Guiding Search in QCSP+ with Back-Propagation

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Abstract. The Quantified Constraint Satisfaction Problem (QCSP) has been introduced to express situations in which we are not able to control the value of some of the variables (the universal ones). Despite the expressiveness of QCSP, many problems, such as two-players games or motion planning of robots, remain difficult to express. Two more modeler-friendly frameworks have been proposed to handle this difficulty, the Strategic CSP and the QCSP$^+$. We define what we name back-propagation on QCSP$^+$. We show how back-propagation can be used to define a goal-driven value ordering heuristic and we present experimental results on board games.

1 Introduction

The Constraint Satisfaction Problem (CSP) consists in finding values for variables such that a set of constraints involving these variables is satisfied. It is a decision problem, in which all variables are existentially quantified (*i.e.,* Is there a value for each variable such that all constraints are satisfied?). This framework is useful to express and solve many real applications.

Problems in which there is a part of uncertainty are hard to model in the CSP formalism, and/or require an exponential number of variables. The uncertainty part may come from weather or any external event, which is out of our control. The Quantified Constraint Satisfaction Problem (QCSP [4]) is a generalisation of the CSP in which variables can be either existentially (as in CSP) or universally quantified. We control the existential variables (we choose their value), but we have no control on universal variables (they can take any value in their domain). Solving such problems is finding values for existential variables according to the values taken by the preceding universal variables in the sequence of variables in order to respect constraints.

The structure of QCSP is such that the domains of universal variables do not depend on values of previous variables. But in many problems, values taken by variables depend on what has been done before. For example, if we want to express a board game like chess, some moves are forbidden, like stacking two pieces on the same cell.

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In [2, 3] this issue has been identified and new formalisms, SCSP and QCSP$^+$, have been proposed to have symmetrical quantifier behaviors. Both in SCSP and QCSP$^+$, the meaning of the universal quantifier has been modified, and the domains of universal variables depend on the values of previous variables. In the field of Quantified Boolean Formulas, this problem has also been identified. Ansótegui *et al.* introduced new QBF formulations and solving strategies for adversarial scenarios [1].

In this paper, we propose a value ordering heuristic for the QCSP$^+$. After preliminary definitions (Section 2) and clues for solving QCSP$^+$ (Section 3), we analyse constraint propagation in QCSP$^+$ in Section 4. Based on this analysis, we derive a value ordering heuristic for QCSP$^+$ in Section 5. Finally, in Section 6, we present experimental results on board-games.

2 Definitions

In this section we focus on the two frameworks that have been proposed to tackle the QCSP modeling issue, Strategic CSPs [3] and QCSP$^+$ [2]. But first of all, we give some background on CSP and QCSP.

2.1 CSP and QCSP

The constraint satisfaction problem. A *constraint network* $N = (X, D, C)$ consists of a finite set of variables $X = \{x_1, \dots, x_n\}$, a set of domains $D = \{D(x_1), \dots, D(x_n)\}$, where the domain $D(x_i)$ is the finite set of values that variable x_i can take, and a set of constraints $C = \{c_1, \dots, c_e\}$. Each constraint c_k is defined by the ordered set $var(c_k)$ of the variables it involves, and by the set $sol(c_k)$ of combinations of values on $var(c_k)$ satisfying it. A *solution* to a constraint network is an assignment of a value from its domain to each variable such that every constraint in the network is satisfied. A value v_i for a variable x_i is *consistent with* a constraint c_j involving x_i iff there exists an assignment I of all the variables in $var(c_j)$ with values from their domain such that x_i is assigned v_i and I satisfies c_j.

Given a constraint network $N = (X, D, C)$, the constraint satisfaction problem (CSP) is the problem of deciding whether there exists an assignment in D for the variables in X such that all constraints in C are satisfied. In a logical formulation, we write, "$\exists x_1 \dots \exists x_n, C$?"

In CSPs, the backtrack algorithm is inefficient when problems are big, and the most common way to solve CSPs is to combine depth-first search and constraint propagation. The aim is to use constraint propagation to reduce the size of the search tree by removing some inconsistent values in domains of variables. An inconsistent value is a value such that if it is assigned to its variable, the CSP is unsatisfiable. That is, removing inconsistent values does not change the set of solutions.

Most CSP solvers use Arc Consistency (AC) as the best compromise between tree pruning and time consumption. A constraint c_j is *arc consistent* iff for any

$x_i \in var(c_j)$, for any $v_i \in D(x_i)$, v_i is consistent with c_j. To propagate AC during the backtrack, after each instantiation, we remove inconsistent values in domains of not yet instantiated variables x_j until all constraints are arc consistent.

The quantified constraint satisfaction problem. The quantified extension of the CSP [4] allows some of the variables to be universally quantified. A *quantified* constraint network consists of variables $X = \{x_1, \ldots, x_n\}$, a set of domains $D = \{D(x_1), \ldots, D(x_n)\}$, a quantifier sequence $\Phi = (\phi_1 x_1, \ldots, \phi_n x_n)$, where $\phi_i \in \{\exists, \forall\}, \forall i \in 1..n$, and a set of constraints C. Given a quantified constraint network, the Quantified CSP (QCSP) is the question "$\phi_1 x_1 \ldots \phi_n x_n, C$?".

Example 1. $\exists x_1 \forall y_1 \exists x_2, (x_1 \neq y_1) \wedge (x_2 < y_1)$ with $x_1, y_1, x_2 \in \{1, 2, 3\}$. This can be read as: is there a value for x_1 such that whatever the value chosen for y_1, there will be a value for x_2 consistent with the constraints?

As in CSP, the backtrack search in QCSP is combined with constraint filtering. Propagation techniques are heavily depending on the quantifiers of variables. In Example 1 the QCSP is unsatisfiable because for each value of x_1, there is a value of y_1 violating $x_1 \neq y_1$. As y_1 is a universal variable and x_1 is an existential variable earlier in the sequence, any value in the domain of x_1 that is not compatible with a value in the domain of y_1 is not part of a solution. Constraint propagation in QCSP has been studied in [4, 8, 7].

2.2 Restricted quantification

One of the advantages that CSPs have on SAT problems (satisfaction of Boolean clauses) is that a CSP model is often close to the intuitive model of a problem, whereas a SAT instance is most of the time an automatic translation of a model to a clausal form, and is not human-readable. QCSP and QBF can be compared as CSP and SAT. To model a problem with a QBF, one needs to translate a model into a formula, and the QBF is not human-readable. QCSP, like CSP, should have the advantage of readability. But modeling a problem, even a simple one, with a QCSP, is a complex task. The prenex form of formulas is counter-intuitive. It would be more natural to have symmetrical behaviors for existential and universal variables. We describe here two frameworks that are more modeler-friendly: Strategic Constraint Satisfaction Problem (SCSP), and QCSP$^+$.

The strategic CSP. In SCSP [3], the meaning of the universal quantifier is different from the universal quantifier in QCSP. It is noted $\mathring{\forall}$. Allowed values for universal variables are values consistent with previous assignments.

Let us change the universal quantifier of Example 1 into the universal quantifier of SCSPs. The problem is now $\exists x_1 \mathring{\forall} y_1 \exists x_2, (x_1 \neq y_1) \wedge (x_2 < y_1)$ with $x_1, y_1, x_2 \in \{1, 2, 3\}$. If x_1 takes the value 1, the domain of y_1 is reduced to $\{2, 3\}$ because of the constraint $(x_1 \neq y_1)$ that prevents y_1 from taking the same value as x_1. This SCSP has a solution. If x_1 takes the value 1, y_1 can take either 2 or 3, and x_2 can always take the value 1 that satisfies the constraint $(x_2 < y_1)$.

Solving a SCSP is quite similar to solving a standard QCSP. The difference is that domains of universal variables are not static, they depend on variables already assigned in the left part of the sequence (before the universal variable we are ready to instantiate).

The quantified CSP$^+$. QCSP$^+$ [2] is based on the same idea as SCSP, which is to modify the meaning of the universal quantifier in order to make it more intuitive than in QCSP. It uses the notion of *restricted quantification*, which is more natural for the human mind than unrestricted quantification used in QCSP. Restricted quantification adds a "such that" right after the quantified variable. A QCSP$^+$ can be written as follows:

$$P = \exists X_1[R_{X_1}], \forall Y_1[R_{Y_1}]\exists X_2[R_{X_2}] \dots \ \exists X_n[R_{X_n}], G$$

All constraints noted R_X are called *rules*. They are the restrictions of the quantifiers. Each existential (*resp.* universal) *scope* X_i (*resp* Y_i) is a set of variables having the same quantifier. The order of variables inside a scope is not important, but two scopes cannot be swapped without changing the problem. The constraint G is called *goal*, it has to be satisfied when all variables are instantiated. The whole problem can be read as "Is there an instantiation of variables in X_1 such that the assignment respects the rule R_{X_1} and that for all tuples of values taken by the set of variables Y_1 respecting R_{Y_1}, there will be an assignment for variables in $X_2 \dots$ such that the goal G is reached?"

A QCSP$^+$ can be expressed as a QCSP. The difference between QCSP and QCSP$^+$ is the prenex form of QCSP. The QCSP$^+$ P is defined by the formula $\exists X_1(R_{X_1} \wedge (\forall Y_1(R_{Y_1} \rightarrow \exists X_2(R_{X_2} \wedge (\dots \ \exists X_n(R_{X_n} \wedge G))))))$. The prenex form of P is $\exists X_1 \forall Y_1 \exists X_2 \dots \ \exists X_n, (R_{X_1} \wedge (\neg R_{Y_1} \vee (R_{X_2} \wedge (\dots \ (R_{X_n} \wedge G)))))$. We see that, in this formula, we lose all the structure of the problem because all information is merged in a big constraint. Furthermore, disjunctions of constraints do not propagate well in CSP solvers. Finally the poor readability of this formula makes it hard to deal with for a human user.

The example of SCSP derived from Example 1 (see above) is modelled as a QCSP$^+$as follows: $\exists x_1 \forall y_1[y_1 \neq x_1]\exists x_2[x_2 < y_1], \top$ with $x_1, y_1, x_2 \in \{1, 2, 3\}$, where $\top$ is the universal constraint.

The difference between SCSP and QCSP$^+$ is mainly the place where constraints are put. A constraint of a SCSP containing a set X of variables should be placed in the rules of the rightmost variable of X in a QCSP$^+$ modeling the same problem.

In the rest of the paper, we will only consider the QCSP$^+$, but minor modifications of our contributions should be enough to adapt them to SCSPs.

3 Solving a QCSP$^+$

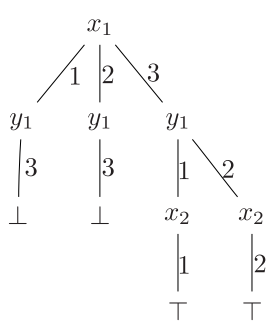
In this section, we focus on solving QCSP$^+$. First, we show how we use a backtracking algorithm with universal and existential variables. Then we focus on constraint propagation in QCSP$^+$.

As with classical CSPs or QCSPs, one can solve a QCSP$^+$ by backtracking. For each variable, we choose a value consistent with the rules attached to the variable, and we go deeper in the search tree. At the very bottom of the tree, we need to check that the assignments are consistent with the goal.

For existential variables, we do as for classical CSPs. If it is possible to assign the current existential variable according to the rules, then we can go deeper in the tree. But if there is no value consistent with the rules, it means that the current branch failed. Then we jump back to the last existential variable we instantiated and make another choice. If there is no previous existential variable, the QCSP$^+$ has no solution. Another case for which the branch can fail is when we instantiated all variables, but the assignments are not consistent with the goal.

On the other hand, if we instantiated all variables such that it is consistent with the goal, we just found a winning branch of the QCSP$^+$. In this case we jump back to the last universal variable instantiated, we assign another value of its domain consistent with its rules and we try to find another winning branch. When all values of the universal variable have been checked and lead to winning branches, we can go back to the previous universal variable. When there is no previous universal variable, the QCSP$^+$ has a solution. If at any moment, the current universal variable we want to instantiate has an empty domain, it is a winning branch. For example if it is a game, it means that the adversary cannot play because we blocked him, or because we just won before his move.

Example 2. Let $P = \exists x_1 \forall y_1[y_1 \neq x_1] \exists x_2, x_2 = y_1, D(x_1) = D(y_1) = \{1, 2, 3\}$, $D(x_2) = \{1, 2\}$ be a QCSP$^+$. In this QCSP$^+$, x_1 can play any of the moves $1, 2$, or 3. Then y_1 can play a move different from the move of x_1, and x_2 can play 1 or 2. At the end, in order to win, x_2 and y_1 must have the same value.



If x_1 plays 1 or 2, y_1 will be able to play 3 and will win because x_2 cannot play 3 (this value is not in its domain). The only way for the $\exists$-player to win is to play 3 at the first move to forbid y_1 to play 3. Then whatever the value taken by y_1 (1 or 2), x_2 will be able to play the same value. The QCSP$^+$ has a solution.

Note that in Example 2, the goal could have been a rule for x_2. When the last variable is an existential one, its rules and the goal have the same meaning.

Propagating the constraints in QCSP$^+$

We try to identify how it is possible to use constraint propagation to reduce the domains of variables.

Example 3. Let $P = \exists x_1 \forall y_1 [y_1 < x_1], \bot$. $D(x_1) = D(y_1) = \{1, 2, 3\}$. x_1 has to choose a value, then y_1 has to take a lower value.

In Example 3, a standard CSP-like propagation of $y_1 < x_1$ would remove the value 1 for x_1 and the value 3 for y_1. Like in CSPs, the value 3 in the domain of y_1 is inconsistent because whatever the value taken by x_1, y_1 will never be able to take this value.[1] But contrary to the CSP-like propagation, the QCSP$^+$ propagation should not remove the value 1 for x_1: if x_1 takes the value 1 he will win because y_1 will have no possible move.

Let x_i be a variable in a QCSP$^+$, with a rule R_{x_i}. If we only propagate the constraint R_{x_i} on the domain of x_i, we remove from its domain all values inconsistent with what is happening before. This propagation is allowed, because values removed this way cannot appear in the current search tree (when the solver will have to instantiate x_i, the allowed values are values consistent with R_{x_i}).

Let x_j be another variable in a scope before the scope of x_i. Now suppose that the rule R_{x_i} involves x_j too (*i.e.,* the values that x_i can take depend on the values taken by x_j). If the CSP-like propagation of the constraint R_{x_i} removes some values in the domain of x_j, it does not mean that x_j cannot take the values removed, but that, if x_j takes one of these values, then when it will be x_i's turn to play, he will not be able to take any value. Hence a QCSP$^+$ propagation should not remove these values.

Briefly speaking, it is allowed to propagate constraints from the left of the sequence to the right, but not to propagate from the right to the left. Benedetti *et al.* proposed the *cascade propagation*, a propagation following this principle.

Cascade propagation. In [2], cascade propagation is proposed as a propagation mechanism. The idea is that propagating a rule can modify the domains of variables of its scope, but not the domains of variables of previous scopes.

In [2], cascade propagation is implemented by creating a sequence of sub-problems. Each sub-problem P_i represents the restriction of problem P to its scopes from the first to the ith. That is, each P_i contains all rules belonging to scopes $1 \dots i$. If there are n scopes in P, P_n will be the problem excluding the goal, and $P_{n+1} = P$. In a sub-problem, propagation is used as in a classical CSP. Each P_i can be considered as representing the fact that we will be able to instantiate variables from the first one of the first scope to the last one of scope i according to the rules. Let P be the problem described in Example 2. Cascade propagation creates 4 sub-problems:

$P_1 = \exists x_1, \top$
$P_2 = \exists x_1 \forall y_1 [y_1 \neq x_1], \top$
$P_3 = \exists x_1 \forall y_1 [y_1 \neq x_1] \exists x_2, \top$
$P_4 = \exists x_1 \forall y_1 [y_1 \neq x_1] \exists x_2, x_2 = y_1$

[1] Note the difference with the QCSP case for which a removal of value in a universal domain means a fail of the whole problem.

For each sub-problem P_1 to P_3, the goal is $\top$ because we say a sub-problem has a solution if we can instantiate all its variables, without thinking of the goal of P. Propagation in each sub-problem can be done independently, but to speed up the process, if a value is removed from the domain of a variable, it can safely be removed from the deeper sub-problems. Furthermore if the domain of any variable in a sub-problem P_k becomes empty with propagation, it means that it is impossible to do the k^{th} move. So, from there it is no longer necessary to check the problems $\{P_{k+1}, \ldots, P_{n+1}\}$ since they are inconsistent too. If the scope k is universal and if P_{k-1} can be completely instantiated, then the current branch is a winning branch. If the scope k is an existential one and if P_{k-1} can be completely instantiated, the current branch is a losing branch.

4 Back-Propagation

In this section we present what we name *back-propagation*, a kind of constraint propagation which uses the information from the right part of the sequence. We also show that this propagation may not work properly in general.

4.1 Removing values in domains

The aim of constraint propagation in CSPs is to remove every value for which we know that, if we assign this value to the variable, it leads to a fail. In QCSP$^+$, the aim of constraint propagation is to remove values in domains too. But values we can remove without loss of solution depend on the quantifiers. In the case of existential variables, the values we can remove are values that do not lead to a solution (as we do in CSPs). Intuitively, it means that the $\exists$-player will not play this value because he knows that he will lose with this move. If all values are removed, it is impossible to win at this point. In the case of universal variables, the values we can remove are values that lead to a solution. Intuitively, it means that the $\forall$-player will not play this value because he knows that it means a loss for him. If all values are removed, it means that the $\forall$-player cannot win, so it is considered as a win for the $\exists$-player.

4.2 An illustrative example

We will see how to propagate information from right to left. Back-propagation adds some redundant constraints inside the rules of variables. These constraints will help to prune domains.

Consider the Example 2 from Section 3. Let us propagate the goal $(x_2 = y_1)$. It removes the value 3 in the domain of y_1. It means that if $y_1 = 3$, we cannot win. So x_1 has to prevent y_1 from taking the value 3. If y_1 is able to take this value, the $\exists$-player will lose. The way to prevent this is to make sure that the rule belonging to y_1, $y_1 \neq x_1$, will force the $\forall$-player not to take the value 3. In other terms, the rule must be inconsistent for $y_1 = 3$. We can express it as $(\neg(y_1 \neq x_1) \wedge y_1 = 3)$, or $\neg(3 \neq x_1)$, that is $(x_1 = 3)$. This constraint can be posted as a rule for x_1. The problem is now $P = \exists x_1[x_1 = 3]\forall y_1[y_1 \neq x_1]\exists x_2, x_2 = y_1, D(x_1) = D(y_1) = \{1, 2, 3\}$, $D(x_2) = \{1, 2\}$. The new rule we just added removes the inconsistent values 1 and 2 for x_1.

4.3 General behavior

First we know that if v is inconsistent with the rules of the scope of x, we can remove it from the domain of x. Then, we can look ahead in the sequence of variables. Consider a QCSP$^+$containing the sequence with $\phi = \forall$ and $\bar{\phi} = \exists$ or $\phi = \exists$ and $\bar{\phi} = \forall$: $\phi x_i[R_{x_i}], \bar{\phi} y_j[C_j(x_i, y_j)], \phi x_k[C_k(x_k, y_j)]$. Suppose that propagating $C_k(x_k, y_j)$ removes the value v_j in the domain of y_j. As we said before, it means that if we assign v_j to y_j, x_k will not be able to play. Then the ϕ-player will have to forbid the $\bar{\phi}$-player to play v_j. If he is not able to forbid it, then he will lose. He can reduce the domain of y_j with the rules belonging to y_j and involving x_i.

To ensure that y_j will not forbid any move for x_k, we have to make sure that the constraint $C_j(x_i, y_j)$ will remove the value v_j. It is equivalent to forcing $\neg C_j(x_i, y_j)$ to let the value v_j in the domain of y_j, or equivalently forcing $(\neg C_j(x_i, y_j) \wedge y_j = v_j)$. Hence, we can add to R_{x_i} the constraint $\neg C_j(x_i, v_j)$,[2] so that the ϕ-player is assured to have chosen a move that prevents the $\bar{\phi}$-player from doing a winning move.

4.4 Back-propagation does not work in general

In the case where there are variables between x_i and x_k, our previous treatment is not correct: imagine that the game always ends before x_k's turn and we are unable to detect it with constraint propagation, we should not take into account the constraints on x_k. For example, consider the following problem:

Example 4. $P = \exists x_1 \forall z_1 z_2[\neq (x_1, z_1, z_2)]\exists t_2 \forall y_1[y_1 \neq x_1]\exists x_2, x_2 = y_1$. $D(x_1) = D(t_2) = D(y_1) = \{1, 2, 3\}, D(z_1) = D(z_2) = \{0, 1, 2\}, D(x_2) = \{1, 2\}$. The constraint $\neq (x_1, z_1, z_2)$ is a clique of binary inequalities between the different variables. The variable t_2 is here only to separate the two scopes of universal variables. This problem is the same as the problem in Example 2 in which we added the variables z_i and the variable t_2.

From Example 4 we can add the same constraint $x_1 = 3$ as we did for Example 2 in Section 4.2. Doing this, we forbid x_1 to take either the value 1 or the value 2. But if x_1 would take any of these two values, we would win since it is not possible to assign values for the different z_i.

In the general case the back-propagation may remove values that are consistent, so it cannot be used as a proper propagation for QCSP$^+$. But from the back-propagation, we can make a value ordering heuristic that will guide search towards a win, or at least prevent the adversary to win.

[2] the constraint C in which we replaced the occurrences of y_j with the value v_j

5 Goal-Driven Heuristic

In this section we present our value ordering heuristic for QCSP$^+$. The behavior of the heuristic is based on the same idea as back-propagation. The difference is that, as it is a heuristic, it does not remove values in domains, but it orders them from the best to the worst in order to explore as few nodes as possible.

In the first part of the section, we discuss value ordering heuristics on QCSP$^+$, and the difference with standard CSP. Afterwards, we present our contribution, a goal-driven value ordering heuristic based on back-propagation.

5.1 Value heuristics

In CSPs, a value ordering heuristic is a function that helps the solver to go towards a solution. When the solver has to make a choice between the different values of a variable, the heuristic gives the value that seems the best for solving the problem. The best value is a value that leads to a solution. If we are able to find a perfect heuristic that always returns a value leading to a solution, it is possible to solve a CSP without backtracking. But, when there is no solution or when we want all the solutions of a CSP, the heuristic, even perfect, does not prevent from backtracking. In the case of QCSP$^+$, value ordering heuristics can be defined too. But the search will not be backtrack-free, even with a good heuristic, because of the universal quantifiers.

In QCSP$^+$, a *good* value for an existential variable (like for CSP) is a value that leads to a solution (*i.e.,* the $\exists$-player wins). A *good* value for a universal variable is a value that leads to a fail (*i.e.,* we quickly prove that the $\forall$-player wins). If the QCSP$^+$ is satisfiable, the heuristic helps to choose values for existential variables, and if it is unsatisfiable, the heuristic helps to choose values for universal variables.

In the rest of the section, we describe the value ordering heuristic we propose for QCSP$^+$.

5.2 The aim of the goal-driven heuristic

Our aim, with the proposal of our heuristic, is to explore the search tree looking ahead to win as fast as possible, to avoid traps from the adversary, and of course not to trap ourselves. For example, in a chess game, if you are able to put your opponent into checkmate this turn, you do not ask yourself if another move would make you win in five moves. Or if your adversary is about to put you into checkmate next turn unless you move your king, you will not move your knight!

In terms of QCSP$^+$ checking that a move is considered as good or bad is a question of constraint satisfaction. We will use the same mechanisms as back-propagation, *i.e.,* checking classical arc consistency of rules.

Let see how to choose *good* values on different examples. In each of these example, the aim is to detect what would be a good value to try first for the first variable. In these examples, ϕ and $\bar{\phi}$ will we the quantifiers $\exists$ and $\forall$ or $\forall$ and $\exists$.

Self-preservation. In Example 5, a rule from a scope with the same quantifier removes some values in the domain of the current variable. The ϕ-player tries not to block himself.

Example 5. $P = \phi x_1 \ldots \phi x_2 [x_2 < x_1] \ldots, D(x_1) = D(x_2) = \{1, 2, 3\}$. AC on the rule of x_2 removes the value 1 in the domain of x_1. As the ϕ-player wants to be able to play again, it could be a better choice to try the values $x_1 = 2$ and $x_1 = 3$ at first. If $x_1 = 1$ is played, the player knows that he will not be able to play for x_2.

Blocking the adversary. In Example 6, a rule from a scope with the opposite quantifier removes some values in the domain of the current variable. The ϕ-player tries to block the $\bar{\phi}$-player.

Example 6. $P = \phi x_1 \ldots \bar{\phi} y_1 [y_1 < x_1] \ldots, D(x_1) = D(y_1) = \{1, 2, 3\}$. AC on the rule of y_1 removes the value 1 in the domain of x_1. As the ϕ-player wants to prevent the $\bar{\phi}$-player from playing, it could be a better choice to try the value $x_1 = 1$ first. If $x_1 = 2$ or $x_1 = 3$ is played, the $\bar{\phi}$-player could be able to keep playing.

If two rules are in contradiction, the heuristic will take into account the leftmost rule, because it is the rule which is the more likely to happen. We will implement this in our algorithm by checking the rules from left to right.

Annoying the adversary. Now imagine the other player finds a good value for his next turn with the same heuristic. Your aim is to prevent him from playing well, so the above process can be iterated.

This point is illustrated with the problem from Example 2:
$P = \exists x_1 \forall y_1 [y_1 \neq x_1] \exists x_2, x_2 = y_1, D(x_1) = D(y_1) = \{1, 2, 3\}, D(x_2) = \{1, 2\}$. The heuristic for finding values for y_1 detects that the value 3 is a good value (x_2 will not be able to win). x_1 is aware of that, and will try to avoid this case. His new problem can be expressed as $P' = \exists x_1 \forall y_1 [y_1 \neq x_1], \perp$ with $D(x_1) = \{1, 2, 3\}, D(y_1) = \{3\}$. (If the $\exists$-player lets y_1 take the value 3, he thinks he will lose. There may be a value for x_1 such that the $\exists$-player will prevent the $\forall$-player from making him lose). The heuristic for finding values for x_1 in P' detects that x_1 should choose to play 3 first to prevent y_1 from playing well.

In the next section, we will discuss on the algorithm for choosing the best values for variables.

5.3 The algorithm

In this section, we describe the algorithm `GDHeuristic` (Goal-Driven Heuristic) used to determine what values are good choices for the current variable. The algorithm takes as input, the current variable and the rightmost scope that we consider. Note that we consider the goal as a scope here. It returns a set of values which are considered better to try first as defined in the above part.

Note that the aim of an efficient heuristic is to make the exploration as short as possible. If we try to instantiate a variable of the ∃-player, this means finding a value that leads to a winning branch. If we try to instantiate a variable of the ∀-player, this means finding a value that proves the ∀-player can win, that is, a losing branch. We see that in both cases, the best value to choose is a value that leads the current player to a win. Thus, in spite of the apparent asymmetry of the process, we can use the same heuristic for both players.

Algorithm 1 implements the goal driven value ordering heuristic. It is called when the solver is about to assign a value to the current variable. The aim is to give the solver the best value to assign to the variable. In fact, it is not more time consuming to return a set of equivalent values than a single value, so we return a set of values for which we cannot decide the best between them.

In this algorithm, we use different functions we explain here:

saveContext() saves the current state (domains of variables)
restoreContext() restores to the last state.
AC(P_i) runs the arc consistency algorithm on the problem P_i
quant(var) returns the quantifier of *var* (∃ or ∀)
scope(var) returns the scope containing *var*
dom(var) returns the current domain of *var*
initDom(var) returns the domain of *var* before the first call to `GDHeuristic()`.

We now describe the algorithm's behavior. The first call to it is done with `GDHeuristic(current variable, goal)`. We try to find the best value for the current variable, for the whole problem (the rightmost scope to consider is the goal). Note that we could bound the depth of analysis by specifying another scope as the last one.

First of all, we save the context (line 1) because we do not want our heuristic to change the domains. The context will be restored each time we return a set of values (lines 5, 14, 17 and 19).

For each future sub-problem (*i.e.,* containing the variables at the right of the current variable), we will try to bring back information in order to select the best values for the current variable. This is the purpose of the loop (line 2). If no information can be used, it will return the whole domain of the current variable (line 20) since all its values seem equivalent.

For each sub-problem $P_{\#_{Scope}}$ containing all variables from scope 1 to scope $\#_{Scope}$, we enforce AC (line 3). If the domain of any variable in $P_{\#_{Scope}-1}$ is reduced, we will decide the aim of our move: *self-preservation* (line 6), *blocking the adversary* (line 7) or *annoying the adversary* (line 13). The heuristic performs at most q^2 calls to AC, where q is the number of scopes. It appears when all recursive calls (line 13) are done with $\texttt{scope}(Var) = LastScope - 1$.

The rest of the main loop is made of two main parts. The first part, from line 4 to line 8, describes the case where the domain of the current variable is modified by the scope $\#_{Scope}$. (Note that we know it is not modified due to an earlier scope because we have not exited from the main loop –line 2– at a previous turn.) If the current variable has the same quantifier as the scope $\#_{Scope}$, the heuristic returns values consistent with the scope $\#_{Scope}$ (*self-preservation* line 6). If the

Algorithm 1: *GDHeuristic*

```
input: CurrentVar, LastScope
Result: set of values
begin
1     saveContext()
2     for #_Scope ← scope(CurrentVar) + 1 to LastScope do
3         AC(P_#Scope)
4         if dom(CurrentVar) ≠ initDom(CurrentVar) then
              ReducedDomain ← dom(CurrentVar)
5             restoreContext()
6             if quant(CurrentVar) = quant(#_Scope) then
                  return ReducedDomain
7             else
                  return initDom(CurrentVar) \ ReducedDomain
8         else
9             if any domain has been reduced before scope #_Scope then
10                Var ← leftmost variable with reduced domain in P_#Scope−1
11                if quant(Var) ≠ quant(#_Scope) then
12                    dom(Var) ← initDom(Var) \ dom(Var)
13                    values ← GDHeuristic(CurrentVar, scope(Var))
14                    restoreContext()
15                    return values
16                else
17                    restoreContext()
18                    return initDom(CurrentVar)
19    restoreContext()
20    return initDom(CurrentVar)
end
```

quantifiers are different, the heuristic returns values that block any move for the scope $\#_{Scope}$, *i.e.,* values inconsistent with the rules of scope $\#_{Scope}$ (*blocking the adversary* line 7). The second part of the main loop, from line 9 to line 18, represents the case where a variable (Var), between the current variable and the scope $\#_{Scope}$ has a domain reduced (line 9). If Var and the scope $\#_{Scope}$ have different quantifiers (*annoying the adversary* line 11), Var will try to block any move for the scope $\#_{Scope}$ by playing one of the values not in its reduced domain (line 12). We then try to find a good value for the current variable according to this new information that we have for Var (line 13). If Var and the scope $\#_{Scope}$ have the same quantifier (*self-preservation* line 16), Var will try to play values in its reduced domain, that is, those that do not block its future move at scope $\#_{Scope}$. But we know that arc consistency has not removed any value in the domain of the current variable. This means that whatever the value selected by the current variable, Var will be able to play in its reduced domain (*i.e.,* good values). As a result, the heuristic cannot discriminate in the initial domain of the current var (line 18).

If no domain is modified (except the domains in scope $\#_{Scope}$), we will have to check the next sub-problem (returning to the beginning of the loop).

Note that at lines 6 and 16, we immediately return a set of values for the current variable. We could imagine continuing deeper to try to break ties among equally good values for the current variable. We tested that variant and we observed that there were not much difference between the two strategies, both in terms of nodes exploration and cpu time. So, we chose the simpler one.

6 Experiments

We implemented a QCSP$^+$ solver on top of the constraint library Choco [5]. Our solver accepts all constraints provided by Choco, and uses the classical constraint propagators already present in the library. In this section, we show some experimental results on using either our goal-driven heuristic or a lexicographical value ordering heuristic. We compare the performance of this solver with QeCode [2], the state-of-the-art QCSP$^+$ solver, built on top of GeCode, which uses a fail-first heuristic by default. This value ordering heuristic first tries the value inconsistent with the earliest future scope. We also tried our solver with the promise heuristic [6], but it was worse than Lexico. The effect of a heuristic trying to ensure that constraints will still be consistent is that a player does not want to win as early as possible (by making the other player's rules inconsistent). In all cases, the variable ordering chosen is the same: we instantiate variables in order of the sequence.

We implemented the same model of the generalized Connect-4 game for our solver and for QeCode.

Connect-4 is a two-player game that is played on a vertical board with 6 rows and 7 columns. The players have 21 pieces each, distinguished by color. The players take turns in dropping pieces in one of the non-full columns. The piece then occupies the lowest empty cell on that column. A player wins by placing 4 of his own pieces consecutively in a line (row, column or diagonal), which ends the game. The game ends in a draw if the board is filled completely without any player winning. The generalized Connect-4 is the same game with a board of m columns and n rows, where the aim is to place k pieces in a line.

At first we tried our solver on 4×4 grids, with alignments of 3 pieces. We ran QeCode and our solver with Lexicographical heuristic (Lexico) or with Goal-Driven heuristic on problems with different number of allowed moves, from 5 to 15. We compare the time taken to solve instances.

The results are presented in Figure 1 (note the log scale). There is a solution for 9 allowed moves and more. As we can see, our solver with lexicographical value ordering solves these instances faster than QeCode. It can be explained by different ways. First, it is possible that Choco works faster in propagating constraints defined as we did. The second reason is that QeCode uses cascade propagation (propagation on the whole problem for each instantiation), whereas our solver propagates only the rules of the current scope. Thus, our solver spends less time in propagation. We can see that the Goal-Driven heuristic speed up the resolution. It is about twice as fast as with Lexico.

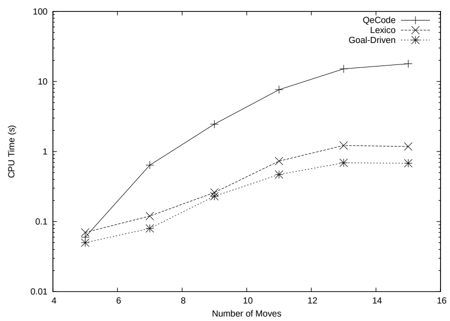


Fig. 1. Connect-3, on 4x4 grid

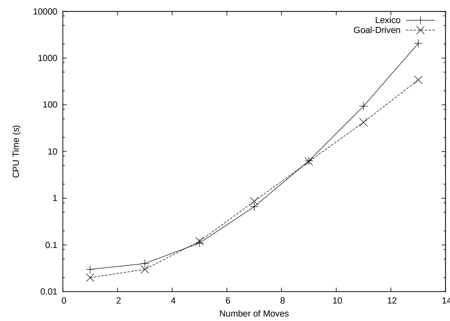


Fig. 2. Connect-4, on 7x6 grid

In next experiments, we only compare our solver with Lexicographical value ordering and with Goal-Driven value ordering because QeCode was significantly slower and the aim is to test the accuracy of our heuristic.

In Figure 2, the real Connect-4 game is solved. We vary the number of moves from 1 to 13 and compare the performance in terms of running time. For a number of moves less than 9, Goal-Driven heuristic does not improve the performance, but from this point, the heuristic seems to be useful. In this problem, all instances we tested are unsatisfiable.

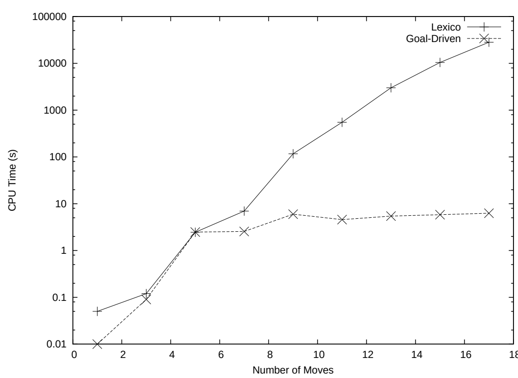


Fig. 3. Noughts and Crosses, on 5x5 grid

In Figure 3, we solve the game of Noughts and Crosses. This is the same problem except the gravity constraint which does not exist in this game. It is possible to put a piece on any free cell in the board. Instead of having n choices for a move, we have $n \times m$ choices. In the problem we tested, the aim is to align 3 pieces. This problem has a solution for 5 moves. We see that the Goal-Driven heuristic is very efficient here for solvable problems. Adding allowed moves (increasing the depth of analysis) has not a big influence on the running time of our solver with the Goal-Driven heuristic. The heuristic seems to be efficient when there are more allowed moves than necessary to finish the game.

Discussion. More generally, when can we expect our Goal-Driven heuristic to work well? As it is based on information computed by AC, it is expected to work well on constraints that propagate a lot, *i.e.,* tight constraints. Furthermore, as it actively uses quantifier alternation and tries to provoke wins/losses before the end of the sequence, it is expected to work well in problems where there exist winning/losing strategies that do not need to reach the end of the sequence.

7 Conclusion

In QCSP$^+$, we cannot propagate constraints from the right of the sequence to the left. Thus, current QCSP$^+$ solvers propagate only from left to right. In this paper, we have analyzed the effect of propagation from right to left. We have derived a value ordering heuristic based on this analysis. We proposed an algorithm implementing this heuristic. Our experimental results on board games show the effectiveness of the approach.

References

1. C. Ansótegui, C. Gomes, and B. Selman. The Achille's heel of QBF. In *Proceedings AAAI'05*, 2005.
2. M. Benedetti, A. Lallouet, and J. Vautard. QCSP made practical by virtue of restricted quantification. In *Proceedings of IJCAI'07*, pages 38–43, 2007.
3. C. Bessiere and G. Verger. Strategic constraint satisfaction problems. In *Proceedings CP'06 Workshop on Modelling and Reformulation*, pages 17,29, 2006.
4. L. Bordeaux and E. Montfroy. Beyond NP: Arc-consistency for quantified constraints. In *Proceedings CP'02*, pages 371–386, 2002.
5. Choco. Java constraint library, http://choco.sourceforge.net/.
6. P.A. Geelen. Dual viewpoint heuristics for binary constraint satisfaction problems. In *Proceedings ECAI'92*, pages 31–35, 1992.
7. I.P. Gent, P. Nightingale, A. Rowley, and K. Stergiou. Solving quantified constraint satisfaction problems. *Artif. Intell.*, 172(6-7):738–771, 2008.
8. N. Mamoulis and K. Stergiou. Algorithms for quantified constraint satisfaction problems. In *Proceedings CP'04*, pages 752–756, 2004.