \in Q_k$ with $d^+_D(x) = \frac{|V_1| - 1}{2}$, then $d^+_D(x) \leq \frac{|V_1| - 1}{2}$, which implies that $|V(T)| \geq 2\delta^+(D) + 1$, a contradiction. So $d^+_D(x) \geq \frac{|V_1| + 1}{2}$ for all $x \in Q_k$, as $|V_1|$ is odd.

Let $G_1$ denote the bipartite graph with partite sets $Q_k$ and $V_1 - Q_k$, and with $E(G_1) = \{uv \mid u \in Q_k, v \in V_1 - Q_k, uv \in E(D)\}$. Note that the following now holds by the above.

$$|Q_k| - \frac{|V_1| + 1}{2} \leq \sum_{u \in Q_k} d^+_D(u) = \left(\frac{|Q_k|}{2}\right) + |E(G_1)|$$

This implies that $|E(G_1)| \geq \frac{|Q_1|(|V_1| - |Q_k|)}{2} + |Q_k|$, which by Corollary 1 implies that there is a component in $G_1$ of size at least $|V_1|/2 + \sqrt{2|Q_k|} \geq |V_1|/2 + \sqrt{2}$. As the size of the maximum component in $G_1$ is an integer it is at least $|V_1|/2 + 3/2$. Two cases can now occur:

- If $|Q_{k-1}| > 1$ or $Q_{k-2} \subseteq V_1$ (or both). If $|Q_{k-1}| > 1$ then let $Z = \{z_1, z_2\}$ be any two distinct vertices in $Q_{k-1}$ otherwise let $Z$ be any two distinct vertices in $Q_{k-1} \cup Q_{k-2}$. By the definition of the $Q_i$’s we note that $Z \cap V_1 = \emptyset$ and there are all arcs from $(V_1 - Q_k)$ to $Z$ and from $Z$ to $Q_k$. We let $X = Y$ be the vertices of a component in $G_1$ of size at least $(|V_1| + 3)/2$ and use Lemma 2 to find a triangle $C$ in $T$, such that the three disjoint triangle-trees, $T_1, T_2$ and $T_3$, of $T - E(C)$ all intersect
X (as X = Y). As X are the vertices of a component in G1 there are edges, u1v1 and u2v2, from G1 such that the following holds. The edge u1v1 connects T3 and Tj, where u2v2 connects T3−j and Tj ∪ T3. Generality assume that u1, u2 ∈ Qk and v1, v2 ∈ V1 − Qk. Now T − E(C) together with the vertices z1 and z2 as well as the 3-circuits v1z1u1v1 and v2z2u2v2 is a triangle-tree in D with more triangles than T, a contradiction.

• If |Qk−1| = 1 and Qk−2 ⊆ V1. Note that k > 3, as otherwise |V(D) \ V(T)| = 1 and we have a contradiction to our assumption. This implies that k > 4 as Q1 ⊆ V1, which implies that Q2 ⊆ V1.

Now let Qk−1 = {z1} and let z2 ∈ Qk−3 be arbitrary. Let G2 denote the bipartite graph with partite sets A = Qk ∪ Qk−2 and B = V1 − A, and with E(G2) = \{uv \mid u ∈ A, v ∈ B, uv ∈ E(D)\}.

Recall that dG2(x) ≥ |V2|/2 + 1 for all x ∈ Qk. Analogously we get that dG2(y) ≥ |V2|/2 + 1 for all y ∈ Qk−2 (as |Qk−1| = 1). This implies the following.

\[ |A||V2|/2 + |Qk−2| ≤ \sum_{u \in A} dG2(u) = \left(\frac{|A|}{2}\right) + |E(G2)| \]

This implies that |E(G2)| ≥ \frac{|A||V2|−|A|}{2} + |A| − |Qk−2|, which by Corollary 1 implies that there is a component in G2 of size at least \frac{|V1|}{2} + 2. As |A| − |Qk−2| = |Qk|. Note that |Qk| > 1, as otherwise the vertex in Qk−1 only has out-degree one, a contradiction. Therefore there is a component in G2 of size at least \frac{|V1|}{2} + 2 and so at least |V1|/2 + 5/2 as V1 is odd.

Let X be the vertices of a component in G1 of size at least |V1|/2 + 3/2 and let Y be the vertices in a connected component of G2 of size at least \frac{|V1|}{2} + 5/2. Now use Lemma 2 to find a triangle C in T, such that the three disjoint triangle-trees, T1, T2 and T3, of T − E(C) have the following property. The set Y intersects T1 and T2 and the set X intersects T2 and T3. Due to the definition of X and Y there exists edges, u1v1 ∈ E(G1) and u2v2 ∈ E(G2), such that the following holds.

The edge u1v1 connects T3 and Tj, where j ∈ \{1, 2\} and u2v2 connects T3−j and Tj ∪ T3. Without loss of generality assume that u1, u2 ∈ Qk and v1, v2 ∈ V1 − Qk. Now T − E(C) together with the vertices z1 and z2 as well as the 3-circuits v1z1u1v1 and v2z2u2v2 is a triangle-tree in D with more triangles than T, a contradiction. This completes the proof.

References