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Hoàng-Reed conjecture holds for tournaments

Frédéric Havet∗ Stéphan Thomassé† Anders Yeo‡

Abstract
Hoàng-Reed conjecture asserts that every digraph $D$ has a collection $C$ of circuits $C_1, \ldots, C_{\delta^+}$, where $\delta^+$ is the minimum outdegree of $D$, such that the circuits of $C$ have a forest-like structure. Formally, $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| \leq 1$, for all $i = 2, \ldots, \delta^+$. We verify this conjecture for the class of tournaments.

1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph $D$ on $n$ vertices and with minimum outdegree $n/k$ has a circuit of length at most $k$. Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A circuit-tree is either a singleton or consists of a set of circuits $C_1, \ldots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| = 1$ for all $i = 2, \ldots, k$, where $V(C_j)$ is the set of vertices of $C_j$. A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique $xy$-directed path for every distinct vertices $x$ and $y$. A vertex-disjoint union of circuit-trees is a circuit-forest. When all circuits have length three, we speak of a triangle-tree. For short, a $k$-circuit-forest is a circuit-forest consisting of $k$ circuits.

Conjecture 1 (Hoàng and Reed [3]) Every digraph has a $\delta^+$-circuit-forest.

This conjecture is not even known to be true for $\delta^+ = 3$. In the case $\delta^+ = 2$, C. Thomassen proved in [6] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e., contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament $T$, that is the 3-uniform hypergraph on vertex set $V$ which edges are the 3-circuits of $T$.

Indeed, if a tournament $T$ has a $\delta^+$-circuit-forest, by the fact that every circuit contains a directed triangle, $T$ also has a $\delta^+$-triangle-forest. Observe that a $\delta^+$-triangle-forest spans exactly $2\delta^+ + c$ vertices, where $c$ is the number of components of the triangle-forest. When $T$ is a regular tournament with outdegree $\delta^+$, hence with $2\delta^+ + 1$ vertices, a $\delta^+$-triangle-forest of $T$ is necessarily a spanning $\delta^+$-triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

Theorem 1 Every tournament has a $\delta^+$-triangle-tree.
2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

**Lemma 1** Let $k \geq 1$ and let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be two sequences of positive reals. Let $A = \sum_{i=1}^{k} a_i$ and $B = \sum_{j=1}^{k} b_j$. If $\sum_{i=1}^{k} a_i b_i = \frac{AB}{2} + q$, where $q \geq 0$, then there is an $i$ such that $a_i + b_i \geq \frac{A+B}{2} + \sqrt{q}$.

**Proof.** If $k = 1$, then the lemma follows immediately as $q = \frac{AB}{2}$ and $A + B \geq \frac{A+B}{2} + \sqrt{AB}$. So assume that $k > 1$. Without loss of generality, we may assume that $(a_1, b_1) \geq (a_2, b_2) \geq \ldots \geq (a_k, b_k)$ in the lexicographical order. Let $r$ be the minimum value such that $b_r = b_i$ for all $i = 1, 2, \ldots, k$. Note that $a_1 \geq |A|/2$, since otherwise $\sum_{i=1}^{k} a_i b_i < \sum_{i=1}^{k} Ab_i/2 = AB/2$. Analogously $b_r \geq |B|/2$. Define $a'$ and $b'$ so that $a_1 = A/2 + a'$ and $b_r = B/2 + b'$.

If $r \neq 1$, then the following holds:
\[
\sum_{i=1}^{k} a_i b_i \leq a_1 b_1 + \sum_{i=2}^{k} a_i b_r \\
\leq a_1 (B - b_r) + (A - a_1) b_r \\
= \left(\frac{A}{2} + a'\right)(\frac{B}{2} - b') + \left(\frac{A}{2} - a'\right)(\frac{B}{2} + b') \\
= \frac{AB}{2} - 2a'b'
\]

As $q \geq 0$, this implies we have equality everywhere above, which means that $b_1 = B - b_r$. As $B = b_1 + b_r$, we must have $k = 2$. As there was equality everywhere above we have $b' = 0$ or $a' = 0$ which implies that $a_1 = a_2 = A/2$ or $b_1 = b_2 = B/2$. In both cases we would have $r = 1$, a contradiction.

Suppose now that $r = 1$. Then
\[
\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = \left(\frac{A}{2} + a'\right)\left(\frac{B}{2} + b'\right) + \left(\frac{A}{2} - a'\right)\left(\frac{B}{2} - b'\right)
\]

This implies that $q \leq 2a'b'$. The minimum value of $a' + b'$ is obtained when $a' = b' = \sqrt{q/2}$. Therefore the minimum value of $a_1 + b_1$ is $A/2 + B/2 + 2\sqrt{q/2}$. This completes the proof of the lemma. 

**Corollary 1** Let $G$ be a bipartite graph with partite sets $A$ and $B$. If $|E(G)| = \frac{|A||B|}{2} + q$, where $q \geq 0$, then there is a component in $G$ of size at least $|V(G)|/2 + \sqrt{q}$.

**Proof.** Let $Q_1, Q_2, \ldots, Q_k$ be the components of $G$. Let $a_i = |A \cap Q_i|$ and $b_i = |B \cap Q_i|$ for all $i = 1, 2, \ldots, k$. We note that $\sum_{i=1}^{k} a_i b_i \geq \frac{|A||B|}{2} + q$. By Lemma 1, we have $a_i + b_i \geq \frac{A+B}{2} + \sqrt{q}$ for some $i$. This completes the proof.

**Lemma 2** Let $T$ be a triangle-tree in a digraph $D$, and let $X \subseteq V(T)$ and $Y \subseteq V(T)$ be such that $|X| + |Y| \geq |V(T)| + 2$. Then there exists a triangle $C$ in $T$ such that the three disjoint triangle-trees in $T - E(C)$ can be named $T_1, T_2, T_3$ such that $Y$ intersects both $T_1$ and $T_2$ and $X$ intersects both $T_2$ and $T_3$.

**Proof.** We show this by induction. As $|X| + |Y| \geq |V(T)| + 2$, we note that $T$ contains at least one triangle. If $T$ only contains one triangle then the lemma holds as either $X$ or $Y$ equals $V(T)$, and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that $T$ contains at least two triangles. Let $T = T_1 \cup C$, where $C$ is a triangle and $T_1$ is a triangle-tree. If $|X \cap V(T_1)| + |Y \cap V(T_1)| \geq |V(T_1)| + 2$, then we are done by induction. So assume that this is not the case. As $|V(T_1)| = |V(T)| - 2$ this implies that $|X \cap V(T_1)| + |Y \cap V(T_1)| \geq 3$.

Without loss of generality assume that $|X \cap V(T_1)| \geq 2$ and $|Y \cap V(T_1)| \geq 1$. Let $T_2$ be the singleton-tree consisting of a vertex in $Y \setminus V(T_1)$ and let $T_3$ be the singleton-tree $X \setminus (V(T_1) \cup V(T_2))$. Note that
Theorem 3

\( T - E(C) \) consists of the triangle-trees \( T_1, T_2 \) and \( T_3 \). By definition, \( X \) intersects both \( T_2 \) and \( T_3 \) and \( Y \) intersects \( T_2 \). If \( Y \) also intersects \( T_1 \), we have our conclusion. If not, since \( |X| + |Y| \geq |V(T)| + 2 \), we have \( Y = T_2 \cup T_3 \) and \( X = V(T) \), and free to rename \( T_1, T_2, T_3 \), we have our conclusion. \( \square \)

3 Proof of Theorem 1.

We will need the following results:

Theorem 2 (Tewes and Volkmann [5]) Let \( D \) be a \( p \)-partite tournament with partite sets \( V_1, V_2, \ldots, V_p \). Then there exists a partition \( Q_1, Q_2, \ldots, Q_k \) of \( D \) such that

- each \( Q_i \) induces an independent set or a strong component,
- there are no arcs from \( Q_j \) to \( Q_i \) for all \( j > i \), and there is an arc from \( Q_i \) to \( Q_{i+1} \) for all \( i = 1, 2, \ldots, k-1 \).

Theorem 3 (Guo and Volkmann [2]) Let \( D \) be a strong \( p \)-partite tournament with partite sets \( V_1, V_2, \ldots, V_p \). For every \( 1 \leq i \leq p \), there exists a vertex \( x \in V_i \) which belongs to a \( k \)-circuit for all \( 3 \leq k \leq p \).

Now, we assume that \( D \) is a strong tournament as otherwise we just consider the terminal strong component. Let \( T \) be a maximum size triangle-tree in \( D \), and assume for the sake of contradiction that \( \left| V(T) \right| < 2d^+(D) + 1 \). Let \( D^{MT} \) be the multipartite tournament obtained from \( D \) by deleting all the arcs with both endpoints in \( V(T) \). Let \( V_1, V_2, \ldots, V_l \) be the partite sets in \( D^{MT} \) such that \( \left| V_i \right| = 1 \) for all \( i \)

Let \( V_1, V_2, \ldots, V_l \) be a partition of \( V(T) \) given by Theorem 2.

If there is a \( Q_i \) with \( Q_i \cap V_1 \neq \emptyset \) and \( Q_i \nsubseteq V_1 \) then we obtain the following contradiction. Since \( Q_i \nsubseteq V_1 \), we observe that \( Q_i \) contains at least two partite set. In addition, note that at least three partite sets intersect \( Q_i \) as \( D^{MT}(Q_i) \) would not be strong if there were only two partite sets since \( \left| V_i \right| = 1 \) for all \( i \). By Theorem 3, in the subgraph of \( D^{MT} \) induced by \( Q_i \), there is a 3-circuit containing exactly one vertex from \( V_1 \). This contradicts the maximality of \( T \). So every set \( Q_i \) is either a subset of \( V_1 \) or is disjoint from \( V_1 \).

Note that \( Q_1 \cap V_1 \neq \emptyset \) and \( Q_k \cap V_1 \neq \emptyset \), as otherwise \( D \) would not be strong. Applying the observation above, we obtain \( Q_1 \cup Q_k \subseteq V_1 \). Let \( D' = D(V_1) \). If there is a vertex \( x \in Q_k \) with \( d^+_D(x) \leq \frac{|V_1|-1}{2} \), then \( d^+_D(x) \leq \frac{|V_1|-1}{2} \), which implies that \( \left| V(T) \right| \geq 2d^+(D) + 1 \), a contradiction. So \( d^+_D(x) \geq \frac{|V_1|+1}{2} \) for all \( x \in Q_k \), as \( \left| V_1 \right| \) is odd.

Let \( G_1 \) denote the bipartite graph with partite sets \( Q_k \) and \( V_1 - Q_k \), and with \( E(G_1) = \{uv \mid u \in Q_k, v \in V_1 - Q_k, uv \in E(D)\} \). Note that the following now holds by the above.

\[
\left| Q_k \right| \left( \frac{|V_1|+1}{2} \right) = \sum_{u \in Q_k} d^+_D(u) = \left( \frac{|Q_k|}{2} \right) \left| E(G_1) \right|
\]

This implies that \( \left| E(G_1) \right| \geq \frac{|Q_k|(|V_1|+1)}{2} + \left| Q_k \right| \), which by Corollary 1 implies that there is a component in \( G_1 \) of size at least \( \frac{|V_1|}{2} + \sqrt{2|Q_k|} \). As the size of the maximum component in \( G_1 \) is an integer it is at least \( \frac{|V_1|}{2} + \sqrt{2} \). Two cases can now occur:

- If \( |Q_{k-1}| > 1 \) or \( Q_{k-2} \nsubseteq V_1 \) (or both). If \( |Q_{k-1}| > 1 \) then first let \( Z = \{z_1, z_2\} \) be any two distinct vertices in \( Q_{k-1} \). Otherwise let \( Z \) be any two distinct vertices in \( Q_{k-1} \cup Q_{k-2} \). By the definition of the \( Q_i \)'s we note that \( Z \cap V_1 = \emptyset \) and there are all arcs from \( (V_1 - Q_k) \) to \( Z \) and from \( Z \) to \( Q_k \). We let \( X = Y \) be the vertices of a component in \( G_1 \) of size at least \( \frac{|V_1|+3}{2} \) and use Lemma 2 to find a triangle \( C \) in \( T \), such that the three disjoint triangle-trees, \( T_1, T_2 \) and \( T_3 \), of \( T - E(C) \) all intersect
X (as $X = Y$). As $X$ are the vertices of a component in $G_1$ there are edges, $u_1v_1$ and $u_2v_2$, from $G_1$ such that the following holds. The edge $u_1v_1$ connects $T_3$ and $T_j$, where $u_2v_2$ connects $T_{3-j}$ and $T_j \cup T_3$. Generality assume that $u_1, u_2 \in Q_k$ and $v_1, v_2 \in V_1 - Q_k$. Now $T - E(C)$ together with the vertices $z_1$ and $z_2$ as well as the 3-circuits $v_1z_1u_1v_1$ and $v_2z_2u_2v_2$ is a triangle-tree in $D$ with more triangles than $T$, a contradiction.

- If $|Q_{k-1}| = 1$ and $Q_{k-2} \subseteq V_1$. Note that $k > 3$, as otherwise $|V(D) \setminus V(T)| = 1$ and we have a contradiction to our assumption. This implies that $k > 4$ as $Q_1 \subseteq V_1$, which implies that $Q_2 \nsubseteq V_1$. Now let $Q_{k-1} = \{z_1\}$ and let $z_2 \in Q_{k-3}$ be arbitrary. Let $G_2$ denote the bipartite graph with partite sets $A = Q_k \cup Q_{k-2}$ and $B = V_1 - A$, and with $E(G_2) = \{uv \mid u \in A, v \in B, uv \in E(D)\}$. Recall that $d_{G_2}^+(x) \geq \frac{|V_1|+1}{2}$ for all $x \in Q_k$. Analogously we get that $d_{G_2}^+(y) \geq \frac{|Q_1|-1}{2}$ for all $y \in Q_{k-2}$ (as $|Q_{k-1}| = 1$). This implies the following.

$$|A| \frac{|V_1|+1}{2} - |Q_{k-2}| \leq \sum_{u \in A} d_{G_2}^+(u) = \left(\frac{|A|}{2}\right) + |E(G_2)|$$

This implies that $|E(G_2)| \geq \frac{|A||V_1|+|A|}{2} + |A| - |Q_{k-2}|$, which by Corollary 1 implies that there is a component in $G_2$ of size at least $\frac{|V_1|}{2} + \sqrt{2|Q_k|}$, as $|A| - |Q_{k-2}| = |Q_k|$. Note that $|Q_k| > 1$, as otherwise the vertex in $Q_{k-1}$ only has out-degree one, a contradiction. Therefore there is a component in $G_2$ of size at least $\frac{|V_1|}{2} + 2$ and so at least $\frac{|V_1|}{2} + 5/2$ as $V_1$ is odd.

Let $X$ be the vertices of a component in $G_1$ of size at least $|V_1|/2 + 3/2$ and let $Y$ be the vertices in a connected component of $G_2$ of size at least $|V_1|/2 + 5/2$. Now use Lemma 2 to find a triangle $C$ in $T$, such that the three disjoint triangle-trees, $T_1, T_2$ and $T_3$, of $T - E(C)$ have the following property. The set $Y$ intersects $T_1$ and $T_2$ and the set $X$ intersects $T_2$ and $T_3$. Due to the definition of $X$ and $Y$ there exists edges, $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, such that the following holds. The edge $u_1v_1$ connects $T_3$ and $T_{2}$, where $j \in \{1, 2\}$ and $u_2v_2$ connects $T_3-j$ and $T_j \cup T_3$. Without loss of generality assume that $u_1, u_2 \in Q_k$ and $v_1, v_2 \in V_1 - Q_k$. Now $T - E(C)$ together with the vertices $z_1$ and $z_2$ as well as the 3-circuits $v_1z_1u_1v_1$ and $v_2z_2u_2v_2$ is a triangle-tree in $D$ with more triangles than $T$, a contradiction. This completes the proof.

\[ \blacksquare \]

References


