A Note on α-Drawable k-Trees

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Abstract

We study the problem of realizing a given graph as an α-complex of a set of points in the plane. We study the realizability problem for trees and 2-trees. In the case of 2-trees, we confine our attention to the realizability of graphs as the α-complex minus faces of dimension two; in other words, realizability of the graph in terms of the 1-skeleton of the α-complex of the point set. We obtain both positive (realizability) and negative (non-realizability) results.

1 Introduction

The problem of characterizing those geometric graphs that satisfy some proximity rule has a long tradition in the computational geometry literature. This tradition is justified in part by the theoretical interest of the associated questions in their own right, and in part by the variety of application areas where proximity graphs are used as descriptors of the shape of a set of points. Extensive surveys about different proximity rules with their applications can be found in [10, 16].

The characterization problem for proximity graphs can naturally be expressed as a graph drawing question. Indeed, for a proximity rule $\mathcal{P}$ and a family of graphs $\mathcal{G}$ we can say that a member $G \in \mathcal{G}$ is *$\mathcal{P}$-drawable* if there exists a set S of distinct points in the plane such that the geometric graph constructed on S by using rule $\mathcal{P}$ is isomorphic to G; we call this geometric graph a *$\mathcal{P}$-drawing* of G. Characterizing $\mathcal{P}$-drawable graphs corresponds to describing the combinatorial properties of the associated $\mathcal{P}$-drawings. Different families of $\mathcal{P}$-drawable graphs have been studied in the literature, including Gabriel drawable graphs, Delaunay drawable triangulations, and sphere of influence drawable graphs (see, *e.g.*, [2, 4, 9, 15]). Those trees that can be drawn as the minimum spanning tree of a set of points in the plane are studied in [5, 11, 14]. The interested reader is referred to [12] for a survey of proximity drawability problems and more references on the topic.

This paper initiates the study of the combinatorial properties of α-complexes of set of points in the plane. α-complexes are a fundamental object in computational topology [8] and have applications in such areas as structural molecular biology [7] and shape analysis [3].

We say that a graph G with n vertices is *α-drawable* if there exists a set S of n distinct points in the plane such that the α-complex of S is a straight-line drawing of G for some value of the parameter α. We call such an α-complex of S an *α-drawing of* G. We present some negative and positive results about those trees and partial 2-trees that are α-drawable. A detailed description of the results in this note follows.

- Regarding trees, we show differences between α-drawable trees and other well-studied families of proximity drawable trees. Namely, we show that that the family of α-drawable trees is a subset of the relative neighborhood drawable trees and a subset of those trees that are drawable as the minimum spanning tree of a set of points. We also prove that there exist α-drawable trees that are not Gabriel-drawable.
- We exhibit a simple partial 2-tree that is not α-drawable. Motivated by the observation that the above counterexample for 2-trees is a series-parallel graph whose planar embeddings all have some interior vertex, we show that all biconnected outerplanar graphs are α-drawable.

Our characterizations of α-drawable graphs are based on constructive proofs that give rise to linear time drawing algorithms, assuming the real RAM model of computation.

2 Preliminaries

Consider a finite set S of points in the plane and a non-negative real number α. For each $p \in S$, let $B_p(\alpha) = \{x : \|x - p\| < \alpha\}$ be a disk centered at p with radius α. Let $B(S, \alpha)$ denote the union of balls $B_p(\alpha)$. We can decompose this union by intersecting each ball

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$B_p(\alpha)$ with the Voronoi cell V_p of p into convex pieces $BV_p(\alpha) = B_p(\alpha) \cap V_p$. Define the *$\alpha$-complex* as the nerve of the decomposition of $B(S, \alpha)$ by $BV_p(\alpha)$ or the set of all simplices $\sigma \subseteq S$ such that $\bigcap_{p\in\sigma} BV_p(\alpha) \neq \emptyset$. For details, see [6, 1]. We use $\alpha(S)$ to refer to the α-complex for the set of points S for this fixed value of the radius α. The 1-skeleton $\alpha_1(S)$ of $\alpha(S)$ is the collection of 1-dimensional faces in $\alpha(S)$.

We shall find the following graph useful: the Gabriel graph of S, $GG(S)$, contains an edge between any pair of points p and q whenever the disk having the line segment pq as its diameter is empty. The edges of the $GG(S)$ are those Delaunay edges that intersect their dual Voronoi edges.

A k-tree is a graph obtained from a k-clique by 0 or more iterations of adding a new vertex joined to exactly k vertices of a k-clique in the old graph. A partial k-tree is a subgraph of a k-tree. Trees are 1-trees.

3 Results on Trees

Lemma 1 *Let (u, v) be an edge of $GG(S)$. If $d(u, v) \geq 2\alpha$, then $(u, v) \notin \alpha(S)$. If $d(u, v) < 2\alpha$, then $(u, v) \in \alpha(S)$.*

Theorem 2 *If $\alpha(S)$ is a tree, it is the Euclidean minimum spanning tree of S.*

Proof. Suppose $\alpha(S)$ were a tree but not the minimum spanning tree, $MST(S)$. Then there would be an edge (u, v) in $\alpha(S)$ not in $MST(S)$, such that $d(u, v) < 2\alpha$. Vertices u and v are connected by a path in $MST(S)$ and adding edge (u, v) to $MST(S)$ creates a cycle. We know that all edges in $MST(S)$ are Gabriel edges (since $MST \subseteq GG$). Suppose that the cycle made by adding (u, v) to $MST(S)$ does not contain an edge of length $\geq 2\alpha$. Then all edges in the path from u to v in $MST(S)$ are Gabriel edges of length $< 2\alpha$. By Lemma 1, these edges are in $\alpha(S)$. But so is (u, v). Thus, $\alpha(S)$ contains a cycle. This is impossible as it is a tree. Therefore, an edge of $MST(S)$ along the path from u to v is of length $\geq 2\alpha$. It may be exchanged with (u, v) to obtain a lighter $MST(S)$. If $\alpha(S)$ is a tree, there cannot be an edge in $\alpha(S)$ that is not in $MST(S)$. □

Corollary 3 *If $\alpha(S)$ is a tree, it contains exactly those Gabriel edges whose length is $< 2\alpha$.*

3.1 Non-realizability results

Lemma 4 *Let T be an α-drawable tree. For two edges (u, v), (w, v) sharing a common vertex v, $\angle uvw > \pi/3$. Moreover, $\angle uvw$ is the largest angle in $\triangle uvw$.*

Proof. Suppose that $\angle uvw \leq \pi/3$. Then (u, w) is not the longest edge of $\triangle uvw$, *i.e.* $d(u, w) \leq d(u, v) \leq 2\alpha$, $d(u, w) \leq d(w, v) \leq 2\alpha$. Since $(u, w) \notin T$ and $d(u, w) < 2\alpha$, by Corollary 3, $(u, w) \notin GG(S)$. Let p be a point inside the diametric disk of u and w. Then $d(u, p) < d(u, v)$ and $d(w, p) < d(w, v)$. One of (u, p), (w, p) is not in T as it is a tree. Either (u, v) may be replaced with the shorter edge (u, p) or (w, v) with (w, p) to make a lighter spanning tree of S than T. This contradicts Theorem 2. We have shown that it is not possible that $\angle uvw \leq \pi/3$.

It is not possible that $\triangle uvw$ has a larger angle than $\angle uvw$ since that would mean that the longest side of $\triangle uvw$ is not (u, w) and that T is not a $MST(S)$. □

Corollary 5 *The maximum vertex degree of an α-drawable tree is at most 5 for any possible value of α.*

This is in contrast with a generic Euclidean minimum spanning tree, which has vertex degree of at most 6 [14].

Lemma 6 *A tree T consisting of two adjacent vertices of degree 5 and additionally only leaf nodes is not α-drawable for any possible value of the parameter α.*

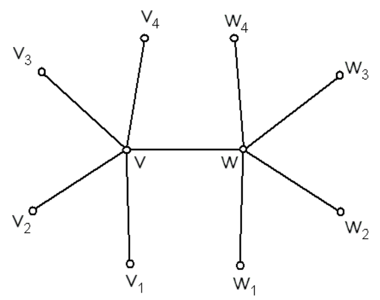

Figure 1: Two adjacent degree 5 vertices v and w.

Sketch of Proof. Suppose $\alpha(S)$ is a realization of T as an α-complex for a given α. Let v and w be adjacent degree 5 vertices in $\alpha(S)$. Let v_i be the leaves adjacent to v and w_i be the leaves adjacent to w. See Fig. 1. Either $\angle v_2vw \geq \angle vww_3$ or $\angle v_2vw < \angle vww_3$. If it is the latter, we rotate the drawing so that w_3 takes the place of v_2 and v_2 the place of w_3. Hereafter, we assume that $\angle v_2vw \geq \angle vwv_3$.

By Lemma 4, $\angle v_2vv_3 > \pi/3$, $\angle v_3vv_4 > \pi/3$ and $\angle w_4ww_3 > \pi/3$. Combining this with the fact that $\angle v_2vw \geq \angle vww_3$, we get $\angle v_4vw + \angle vww_4 < \pi/2$. Assume that $\angle v_4vw < \pi/2$ (an analogous argument can be made in case $\angle vww_4 < \pi$). We know that $\alpha(S)$ must not contain (v_4, w_4). By Corollary 3, if this edge is not present, it is either because (v_4, w_4) is not a Gabriel edge or because $d(v_4, w_4) \geq 2\alpha$.

First, we show that (v_4, w_4) must be a Gabriel edge. Suppose the contrary. Then $\angle v_4ww_4 \geq \pi/2$. Write $\angle v_4vw$ as $\pi/2 - \beta$ and $\angle w_4wv$ as $\pi/2 + \gamma$, where $\gamma < \beta$. Then $\angle vv_4w \geq \pi - \pi/2 + \beta - \gamma > \pi/2$. This implies that

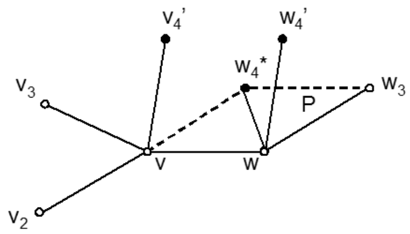

Figure 2: v_4 lies on (v, v_4'), w_4 lies on (w, w_4').

the Gabriel disk of (v, w) is not empty and that (v, w) is not in $\alpha(S)$, a contradiction.

Last, we show that necessarily $d(v_4, w_4) < 2\alpha$. Let L be the length of the longest edge in $\alpha(S)$. Let v_4' be a point such that $\angle v_4' vw = \angle v_4 vw < \pi/2$ and $d(v_4', v) = L$. We then choose point w_4', satisfying $\angle v_4' vw + \angle vww_4' < \pi$ and $d(w, w_4') = L$, such that w_4 lies along (w, w_4'), see Fig. 2. We try to choose v_4 and w_4 as far apart as possible.

One possibility is to place w_4 so that $d(v, w_4) = d(w, w_3)$ and $d(v, w) = d(w_4, w_3)$. In this case w_4 is a vertex of a parallelogram P, w_4^*, see Fig. 2. Since $\angle v_2 vw \geq \angle vww_3, \angle v_2 vv_3 > \pi/3$ and $\angle v_3 vv_4 > \pi/3$, it follows that $\angle v_4 vw_4^* < \pi/3$. Since $d(v, v_4) \leq L$ and $d(v, w_4^*) \leq L$, this implies that $d(v_4, w_4^*) < L$. Thus, this placement for w_4 is not sufficiently far from v_4. Note that neither $\angle w_4^* ww_3 > \pi/2$ nor $\angle vww_4^* > \pi/2$ for that would mean that one of (v, w) or (w, w_3) is not a Gabriel edge and may not be in $\alpha(S)$.

Consider placing w_4 inside P. Note that $\angle vww_3 \leq \angle v_2 vw < \pi$, since $\angle v_2 vv_3 + \angle v_3 vv_4 + \angle v_4 vw > \pi$. Then we have the following 3 cases: (1) Suppose $\angle vww_4 < \pi/2$ and $\angle w_4 ww_3 < \pi/2$. Then (v, w_4) and (w_4, w_3) are Gabriel edges and since they are not in $\alpha(S)$, $d(v, w_4) > L$, $d(w_4, w_3) > L$ by Corollary 3. No such placement of w_4 is possible inside P; (2) Suppose $\angle vww_4 \geq \pi/2$ and $\angle w_4 ww_3 < \pi/2$. Then (w_4, w_3) is a Gabriel edge and since it is not in $\alpha(S)$, $d(w_4, w_3) > L$, while (v, w_4) is not a Gabriel edge. As $\angle vww_4 > \pi/2$, this angle is greater than $\angle vww_4^*$. Increasing $\angle vww_4$ decreases $\angle w_4 ww_3$. Thus, w_4 lies inside $\triangle vw_4^* w$ and $d(v, w_4) \leq L$. As (v, w_4) is a Gabriel edge and $d(v, w_4) \leq L$, $(v, w_4) \in \alpha(S)$, a contradiction; (3) Suppose $\angle vww_4 < \pi/2$ and $\angle w_4 ww_3 \geq \pi/2$. Then (v, w_4) is a Gabriel edge and since it is not in $\alpha(S)$, $d(v, w_4) > L$, while (w_4, w_3) is not a Gabriel edge. This is impossible, by the same argument as in (2). Thus, w_4 may not be placed in P.

Consider moving w_4 outside of P so that $d(w_4, w_3) = d(w_4^*, w_3)$. Recall that (w_4^*, w_3) is a Gabriel edge in P. As $d(w, w_4)$ increases, $\angle w_4 ww_3$ decreases, so (w_4, w_3) remains a Gabriel edge. Thus w_4 may not be placed closer to w_3. As $d(w, w_4)$ increases, $d(v_4, w_4)$ first decreases and then increases. It can be shown (using similar arguments as those presented already) that when $d(w, w_4) = L$, $d(v_4, w_4) \leq L$. Thus, no choice of w_4 such that $d(w_4, w_3) = d(w_4^*, w_3)$ is sufficiently far from v_4, and therefore, the same is true if $d(w_4, w_3) > d(w_4^*, w_3)$.

Therefore, (v_4, w_4) is a Gabriel edge such that $d(v_4, w_4) \leq L < 2\alpha$ and must be in $\alpha(S)$. □

3.2 Realizability results

Lemma 7 *$\alpha(S)$ can be a tree with arbitrarily many adjacent degree four vertices.*

Sketch of Proof. See Fig. 3. □

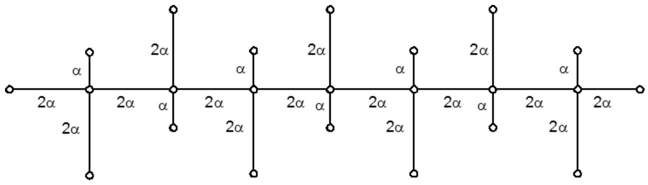

Figure 3: A degree four "caterpillar graph" shows that $\alpha(S)$ can be a tree and have arbitrarily many adjacent degree four vertices.

We conclude this section by comparing α-drawable trees with other well-known families of proximity drawable trees.

Theorem 8 *The family of α-drawable trees is a proper subset of the family of trees that have a minimum spanning tree realization. It is also a proper subset of the relative neighborhood drawable trees. Also, there exist α-drawable trees that are not Gabriel drawable.*

Sketch of Proof. All trees whose maximum vertex degree is at most five are relative neighborhood drawable and also admit a realization as the Euclidean minimum spanning tree of a set of points in the plane [2, 14]. On the other hand, the tree in Fig. 1 containing two adjacent degree five vertices is not α-drawable by Lemma 6. Also, no tree having two adjacent vertices of degree four is Gabriel drawable [2], while, by Lemma 7 it may be α-drawable. □

4 Results on 2-Trees

4.1 Non-realizability results

Lemma 9 *There are 2-trees that are not α-drawable for any possible value of α.*

Sketch of Proof. Consider the partial 2-tree ABU given in Fig. 4.

It can be shown that ABU is not α-drawable for any possible value of α (proof omitted). The only completion of ABU in a 2-tree with 5 vertices is ABU augmented by the edge (a, c). This 2-tree is well-known as the *3-sun*. Therefore, a 3-sun cannot be realized as $\alpha_1(S)$ for any point set S. □

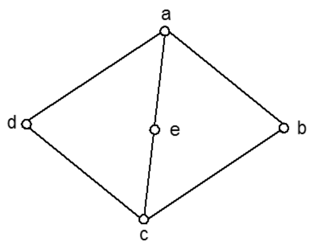

Figure 4: The partial 2-tree ABU.

4.2 Realizability results

Theorem 10 *Every biconnected outerplanar graph is α-drawable for any possible value of α.*

Sketch of Proof. Let G be a biconnected outerplanar graph. Construct a special dual for G by adding a vertex for each bounded face of the graph and a vertex for each edge on the unbounded face. Connect with edges those vertices corresponding to the bounded faces that share an edge and also those vertices corresponding to the bounded faces that have an edge on the unbounded face with the vertex corresponding to that edge. This dual is a tree with no degree 2 vertices. By the results of [13], this tree may be realized as a Voronoi diagram of a set of points S. If we set the scale of this drawing to be sufficiently small, for the given α, the α-balls touch the Voronoi edges for any pair of primal edges of G. □

5 Open Problems

1. Are all binary trees α-drawable? Are all binary trees up to some maximum depth k α-drawable?
2. Is a Gabriel drawable tree always α-drawable?
3. Which partial 2-trees are α-drawable?
4. If $\alpha(S)$ is a tree and we consider any subtree of $\alpha(S)$, is it true that the subtree is $\alpha(S')$ on the restricted set of vertices S'? If true, this would immediately settle the following problem, generalizing Lemma 6:
5. Is any tree containing two adjacent degree 5 vertices α-drawable?

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