# SLIDE: A Useful Special Case of the CARDPATH Constraint

Christian Bessiere, Emmanuel Hébrard, Brahim Hnich, Zeynep Kiziltan, Toby Walsh

# SLIDE: A Useful Special Case of the CARDPATH Constraint

**Christian Bessiere**[1] and **Emmanuel Hebrard**[2] and **Brahim Hnich**[3] and **Zeynep Kiziltan**[4] and **Toby Walsh**[5]

**Abstract.** We study the CARDPATH constraint. This ensures a given constraint holds a number of times down a sequence of variables. We show that SLIDE, a special case of CARDPATH where the slid constraint must hold always, can be used to encode a wide range of sliding sequence constraints including CARDPATH itself. We consider how to propagate SLIDE and provide a complete propagator for CARDPATH. Since propagation is NP-hard in general, we identify special cases where propagation takes polynomial time. Our experiments demonstrate that using SLIDE to encode global constraints can be as efficient and effective as specialised propagators.

## 1 INTRODUCTION

In many scheduling problems, we have a sequence of decision variables and a constraint which applies down the sequence. For example, in the car sequencing problem, we need to decide the sequence of cars on a production line. We might have a constraint on how often a particular option is met (e.g. 1 out of 3 cars can have a sun-roof). As a second example, in a nurse rostering problem, we need to decide the sequence of shifts worked by nurses. We might have a constraint on how many consecutive night shifts any nurse can work. Such constraints have been classified as sliding sequence constraints [7]. To model such constraints, we can use the CARDPATH constraint. This ensures that a given constraint holds a number of times down a sequence of variables [5]. We identify a special case of CARDPATH which we call SLIDE, that is interesting for several reasons. First, many sliding sequence constraints, including CARDPATH, can easily be encoded using this special case. SLIDE is therefore a "general-purpose" constraint for encoding sliding sequencing constraints. This is an especially easy way to provide propagators for such global constraints within a constraint toolkit. Second, we give a propagator for enforcing generalised arc-consistency on SLIDE. By comparison, the previous propagator for CARDPATH given in [5] does not prune all possible values. Third, SLIDE can be as efficient and effective as specialised propagators in solving sequencing problems.

[1] LIRMM (CNRS / U. Montpellier), France, email: bessiere@lirmm.fr. Supported by the ANR project ANR-06-BLAN-0383-02.

[2] 4C, UCC, Ireland, email: ehebrard@4c.ucc.ie.

[3] Izmir Uni. of Economics, Turkey, email: brahim.hnich@ieu.edu.tr. Supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under Grant No. SOBAG-108K027.

[4] CS Department, Uni. of Bologna, Italy, email: zeynep@cs.unibo.it.

[5] NICTA and UNSW, Sydney, Australia, email: toby.walsh@nicta.com.au. Funded by the Australian Government's Department of Broadband, Communications and the Digital Economy, and the ARC.

## 2 CARDPATH AND SLIDE CONSTRAINTS

A constraint satisfaction problem consists of a set of variables, each with a finite domain of values, and a set of constraints specifying allowed combinations of values for given sets of variables. We use capital letters for variables (e.g. $X$), and lower case for values (e.g. $d$). We write $D(X)$ for the domain of variable $X$. Constraint solvers typically explore partial assignments enforcing a local consistency property. A constraint is *generalised arc consistent* (*GAC*) iff when a variable is assigned any value in its domain, there exist compatible values in the domains of all the other variables of the constraint.

The CARDPATH constraint was introduced in [5]. If $C$ is a constraint of arity $k$ then CARDPATH$(N, [X_1, \ldots, X_n], C)$ holds iff $C(X_i, \ldots, X_{i+k-1})$ holds $N$ times for $1 \leq i \leq n-k+1$. For example, we can count the number of changes in the type of shift with CARDPATH$(N, [X_1, \ldots, X_n], \neq)$. Note that CARDPATH can be used to encode a range of Boolean connectives since $N \geq 1$ gives disjunction, $N = 1$ gives exclusive or, and $N = 0$ gives negation. We shall focus on a special case of the CARDPATH constraint where the slid constraint holds always. SLIDE$(C, [X_1, \ldots, X_n])$ holds iff $C(X_i, \ldots, X_{i+k-1})$ holds for all $1 \leq i \leq n-k+1$. That is, a CARDPATH constraint in which $N = n-k+1$. We also consider a more complex form of SLIDE that applies only every $j$ variables. More precisely, SLIDE$_j(C, [X_1, \ldots, X_n])$ holds iff $C(X_{ij+1}, \ldots, X_{ij+k})$ holds for $0 \leq i \leq \frac{n-k}{j}$. By definition SLIDE$_j$ for $j = 1$ is equivalent to SLIDE.

Beldiceanu and Carlsson have shown that CARDPATH can encode a wide range of constraints like CHANGE, SMOOTH, AMONGSEQ and SLIDINGSUM [5]. As we discuss later, SLIDE provides a simple way to encode such sliding sequencing constraints. It can also encode many other more complex sliding sequencing constraints like REGULAR [16], STRETCH [13], and LEX [7], as well as many types of chanelling constraints like ELEMENT [19] and optimisation constraints like the soft forms of REGULAR [20]. More interestingly, CARDPATH can itself be encoded into a SLIDE constraint. In [5], a propagator for CARDPATH is proposed that greedily constructs upper and lower bounds on the number of (un)satisfied constraints by posting and retracting (the negation of) each of the constraints. This propagator does not achieve GAC. We propose here a complete propagator for enforcing GAC on SLIDE. SLIDE thus provides a GAC propagator for CARDPATH. In addition, SLIDE provides a GAC propagator for any of the other global constraints it can encode. As our experimental results reveal, SLIDE can be as efficient and effective as specialised propagators.

We illustrate the usefulness of SLIDE with the AMONGSEQ constraint which ensures that values occur with some given frequency. For instance, we might want that no more than 3 out of every sequence of 7 shift variables are a "night shift". More precisely, AMONGSEQ$(l, u, k, [X_1, \ldots, X_n], v)$ holds iff between $l$ and $u$ variables in every sequence of $k$ variables take value in the ground set $v$ [8]. We can encode this using SLIDE. More precisely, AMONGSEQ$(l, u, k, [X_1, \ldots, X_n], v)$ can be encoded as SLIDE$(D^{k,v}_{l,u}, [X_1, \ldots, X_n])$ where $D^{k,v}_{l,u}$ is an instance of the AMONG constraint [8]. $D^{k,v}_{l,u}(X_i, \ldots, X_{i+k-1})$ holds iff between $l$ and $u$ variables take values in the set $v$. For example, suppose 2 of every 3 variables along a sequence $X_1 \ldots X_5$ should take the value $a$, where $X_1 = a$ and $X_2, \ldots, X_5 \in \{a, b\}$. This can be encoded as SLIDE$(E, [X_1, X_2, X_3, X_4, X_5])$ where $E(X_i, X_{i+1}, X_{i+2})$ ensures two of its three variables take $a$. This SLIDE constraint ensures that $E(X_1, X_2, X_3)$, $E(X_2, X_3, X_4)$ and $E(X_3, X_4, X_5)$ all hold. Note that each ternary constraint is GAC. However, enforcing GAC on the SLIDE constraint sets $X_4 = a$ as there are only two satisfying assignments and neither have $X_4 = b$.

## 3 SLIDE WITH MULTIPLE SEQUENCES

We often wish to slide a constraint down two or more sequences of variables at once. For example, suppose we want to ensure that two vectors of variables, $X_1$ to $X_n$ and $Y_1$ to $Y_n$ differ at *every* index. We can encode such a constraint by interleaving the two sequences and sliding a constraint down the single sequence with a suitable offset. In our example, we simply post SLIDE$_2(\neq, [X_1, Y_1, \ldots, X_n, Y_n])$. As a second example of sliding down multiple sequences of variables, consider the constraint REGULAR$(\mathcal{A}, [X_1, \ldots, X_n])$. This ensures that the values taken by a sequence of variables form a string accepted by a deterministic finite automaton $\mathcal{A}$ [16]. This global constraint is useful in scheduling, rostering and sequencing problems to ensure certain patterns do (or do not) occur over time. It can be used to encode a wide range of other global constraints including: AMONG [8], CONTIGUITY [15], LEX and PRECEDENCE [14].

To encode the REGULAR constraint with SLIDE, we introduce variables, $Q_i$ to record the state of the automaton. We then post SLIDE$_2(F, [Q_0, X_1, Q_1, \ldots, X_n, Q_n])$ where $Q_0$ is set to the starting state, $Q_n$ is restricted to accepting states, and $F(Q_i, X_{i+1}, Q_{i+1})$ holds iff $Q_{i+1} = \delta(X_i, Q_i)$ where $\delta$ is the transition function of the automaton. If we decompose this encoding into the conjunction of slid constraints, we get a set of constraints similar to [6]. Enforcing GAC on this encoding ensures GAC on REGULAR and, by exploiting functionaliy of $F$, takes $O(ndq)$ time where $d$ is the number of values for $X_i$ and $q$ is the number of states of the automaton. This is asymptotically identical to the specialised REGULAR propagator [16]. This encoding is highly competitive in practice with the specialized propagator [2].

One advantage of this encoding is that it gives explicit access to the states of the automaton. Consider, for example, a rostering problem where workers are allowed to work for up to three consecutive shifts. This can be specified with a simple REGULAR constraint. Suppose now we want to minimise the number of times a worker has to work for three consecutive shifts. To encode this, we can post an AMONG constraint on the state variables to count the number of times we visit the state representing three consecutive shifts, and minimise the value taken by this variable. As we shall see later in the experiments, the encoding also gives an efficient *incremental* propagator. In fact, the complexity of repeatedly enforcing GAC on this encoding of the REGULAR constraint down the whole branch of a backtracking search tree is just $O(ndq)$ time.

## 4 SLIDE WITH COUNTERS

We may want to slide a constraint on a sequence of variables computing a count. We can use SLIDE to encode such constraints by incrementally computing the count in an additional sequence of variables. Consider, for example, CARDPATH$(N, [X_1, \ldots, X_n], C)$. For simplicity, we consider $k = 2$ (i.e., $C$ is binary). The generalisation to other $k$ is straightforward. We introduce a sequence of integer variables $M_i$ in which to accumulate the count. We encode CARDPATH as SLIDE$_2(G, [M_1, X_1, \ldots, M_n, X_n])$ where $M_1 = 0$, $M_n = N$, and $G(M_i, X_i, M_{i+1}, X_{i+1})$ is defined as: if $C(X_i, X_{i+1})$ holds then $M_{i+1} = M_i + 1$, otherwise $M_{i+1} = M_i$. GAC on SLIDE ensures GAC on CARDPATH.

As a second example, consider the STRETCH constraint [13]. Given variables $X_1$ to $X_n$ taking values from a set of shift types $\tau$, a set $\pi$ of ordered pairs from $\tau \times \tau$, and functions $shortest(t)$ and $longest(t)$ giving the minimum and maximum length of a stretch of type $t$, STRETCH$([X_1, \ldots, X_n])$ holds iff each stretch of type $t$ has length between $shortest(t)$ and $longest(t)$; and consecutive types of stretches are in $\pi$. We can encode STRETCH as SLIDE$_2(H, [X_1, Q_1, \ldots, X_n, Q_n])$ where $Q_1 = 1$ and $H(X_i, X_{i+1}, Q_i, Q_{i+1})$ holds iff (1) $X_i = X_{i+1}$, $Q_{i+1} = 1 + Q_i$, and $Q_{i+1} \leq longest(X_i)$; or (2) $X_i \neq X_{i+1}$, $\langle X_i, X_{i+1} \rangle \in \pi$, $Q_i \geq shortest(X_i)$ and $Q_{i+1} = 1$. GAC on SLIDE ensures GAC on STRETCH.

## 5 OTHER EXAMPLES OF SLIDE

There are many other examples of global constraints which we can encode using SLIDE. For example, we can encode LEX [7] using SLIDE. LEX holds iff a vector of variables $[X_1..X_n]$ is lexicographically smaller than another vector of variables $[Y_1..Y_n]$. We introduce a sequence of Boolean variables $B_i$ to indicate if the vectors have been ordered by position $i - 1$. Hence $B_1 = 0$. We then encode LEX as SLIDE$_3(I, [B_1, X_1, Y_1, \ldots, B_n, X_n, Y_n])$ where $I(B_i, X_i, Y_i, B_{i+1})$ holds iff $(B_i = B_{i+1} = 0 \wedge X_i = Y_i)$ or $(B_i = 0 \wedge B_{i+1} = 1 \wedge X_i < Y_i)$ or $(B_i = B_{i+1} = 1)$. This gives us a linear time propagator as efficient and incremental as the specialised algorithm in [12]. As a second example, we can encode many types of channelling constraints using SLIDE like DOMAIN [17], LINKSET2BOOLEANS [7] and ELEMENT [19]. As a final example, we can encode "optimisation" constraints like the soft form of the REGULAR constraint which measures the Hamming or edit distance to a regular string [20]. There are, however, constraints that can be encoded using SLIDE which do not give as efficient and effective propagators as specialised algorithms (e.g. the global ALLDIFFERENT constraint [18]).

## 6 PROPAGATING SLIDE

A constraint like SLIDE is only really useful if we can propagate it efficiently and effectively. The simplest possible way to propagate SLIDE$_j(C, [X_1, \ldots, X_n])$ is to decompose it into a sequence of constraints, $C(X_{ij+1}, \ldots, X_{ij+k})$ for $0 \leq i \leq \frac{n-k}{j}$ and let the constraint solver propagate the decomposition. Surprisingly, this is enough to achieve GAC in many cases. For example, we can achieve GAC in this way on the SLIDE encoding of the REGULAR constraint. If the constraints in the decomposition overlap on just one variable then the constraint graph is Berge acyclic [4], and enforcing GAC on the decomposition of SLIDE$_j$ achieves GAC on SLIDE$_j$. Similarly, enforcing GAC on the decomposition achieves GAC on SLIDE$_j$ if the constraint being slid is monotone. A constraint $C$ is monotone iff there exists a total ordering $\prec$ of the values such that for any two values $v, w$, if $v \prec w$ then $v$ can replace $w$ in any support for $C$. For instance, the constraints AMONG and SUM are monotone if either no upper bound, or no lower bound is given.

**Theorem 1** *Enforcing GAC over each constraint in the decomposition of* $\text{SLIDE}_j$ *achieves GAC on* $\text{SLIDE}_j$ *if the constraint being slid is monotone.*

**Proof:** For an arbitrary value $v \in D(X)$, we show that if every constraint is GAC, then we can build a support for $X = v$ on $\text{SLIDE}_j$. For any variable other than $X$, we choose the smallest value in the total order. This is the value that can be substituted for any other value in the same domain. A tuple built this way satisfies all the constraints being slid since we know that there exists a support for each (they are GAC), and the values we chose can be substituted for this support. □

In the general case, when constraints overlap on more than one variable (e.g. in the SLIDE encoding of AMONGSEQ), we need to do more work to achieve GAC. We distinguish two cases: when the arity of the constraint being slid is not fixed, and when the arity is fixed. We show that enforcing GAC in the former case is NP-hard.

**Theorem 2** *Enforcing GAC on* $\text{SLIDE}(C, [X_1, \ldots, X_n])$ *is NP-hard when the arity of* $C$ *is not fixed even if enforcing GAC on* $C$ *is itself polynomial.*

**Proof:** We give a reduction from 3-SAT in $N$ variables and $M$ clauses. We introduce variables $X_i^j$ for $1 \leq i \leq N+1$ and $1 \leq j \leq M$. For each clause $j$, if the clause is $x_a \vee \neg x_b \vee x_c$, then we set $X_1^j \in \{x_a, \neg x_b, x_c\}$ to represent the values that make this clause true. For each clause $j$, we set $X_{i+1}^j \in \{0, 1\}$ for $1 \leq i \leq N$ to represent a truth assignment. Hence, we duplicate the truth assignment for each clause. We now build the following constraint $\text{SLIDE}(C, [X_1^1, .., X_{N+1}^1, .., X_1^j, .., X_{N+1}^j, .., X_1^M, .., X_{N+1}^M])$ where $C$ has arity $N+1$. We construct $C(Y_1, \ldots, Y_{N+1})$ to hold iff $Y_1 = x_d$ and $Y_{1+d} = 1$, or $Y_1 = \neg x_d$ and $Y_{1+d} = 0$. (in these two cases, the value assigned to $Y_1$ represents the literal that makes clause $j$ true), or $Y_i \in \{0, 1\}$ and $Y_i = Y_{i+N+1}$ (in this case, the truth assignment is passed down the sequence). Enforcing GAC on $C$ is polynomial and an assignment satisfying the SLIDE constraint corresponds to a satisfying assignment for the original 3-SAT problem. □

When the arity of the constraint being slid is not great, we can enforce GAC on SLIDE using dynamic programming (DP) in a similar way to the DP-based propagators for the REGULAR and STRETCH constraints [16, 13]. A much simpler method, however, which is just as efficient and effective as dynamic programming is to exploit a variation of the dual encoding into binary constraints [10] based on tuples of support. Such an encoding was proposed in [1] for a particular sliding constraint. Here we show that this method is more general and can be used for arbitrary SLIDE constraints. Using such an encoding, SLIDE can be easily added to any constraint solver. We illustrate the intersection encoding by means of an example.

Consider again the AMONGSEQ example in which 2 of every 3 variables of $X_1 \ldots X_5$ should take the value $a$, where $X_1 = a$ and $X_2, \ldots, X_5 \in \{a, b\}$. We can encode this as $\text{SLIDE}(E, [X_1, X_2, X_3, X_4, X_5])$ where $E(X_i, X_{i+1}, X_{i+2})$ is an instance of the AMONG constraint that ensures two of its three variables take $a$. If the sliding constraint has arity $k$, we introduce an *intersection* variable for each subsequence of $k-1$ variables of SLIDE. The first intersection variable $V_1$ has a domain containing all tuples from $D(X_1) \times \ldots \times D(X_{k-1})$. The $j$th intersection variable $V_j$ has domain containing $D(X_j) \times \ldots \times D(X_{j+k-2})$. And so on until $V_{n-k+2}$. In our example in Fig 1, this gives $D(V_1) = D(X_1) \times D(X_2), \ldots, D(V_4) = D(X_4) \times D(X_5)$. We then post binary compatibility constraints between consecutive intersection variables. These constraints ensure that the two intersection variables assign $(k-1)$-tuples that agree on the values of their $k-2$ common original variables (like constraints in the dual encoding). They also ensure that the $k$-tuple formed by the two $(k-1)$-tuples satisfies the corresponding instance of the slid constraint. For instance, in Fig 1, the binary constraint between $V_1$ and $V_2$ does not allow the pair $\langle ab, aa \rangle$ because the second argument of $ab$ for $V_1$ (value $b$ for $X_2$) is in conflict with the first argument of $aa$ for $V_2$ (value $a$ for $X_2$). That same constraint between $V_1$ and $V_2$ does not allow the pair $\langle ab, bb \rangle$ because the tuple $abb$ is not allowed by $E(X_1, X_2, X_3)$.

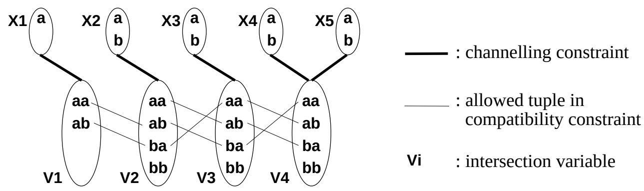


**Figure 1.** Intersection encoding

Enforcing AC on such compatibility constraints prunes $aa$ and $bb$ from $V_2$, $ab$ and $bb$ from $V_3$, and $ba$ and $bb$ from $V_4$. Finally, we post binary channelling constraints to link the tuples to the original variables. One such constraint for each original variable is sufficient. For example, we can have a channelling constraint between $V_4$ and $X_4$ which ensures that the first argument of the tuple assigned to $V_4$ equals the value assigned to $X_4$. Enforcing AC on this channelling constraint prunes $b$ from the domain of $X_4$. We could instead post a channelling constraint between $V_3$ and $X_4$ ensuring that the second argument in $V_3$ equals $X_4$. The AMONGSEQ constraint is now GAC.

**Theorem 3** *Enforcing AC on the intersection encoding of* SLIDE *achieves GAC in* $O(nd^k)$ *time and* $O(nd^{k-1})$ *space where* $k$ *is the arity of the constraint to slide and* $d$ *is the maximum domain size.*

**Proof:** The constraint graph associated with the intersection encoding is a tree. Enforcing AC on this therefore achieves GAC. Enforcing AC on the channelling constraints then ensures that the domains of the original variables are pruned appropriately. As we introduce $O(n)$ intersection variables, and each can contain $O(d^{k-1})$ tuples, the intersection encoding requires $O(nd^{k-1})$ space. Enforcing AC on a compatibility constraint between two intersection variables $V_i$ and $V_{i+1}$ takes $O(d^k)$ time as each tuple in the intersection variable $V_i$ has at most $d$ supports which are the tuples of $V_{i+1}$ that are equal to $V_i$ on their $k-2$ common arguments. Enforcing AC on $O(n)$ such constraints therefore takes $O(nd^k)$ time. Finally, enforcing AC on each of the $O(n)$ channelling constraints takes $O(d^{k-1})$ time as they are functional. Hence, the total time complexity is $O(nd^k)$. □

Arc consistency on the intersection encoding simulates pairwise consistency on the decomposition. It does this efficiently as intersection variables represent in extension 'only' the intersections. This is sufficient because the constraint graph is acyclic. This encoding is also very easy to implement in any constraint solver. It has good incremental properties. Only those constraints associated with a variable which changes need to wake up.

The intersection encoding of $\text{SLIDE}_j$ for $j > 1$ is less expensive to build than for $j = 1$ as we need intersection variables for subsequences of less than $k-1$ variables. For $1 \leq j \leq k/2$, we introduce intersection variables for subsequences of variables of length $k - j$ starting at indices $1, j+1, 2j+1...$ whose domains contain $(k-j)$-tuples of assignments. Compatibility and channelling constraints are defined as with $j = 1$. If $j > k/2$, two consecutive intersection variables (for two subsequences of $k - j$ variables) involve less than $k$ variables of the $\text{SLIDE}_j$. The compatibility constraint between them cannot thus ensure the satisfaction of the slid constraint. We therefore introduce intersection variables for subsequences of length $\lceil k/2 \rceil$ starting at indices $1, j+1, 2j+1...$ and for subsequences of length $\lceil k/2 \rceil$ finishing at indices $k, j+k, 2j+k...$ The compatibility constraint between two consecutive intersection variables representing the subsequence starting at index $pj + 1$ and the subsequence finishing at index $pj + k$ ensures satisfaction of the $(p+1)$th instance of the slid constraint. The compatibility constraint between two consecutive intersection variables representing subsequence finishing at index $pj + k$ and the subsequence starting at index $(p+1)j + 1$ ensures the consistency of the arguments in the intersection of two instances of the slid constraint.

## 7 EXPERIMENTS

We now demonstrate the practical value of SLIDE. Due to space limits, we only report detailed results on a nurse scheduling problem, and summarise the results on balanced incomplete block design generation and car sequencing problems. Experiments are performed with ILOG Solver 6.2 on a 2.8GHz Intel computer running Linux.

We consider a Nurse Scheduling Problem [9] in which we generate a schedule of shift duties for a short-term planning period. There are three types of shifts (day, evening, and night). We ensure that (1) each nurse takes a day off or is assigned to an available shift; (2) each shift has a minimum required number of nurses; (3) each nurse's work load is between specific lower and upper bounds; (4) each nurse works at most 5 consecutive days; (5) each nurse has at least 12 hours of break between two shifts; (6) the shift assigned to a nurse does not change more than once every three days. We construct four different models, all with variables indicating what type of shift, if any, each nurse is working on each day. We break symmetry between the nurses with lex concstraints. The constraints (1)-(3) are enforced using global cardinality constraints. Constraints (4), (5) and (6) form sequences of respectively 6-ary, binary and ternary constraints. Since (4) is monotone, we simply post the decomposition in the first three models. This achieves GAC by Theorem 1. The models differ in how (5) and (6) are propagated. In `decomp`, they are decomposed into conjunction of slid constraints. In `amongseq`, (5) is decomposed and (6) is enforced using the AMONGSEQ constraint of ILOG Solver (called `IloSequence`). The combination of (5) and (6) are enforced by SLIDE in `slide`. Finally, in $\texttt{slide}_c$, we use SLIDE for the combination of (4), (5), and (6).

We test the models using the instances available at *http://www.projectmanagement.ugent.be/nsp.php* in which nurses have no maximum workload, but a set of preferences to optimise. We ignore these preferences and post a constraint bounding the maximum workload to at most 5 day shifts, 4 evening shifts and 2 night shifts per nurse and per week. Similarly, each nurse must have at least 2 rest days per week. We solve three samples of instances involving 25, 30 and 60 nurses to schedule over 28 days.

We use the same variable ordering for all models so that heuristic choices do not affect results. We schedule the days in chronological order and within each day we allocate a shift to every nurse in lexicographical order. Initial experiments show that this is more efficient than the minimum domain heuristic. However, it restricts the variety of domains passed to the propagators, and thus hinders any demonstration of differences in pruning. We therefore also use a more random heuristic. We allocate within each day a shift to every nurse randomly with 20% frequency and lexicographically otherwise.

| | #solved | bts[1] | time[1] | bts[2] | time[2] |
|---|---|---|---|---|---|
| 25 nurses, 28 days (99 instances) | | | | | |
| decomp | 99 | 301 | 0.13 | 301 | 0.13 |
| amongseq | 99 | 301 | 0.19 | 301 | 0.19 |
| slide | 99 | 301 | 0.19 | 301 | 0.19 |
| $\texttt{slide}_c$ | 99 | 295 | 0.68 | 295 | 0.68 |
| 30 nurses, 28 days (99 instances) | | | | | |
| decomp | 68 | 7101 | 2.80 | 15185 | 5.29 |
| amongseq | 67 | 7101 | 4.31 | 7150 | 4.33 |
| slide | 70 | 3303 | 1.99 | 4319 | 2.53 |
| $\texttt{slide}_c$ | 75 | 1047 | 2.13 | 11014 | 10.02 |
| 60 nurses, 28 days (100 instances) | | | | | |
| decomp | 51 | 5999 | 4.38 | 5999 | 4.38 |
| amongseq | 51 | 5999 | 7.10 | 5999 | 7.10 |
| slide | 52 | 5300 | 5.61 | 8479 | 7.21 |
| $\texttt{slide}_c$ | 58 | 2157 | 7.52 | 4501 | 12.07 |

**Table 1.** Nurse scheduling with lexicographical variable ordering ([1] on instances solved by all methods, [2] on instances solved by the method).

| | #solved | bts[1] | time[1] | bts[2] | time[2] |
|---|---|---|---|---|---|
| 25 nurses, 28 days (99 instances) | | | | | |
| decomp | 86 | 35084 | 7.69 | 41892 | 10.06 |
| amongseq | 85 | 35401 | 14.43 | 35401 | 14.43 |
| slide | 97 | 1699 | 1.00 | 1547 | 0.92 |
| $\texttt{slide}_c$ | 97 | 457 | 0.58 | 438 | 0.56 |
| 30 nurses, 28 days (99 instances) | | | | | |
| decomp | 20 | 68834 | 11.94 | 69550 | 12.75 |
| amongseq | 20 | 68834 | 18.89 | 69550 | 19.83 |
| slide | 42 | 378 | 0.18 | 8770 | 7.29 |
| $\texttt{slide}_c$ | 43 | 365 | 0.95 | 12857 | 6.76 |
| 60 nurses, 28 days (100 instances) | | | | | |
| decomp | 3 | 122406 | 71.06 | 250427 | 142.90 |
| amongseq | 2 | 122406 | 119.40 | 122406 | 119.40 |
| slide | 27 | 562 | 0.65 | 2367 | 2.19 |
| $\texttt{slide}_c$ | 34 | 542 | 3.96 | 1368 | 6.38 |

**Table 2.** Nurse scheduling with random variable ordering ([1] on instances solved by all methods, [2] on instances solved by the method).

Tables 1 and 2 report the mean runtime and fails to solve the instances with 5 minutes cutoff. Between the first three models, the best results are due to `slide`. We solve more instances with `slide`, as well as explore a smaller tree. By developing a propagator for a generic constraint like SLIDE, we can increase pruning without hurting efficiency. Note that `slide` always performs better than `amongseq`. A possible reason is that AMONGSEQ cannot encode constraint (6) as directly as SLIDE. As in previous models, we need to channel into Boolean variables and post AMONGSEQ on them. This may not give as effective and efficient pruning. SLIDE thus offers both modelling and solving advantages over existing sequencing constraints. Note also that $\texttt{slide}_c$ solves additional instances in the time limit. This is not suprising as the model slides the combination of the constraints (4), (5), and (6). Recall that the sliding constraint of (4) is 6-ary. It is pleasing to note that the intersection encoding performs well even in the presence of such a high arity constraint.

We also ran experiments on Balanced Incomplete Block Designs (BIBDs) and car sequencing. For BIBD, we use the model in [12] which contains LEX constraints. We propagate these either using the specialised algorithm of [12] or the SLIDE encoding. As both propagators maintain GAC, we only compare runtimes. Results on large instances show that the SLIDE model is as efficient as the LEX model. For car sequencing, we test the scalability of SLIDE on large arity constraints and large domains using 80 instances from CSPLib. Unlike a model using `IloSequence`, our SLIDE model does not combine reasoning about overall cardinality of a configuration with the sequence of AMONG constraints. Hence, it is not as efficient: 26 instances were solved with SLIDE within the five minute cutoff, compared to 39 with `IloSequence`. However, 9 of the instances solved with SLIDE were not solved by `IloSequence`. The memory overhead of the SLIDE propagator was not excessive despite the slid constraints having arity 5 and domains of size 30. The SLIDE model used on average 22Mb of space, compared to 5Mb for `IloSequence`.

## 8 RELATED WORK

Pesant introduced the REGULAR constraint, and gave a propagator based on dynamic programming to enforce GAC [16]. As we saw, the REGULAR constraint can be encoded using a simple SLIDE constraint. In this simple case, the dynamic programming machinery of Pesant's propagator is unnecessary as the decomposition into ternary constraints does not hinder propagation. We have found that SLIDE is as efficient as REGULAR in practice [2]. Furthermore, our encoding introduces variables for representing the states. Access to the state variables may be useful (e.g. for expressing objective functions). Although an objective function can be represented with the COSTREGULAR constraint [11], this is limited to the sum of the variable-value assignment costs. Our encoding is more flexible, allowing different objective functions like the min function used in the example in Section 3.

Beldiceanu, Carlsson, Debruyne and Petit have proposed specifying global constraints by means of deterministic finite automata augmented with counters [6]. They automatically construct propagators for such automata by decomposing the specification into a sequence of signature and transition constraints. This gives an encoding similar to our SLIDE encoding of the REGULAR constraint. There are, however, a number of advantages of SLIDE over using an automaton. If the automaton uses counters, pairwise consistency is needed to guarantee GAC (and most constraint toolkits do not support pairwise consistency). We can encode such automata using a SLIDE where we introduce an additional sequence of variables for each counter. SLIDE thus provides a GAC propagator for such automata. Moreover, SLIDE has a better complexity than a brute-force pairwise consistency algorithm based on the dual encoding as it considers only the intersection variables, reducing the space complexity by a factor of $d$.

Hellsten, Pesant and van Beek developed a GAC propagator for the STRETCH constraint based on dynamic programming similar to that for the REGULAR constraint [13]. As we have shown, we can encode the STRETCH constraint and maintain GAC using SLIDE. Several propagators for the AMONGSEQ are proposed and compared in [21, 3]. Among these propagators, those based on the REGULAR constraint do the most pruning and are often fastest. Finally, Bartak has proposed a similar intersection encoding for propagating a sliding scheduling constraint [1] We have shown that this method is more general and can be used for arbitrary SLIDE constraints.

## 9 CONCLUSIONS

We have studied the CARDPATH constraint. This slides a constraint down a sequence of variables. We considered SLIDE a special case of CARDPATH in which the slid constraint holds at every position. We demonstrated that this special case can encode many global sequencing constraints including AMONGSEQ, CARDPATH, REGULAR in a simple way. SLIDE can therefore serve as a "general-purpose" constraint for decomposing a wide range of global constraints, facilitating their integration into constraint toolkits. We proved that enforcing GAC on SLIDE is NP-hard in general. Nevertheless, we identified several useful and common cases where it is polynomial. For instance, when the constraint being slid overlaps on just one variable or is monotone, decomposition does not hinder propagation. Dynamic programming or a variation of the dual encoding can be used to propagate SLIDE when the constraint being slid overlaps on more than one variable and is not monotone. Unlike the previous proposed propagator for CARDPATH, this achieves GAC. Our experiments demonstrated that using SLIDE to encode constraints can be as efficient and effective as specialised propagators. There are many directions for future work. One promising direction is to use binary decision diagrams to store the supports for the constraints being slid when they have many satisfying tuples. We believe this could improve the efficiency of our propagator in many cases.

## REFERENCES


[1] R. Bartak, 'Modelling resource transitions in constraint-based scheduling', in *Proc. of SOFSEM 2002: Theory and Practice of Informatics*. (2002).

[2] C. Bessiere, E. Hebrard, B. Hnich, Z. Kiziltan, C.-G. Quimper and T. Walsh, 'Reformulating global constraints: the SLIDE and REGULAR constraints', in *Proc. of SARA'07*. (2007).

[3] S. Brand, N. Narodytska, C.-G. Quimper, P. Stuckey and T. Walsh, 'Encodings of the SEQUENCE Constraint', in *Proc. of CP'07*. (2007).

[4] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis, 'On the desirability of acyclic database schemes', *Journal of the ACM*, **30**, 479–513, (1983).

[5] N. Beldiceanu and M. Carlsson, 'Revisiting the cardinality operator and introducing cardinality-path constraint family', in *Proc. of ICLP'01*. (2001).

[6] N. Beldiceanu, M. Carlsson, R. Debruyne, and T. Petit, 'Reformulation of global constraints based on constraints checkers', *Constraints*, **10**(4), 339–362, (2005).

[7] N. Beldiceanu, M. Carlsson, and J-X. Rampon, 'Global constraints catalog', Technical report, SICS, (2005).

[8] N. Beldiceanu and E. Contejean, 'Introducing global constraints in CHIP', *Mathl. Comput. Modelling*, **20**(12), 97–123, (1994).

[9] E.K. Burke, P.D. Causmaecker, G.V. Berghe and H.V. Landeghem, 'The state of the art of nurse rostering', *Mathl. Journal of Scheduling*, **7**(6), 441–499, (2004).

[10] R. Dechter and J. Pearl, 'Tree clustering for constraint networks', *Artificial Intelligence*, **38**, 353–366, (1989).

[11] S. Demassey, G. Pesant, and L.-M. Rousseau, 'A cost-regular based hybrid column generation approach', *Constraints*, **11**(4), 315–333, (2006).

[12] A. Frisch, B. Hnich, Z. Kiziltan, I. Miguel, and T. Walsh, 'Global constraints for lexicographic orderings', in *Proc. of CP'02*. (2002).

[13] L. Hellsten, G. Pesant, and P. van Beek, 'A domain consistency algorithm for the stretch constraint', in *Proc. of CP'04*. (2004).

[14] Y.C. Law and J.H.M. Lee, 'Global constraints for integer and set value precedence', in *Proc. of CP'04*. (2004).

[15] M. Maher, 'Analysis of a global contiguity constraint', in *Proc. of the CP'02 Workshop on Rule Based Constraint Reasoning and Programming*, (2002).

[16] G. Pesant, 'A regular language membership constraint for finite sequences of variables', in *Proc. of CP'04*. (2004).

[17] P. Refalo, 'Linear formulation of constraint programming models and hybrid solvers', in *Proc. of CP'00*. (2000).

[18] J-C. Régin, 'A filtering algorithm for constraints of difference in CSPs', in *Proc. of AAAI'94*. (1994).

[19] P. Van Hentenryck and J.-P. Carillon, 'Generality versus specificity: An experience with AI and OR techniques', in *Proc. of AAAI'88*. (1988).

[20] W-J. van Hoeve, G. Pesant, and L-M. Rousseau, 'On global warming : Flow-based soft global constaints', *Journal of Heuristics*, **12**(4-5), 347–373, (2006).

[21] W-J. van Hoeve, G. Pesant, L-M. Rousseau, and A. Sabharwal, 'Revisiting the sequence constraint' in *Proc. of CP'06*. (2006).