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Variational approximation of a constraint signal by a Mumford-Shah type energy functional

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Abstract—In this paper, the main objective is to establish an existence result of a variational model in image segmentation constrained by a given vector field. In the one dimensional case, we give a discrete version converging in a variational way to the continuous model. We finally describe the numerical analysis of this model with application in image segmentation.

Keywords—Image processing theory, image segmentation.

I. INTRODUCTION

In this paper, we aim to describe a theoretical and numerical treatment of the following problem stemming from the theory of image segmentation:

$$E(u) = \inf_{v \in SBV(\Omega)} E(v),$$

$$E(v) = \int_{\Omega} |\nabla v|^2 dx + \int_{S_v} f d\mathcal{H}^{N-1} + \int_{\Omega} |g - v|^2 dx,$$

where Ω is a domain in \mathbf{R}^N and $g : \Omega \rightarrow \mathbf{R}$.

The density f is the extended real-valued function $v \mapsto 1 + I_A(v)$ where A is the set of all the functions v in $SBV(\Omega)$ satisfying, for a given $a \in \mathbf{R}$, the condition:

$$[v](x)\nu_v(x) \cdot \Phi(x) \geq a,$$

for $\mathcal{H}^{N-1} \llcorner S_v$ a.e. x in Ω . We point out that the energy E is the perturbation of the so called *Mumford-Shah energy* [6] ($f \equiv 1$) by the indicator function I_A of the set A (i.e. $I_A(v) = 0$ if $v \in A$, $+\infty$ otherwise). The vector valued function Φ belonging to $\mathcal{C}(\bar{\Omega}, \mathbf{R}^N)$, may be considered, when $N = 2$, as a constraint on the outline of an image g in computer vision: the outlines of the image having a jump of grey level in the direction of Φ more than a are selected by the model. In the one dimensional case it leads to detect some selected discontinuities of the signal g .

In the paper \mathcal{L} denotes the Lebesgue measure restricted to Ω , $SBV(\Omega)$ the space of all the functions v of bounded variation whose distributional gradient is of the form $Dv = \nabla v \mathcal{L} + [v]\nu_v \mathcal{H}^{N-1} \llcorner S_v$. We denote by $[v] := u^+ - u^-$ the jump of v through the jump set S_v , ν_v is the unit normal to S_v and $\mathcal{H}^{N-1} \llcorner S_v$ is the restriction to S_v of the $N - 1$ -dimensional

Hausdorff measure. In the one dimensional case, we adopt the following notation for the structure of the distributional derivative of any v in $SBV(0, 1)$: $v' = \dot{v} dt + [v]\mathcal{H}^0 \llcorner S_v$. In this specific case, $[v](t)$ is nothing but the classical jump of v at $t \in S_v$ and \mathcal{H}^0 is the counting measure so that $\int [v]\mathcal{H}^0 \llcorner S_v = \sum_{t \in S_v} [v]\delta_t$. For more about SBV spaces, we refer the reader to [4].

In Section 2 we establish the existence of a solution u of the problem. In Section 3 we define a variational discrete model in the case $N = 1$. Let E_h be some suitable functional expected to describe the discrete energy associated with the above functional E , h denoting the step of discretization. We say that the problem $\inf E_h$ is a *variational discrete model* of $\inf E$ if $\min E_h$ converges to $\min E$ when h goes to zero and if every minimizer u_h of E_h tends in $L^1(0, 1)$ to a minimizer of E . Provided that $(u_h)_{h>0}$ be compact, an appropriate convergence for the sequence of functionals $(E_h)_{h>0}$ leading to this objective, is the so called Γ -convergence of the sequence $(E_h)_{h>0}$ to the functional E . Section 4 is devoted to the description of an algorithm giving a solution u_h of the discrete problem $\min E_h$. We conclude the paper by giving some numerical experiments with application in image segmentation. Previous work have all ready applied variational models to image segmentation [8], [9], [10], [2].

II. EXISTENCE OF A SOLUTION

We equip $SBV(\Omega)$ with the norm $\|\cdot\|_{SBV(\Omega)}$ defined by:

$$\|v\|_{BV(\Omega)} = \int_{\Omega} |v| dx + \int_{\Omega} |Dv|,$$

where $\int_{\Omega} |Dv|$ is the total mass of the total variation $|Dv|$ of the measure Dv .

We say that a sequence $(u_n)_{n \in \mathbf{N}}$ weakly converges to u in $BV(\Omega)$ iff $u_n \rightarrow u$ strongly in $L^1(\Omega)$ and $Du_n \rightarrow Du$ weakly in the sense of measures in $\mathcal{M}(\Omega, \mathbf{R}^N)$.

Let $(u_n)_{n \in \mathbf{N}}$ be a minimizing sequence of $\inf E(v)$.

From Ambrosio's compactness theorem (see [1], [4]), one can establish the weak lower-semi-continuity of the functional E together with existence of a weak cluster point \bar{u} of $(u_n)_{n \in \mathbf{N}}$. Combining these two arguments gives $E(\bar{u}) = \inf E(v)$. We have established

Theorem 1: There exists at least one minimizer of the problem

$$\inf_{v \in A} \left\{ \int_{\Omega} |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v) + \int_{\Omega} |g - v|^2 dx \right\}$$

The following compactness theorem is due to L. Ambrosio (see [1], [4]).

Theorem 2: Let $(u_n)_n$ be a sequence of elements in $SBV(\Omega)$ satisfying

- (i) $\sup_{n \in \mathbf{N}} \{\|u_n\|_{BV(\Omega)}\} < +\infty$,
- (ii) the sequence $(\nabla u_n)_{n \in \mathbf{N}}$ is equi-integrable,
- (iii) the sequence $(\mathcal{H}^{N-1}(S_{u_n}))_{n \in \mathbf{N}}$ is uniformly bounded.

Then one can extract a subsequence $(u_{n_k})_{k \in \mathbf{N}}$ which converges to some $u \in SBV(\Omega)$ such that

$$\begin{aligned} u_{n_k} &\rightarrow u \text{ in } L^1_{loc}(\Omega), \\ \nabla u_{n_k} &\rightharpoonup \nabla u \text{ weakly in } L^1(\Omega, \mathbf{R}^N), \\ J u_{n_k} &\rightharpoonup J u \text{ weakly in } \mathcal{M}(\Omega, \mathbf{R}^N), \\ \mathcal{H}^{N-1}(S_{u_{n_k}}) &\leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{N-1}(S_{u_{n_k}}). \end{aligned} \quad (1)$$

Let $(u_n)_{n \in \mathbf{N}}$ be a minimizing sequence of $\inf E(v)$. From classical argument stemming from measure theory, one can establish the weak lower-semicontinuity of the functional E . On the other hand, applying Theorem 2, one can easily see that there exists a weak cluster point \bar{u} of $(u_n)_{n \in \mathbf{N}}$. Combining these two arguments gives $E(\bar{u}) = \inf E(v)$. We have established

Theorem 3: There exists at least one minimizer of the problem

$$\inf_{v \in A} \left\{ \int_{\Omega} |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v) + \int_{\Omega} |g - v|^2 dx \right\}$$

III. DISCRETE MODEL IN ONE DIMENSIONAL CASE

We assume here that $\Phi > 0$ on $[0, 1]$ and $a > 0$. For $h = \frac{1}{N}$ we consider the following density defined for every $(t, e) \in (0, 1) \times \mathbf{R}$ by:

$$W_h(t, e) := \begin{cases} e^2 & \text{if } e \leq e_h(t) \\ \frac{1}{h} & \text{if } e > e_h(t), \end{cases}$$

where $e_h(t) = \frac{a}{h\Phi(t)}$ and set $\tilde{W}_h(t, e) = W_h(ih, e)$ if $t \in [ih, (i+1)h[$.

Let $\mathcal{A}_h(0, 1)$ denote the set of all the continuous functions in $(0, 1)$, affine on each interval $(ih, (i+1)h)$ of $(0, 1)$ and bounded by $\|g\|_{\infty}$. We define the functional:

$$F_h(v) := \begin{cases} \int_0^1 \tilde{W}_h(t, v'(t)) dt & \text{if } v \in \mathcal{A}_h(0, 1), \\ +\infty & \text{if } v \in L^1(0, 1) \setminus \mathcal{A}_h(0, 1). \end{cases}$$

Our objective is to establish the Γ -convergence of the functional $v \mapsto E_h(v) := F_h(v) + \int_0^1 |g(ih) - v|^2 dx$ to the energy functional E when $L^1(0, 1)$ is equipped with its strong

topology. In order to make relevant our variational approximating scheme, we must establish the following compactness result whose proof is a straightforward consequence of the behavior of W_h and of the compactness of the embedding $W^{1,1}(0, 1) \subset L^1(0, 1)$.

Proposition 1: Let $(u_h)_{h>0}$ be a sequence of $L^1(0, 1)$ satisfying

$$E_h(u_h) = \inf\{E_h(v) : v \in L^1(0, 1)\}.$$

Then, there exist a subsequence (not relabeled) and $u \in L^1(0, 1)$ such that $u_h \rightarrow u$ strongly in $L^1(0, 1)$.

Existence of u_h above is obtained by arguments similar to those of the proof of Theorem 3. We now establish the Γ -convergence of E_h to E , a notion of convergence introduced by De Giorgi and Franzoni. For overview, we refer the reader to [3] and references therein.

Theorem 4: Let $u \in L^1(0, 1)$. Then the sequence $(E_h)_{h>0}$ Γ -converges to E , i.e., the two following statements hold:

(Γ_1) there exists a sequence $(u_h)_{h>0}$ strongly converging to u in $L^1(0, 1)$ such that:

$$\limsup_{h \rightarrow 0} E_h(u_h) \leq E(u);$$

(Γ_2) for every $u \in L^1(0, 1)$ and every sequence $(u_h)_{h>0}$ strongly converging to u , we have:

$$E(u) \leq \liminf_{h \rightarrow 0} E_h(u_h).$$

We should point out that this convergence is variational, i.e., every cluster point of the sequence $(u_h)_{h>0}$ given in Proposition 1 is a minimizer of E , and $\min\{E_h(v) : v \in L^1(0, 1)\}$ converges to $\min\{E(v) : v \in L^1(0, 1)\}$. For a proof of (Γ_1) and (Γ_2), we refer the reader to [5].

IV. DESCRIPTION OF AN ALGORITHM FOR THE DISCRETE MODEL

In this section, we would like to describe an algorithm for the computation of a solution of the discrete variational model stated in the previous section :

$$E_h(u) = \min_{v \in \mathcal{A}_h(0, 1)} \left(F_h(v) + \int_0^1 |g(ih) - v|^2 dx \right).$$

Obviously, one can rewrite the energy E_h as a functional defined in \mathbf{R}^{N+1} and the discrete problem becomes:

$$\begin{aligned} E_h(u) &= \min_{v \in \mathbf{R}^{N+1}} \left(\sum_{\{i: v_{i+1} - v_i \leq \frac{a}{\Phi(ih)}\}} \frac{|v_{i+1} - v_i|^2}{h} \right. \\ &\quad \left. + \mathcal{H}^0(\{i : v_{i+1} - v_i > \frac{a}{\Phi(ih)}\}) \right), \end{aligned}$$

where $v = (v_0, \dots, v_N)$.

The strategy consists now in conditioning the minimization by fixing:

a) firstly the cardinal of the set $(\{i : v_{i+1} - v_i > \frac{a}{\Phi(ih)}\})$ that we will call *fracture set*,

b) secondly, for each fixed cardinal, the site of such fracture set. The problem may be written : find u in \mathbf{R}^{N+1} solution of:

$$\min_{k=0,\dots,N-1} \min_{F_k} \min_{v \in \mathbf{R}^{N-k}, v_{i+1}-v_i \leq \frac{\alpha}{\Phi(ih)}} \left(\sum_{i \notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + k + \sum_{i \notin F_k} h|g(ih) - v_i|^2 \right).$$

Note that we have replaced the sum $\sum_{i=0}^N h|g(ih) - v_i|^2$ by the sum $\sum_{i \notin F_k} h|g(ih) - v_i|^2$ in the definition of the discrete energy. The first minimization problem:

$$v \in \mathbf{R}^{N-k}, v_{i+1}-v_i \leq \frac{\alpha}{\Phi(ih)} \left(\sum_{i \notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + k + \sum_{i \notin F_k} h|g(ih) - v_i|^2 \right)$$

consists in solving the classical quadratic optimization problem

$$\min_{\mathbf{R}^{N-k}} \left(\sum_{i \notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + \sum_{i \notin F_k} h|g(ih) - v_i|^2 \right), \quad (2)$$

and to take into account the solution u_{k, F_k} if and only if its components satisfy the constraint $v_{i+1} - v_i \leq \frac{\alpha}{\Phi(ih)}$.

It is easily seen that (2) is equivalent to the linear problem:

$$\left(A^T A + h^2 I \right) v = h^2 \tilde{g}_h,$$

where I is the $(N-k) \times (N-k)$ identity matrix and A is a $(N-k) \times (N-k)$ matrix whose entries are -1 or 1 on the diagonal $i = j$ and 1 or 0 on the diagonal $j = i + 1$.

V. NUMERICAL EXPERIMENTS

The problem is equivalent to the linear problem:

$$\begin{pmatrix} b_0 & a_0 & \cdots & 0 \\ a_0 & b_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}$$

where

$$\begin{cases} b_0 = b_{n-1} & = 1 + \frac{1}{h^2} \\ b_i & = 1 + \frac{2}{h^2} \quad \text{for } 1 \leq i \leq n-1 \\ a_i & = -\frac{1}{h^2} \quad \text{for all } i \end{cases}$$

Set :

$$\begin{cases} w_0 & = \frac{a_0}{b_0} \\ y_0 & = \frac{g_0}{b_0} \end{cases}$$

and replace b_1 by $b_1 - a_0 w_0$ and g_1 by $g_1 - a_0 y_0$.

then, for all $k > 1$

do :

$$\begin{cases} w_k & = \frac{a_k}{b_k - a_k w_{k-1}} \\ y_k & = \frac{g_k - a_k y_{k-1}}{b_k - a_k w_{k-1}} \end{cases}$$

and replace b_k by $b_k - a_{k-1} w_{k-1}$ and g_k by $g_k - a_{k-1} y_{k-1}$.

So, we obtain the solution:

$$\begin{cases} v_n & = y_n \\ v_k & = y_k - w_k v_{k+1} \quad \text{for } 0 \leq k \leq n-1 \end{cases}$$



Fig. 1. Original image of Lena.

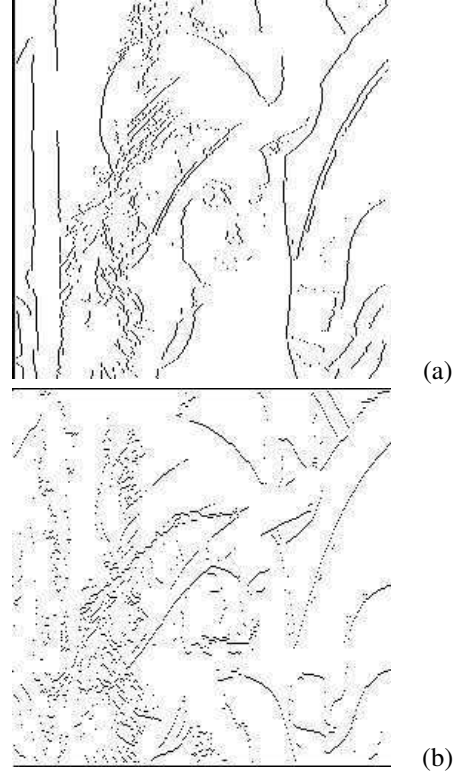


Fig. 2. Without filter, a) On line, b) On column.

To generalize, we introduce coefficients α, β and γ in $E(v)$ and put:

$$E(v) = \alpha \int_{\Omega} |\nabla v|^2 dx + \beta \int_{\Omega} |g - v|^2 dx + \gamma \int_{S_v} f d\mathcal{H}^{N-1}.$$

We now give some reconstructed pictures obtained with various values of coefficients α, β and γ and various values of the number $\frac{\alpha}{\Phi(ih)}$ that we call *filter*. For working in one dimension, we scan the picture on line and on column in order to superpose the both. We limit the computation by taking $\#(F_k) = 1$.

Without filter, from the original image of Lena, Fig.1, we obtain the detection on line, Fig.2.a, and on column, Fig.2.b.

By superposition of the results of the Fig.2.a and b we have a full detection illustrated Fig.3. For the on line detection, Fig.4 and 5, we have applied several values for the filter and for α , β and γ . In particular, for the Fig. 4.c, 4.d and 5, we have separated the pixels in two parts. The values of the parameters α , β and γ have been fixed experimentally.

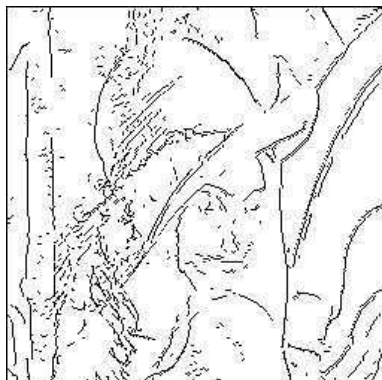


Fig. 3. By superposition of Fig.2.a and 2.b

For the on column detection, Fig.6, we have also applied several values for the filter and for α , β and γ . Finally, we illustrate Fig. 7 the superposition of the on line and on column detections.

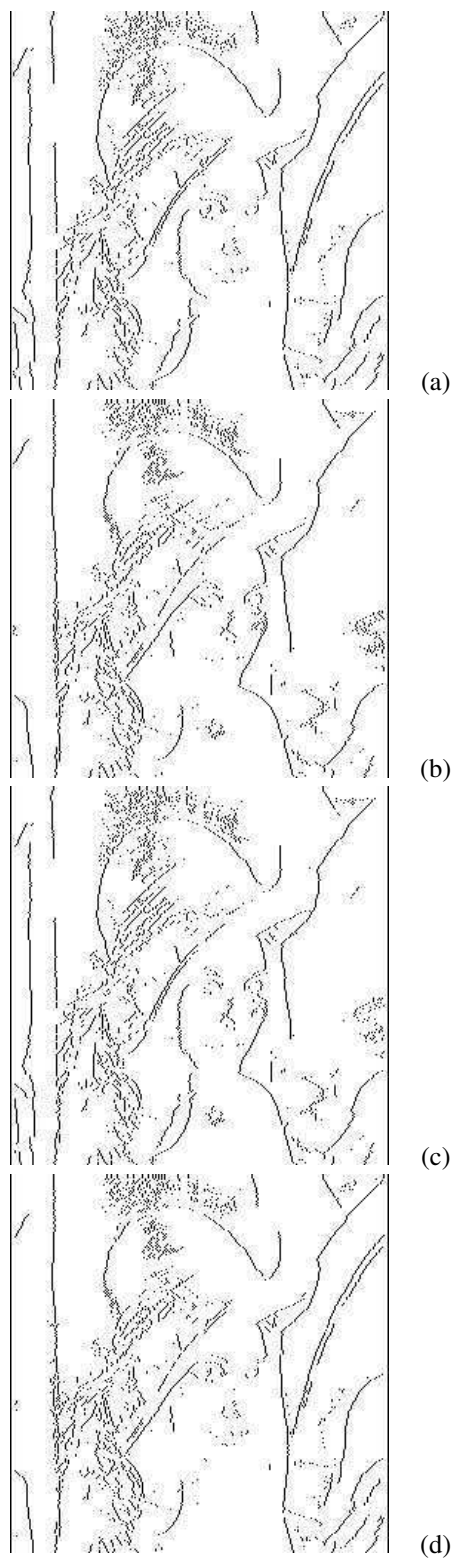


Fig. 4. On line, a) filter = 30 and $\alpha = \beta = \gamma = 1$, b) filter = 0.001, $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$. c) With two filters: filter1 = 0.001 for pixels between 0 and 128 with $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$, filter2 = 30 for pixels between 129 and 255 with $\alpha = \beta = \gamma = 1$, d) With two filters: filter1 = 30 for pixels between 0 and 128 with $\alpha = \beta = \gamma = 1$, filter2 = 0.001 for pixels between 129 and 255 with $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$.

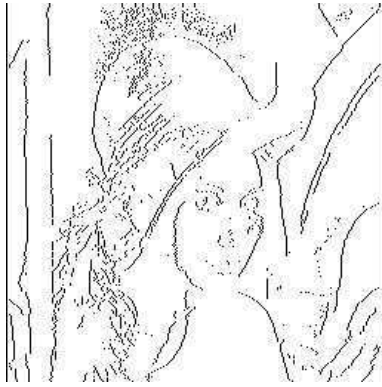


Fig. 5. On line with two filters: filter1 = 0.001 for pixels between 154 and 196 with $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$, filter2 = 30 for pixels between 0 and 153, and from 197 until 255, with $\alpha = \beta = \gamma = 1$.

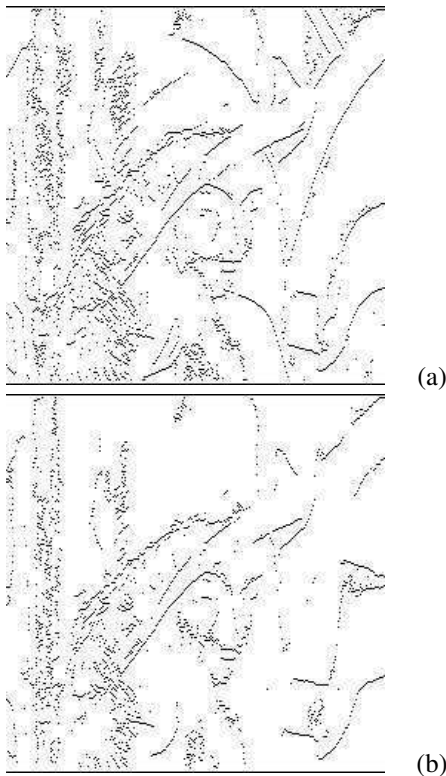


Fig. 6. On column, a) filter = 30, $\alpha = \beta = \gamma = 1$, b) filter = 0.001, $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$.

VI. CONCLUSION

In this paper we have generalized the well known one dimensional discretization of the Mumford-Shah functional by taking into account a constraint on the outline (i.e. the filter). We have illustrated the method by scanning a picture on line and on column in order to superpose the both. Actually, the used filters in line and column could be considered as the projections of a given two dimensional vector field on the picture. In perspective of this work, we are planning to compare our method with level set methods [7].

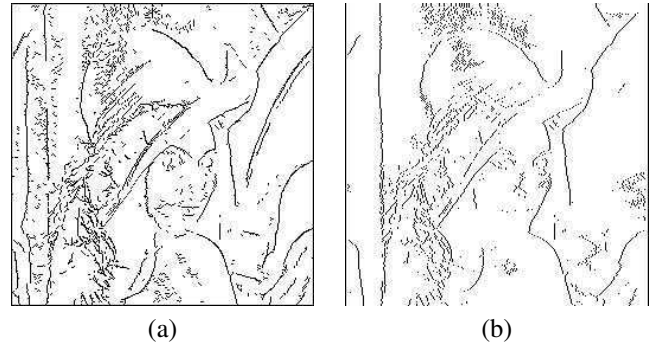


Fig. 7. a) Superposition of Fig.5 and Fig.6.a, b) Superposition of Fig.5 and Fig.6.b.

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