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► **To cite this version:**

Joseph Rakotondralambo, Gérard Michaille, Robert Brouzet, William Puech. Variational Approximation of a Constraint Signal by a Mumford-Shah Type Energy Functional. IPTA: Image Processing Theory, Tools and Applications, Nov 2008, Sousse, Tunisia. 1st International Workshop on Image Processing Theory, Tools and Applications, 2008, <<http://ipta08.ibisc.univ-evry.fr/>>. <lirmm-00371823>

**HAL Id: lirmm-00371823**

**<https://hal-lirmm.ccsd.cnrs.fr/lirmm-00371823>**

Submitted on 30 Mar 2009

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# Variational approximation of a constraint signal by a Mumford-Shah type energy functional

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**Abstract**—In this paper, the main objective is to establish an existence result of a variational model in image segmentation constrained by a given vector field. In the one dimensional case, we give a discrete version converging in a variational way to the continuous model. We finally describe the numerical analysis of this model with application in image segmentation.

**Keywords**—Image processing theory, image segmentation.

## I. INTRODUCTION

In this paper, we aim to describe a theoretical and numerical treatment of the following problem stemming from the theory of image segmentation:

$$E(u) = \inf_{v \in SBV(\Omega)} E(v),$$

$$E(v) = \int_{\Omega} |\nabla v|^2 dx + \int_{S_v} f d\mathcal{H}^{N-1} + \int_{\Omega} |g - v|^2 dx,$$

where  $\Omega$  is a domain in  $\mathbf{R}^N$  and  $g : \Omega \rightarrow \mathbf{R}$ .

The density  $f$  is the extended real-valued function  $v \mapsto 1 + I_A(v)$  where  $A$  is the set of all the functions  $v$  in  $SBV(\Omega)$  satisfying, for a given  $a \in \mathbf{R}$ , the condition:

$$[v](x)\nu_v(x) \cdot \Phi(x) \geq a,$$

for  $\mathcal{H}^{N-1} \llcorner S_v$  a.e.  $x$  in  $\Omega$ . We point out that the energy  $E$  is the perturbation of the so called *Mumford-Shah energy* [6] ( $f \equiv 1$ ) by the indicator function  $I_A$  of the set  $A$  (i.e.  $I_A(v) = 0$  if  $v \in A$ ,  $+\infty$  otherwise). The vector valued function  $\Phi$  belonging to  $\mathcal{C}(\bar{\Omega}, \mathbf{R}^N)$ , may be considered, when  $N = 2$ , as a constraint on the outline of an image  $g$  in computer vision: the outlines of the image having a jump of grey level in the direction of  $\Phi$  more than  $a$  are selected by the model. In the one dimensional case it leads to detect some selected discontinuities of the signal  $g$ .

In the paper  $\mathcal{L}$  denotes the Lebesgue measure restricted to  $\Omega$ ,  $SBV(\Omega)$  the space of all the functions  $v$  of bounded variation whose distributional gradient is of the form  $Dv = \nabla v \mathcal{L} + [v]\nu_v \mathcal{H}^{N-1} \llcorner S_v$ . We denote by  $[v] := u^+ - u^-$  the jump of  $v$  through the jump set  $S_v$ ,  $\nu_v$  is the unit normal to  $S_v$  and  $\mathcal{H}^{N-1} \llcorner S_v$  is the restriction to  $S_v$  of the  $N - 1$ -dimensional

Hausdorff measure. In the one dimensional case, we adopt the following notation for the structure of the distributional derivative of any  $v$  in  $SBV(0, 1)$ :  $v' = \dot{v} dt + [v]\mathcal{H}^0 \llcorner S_v$ . In this specific case,  $[v](t)$  is nothing but the classical jump of  $v$  at  $t \in S_v$  and  $\mathcal{H}^0$  is the counting measure so that  $\int [v]\mathcal{H}^0 \llcorner S_v = \sum_{t \in S_v} [v]\delta_t$ . For more about  $SBV$  spaces, we refer the reader to [4].

In Section 2 we establish the existence of a solution  $u$  of the problem. In Section 3 we define a variational discrete model in the case  $N = 1$ . Let  $E_h$  be some suitable functional expected to describe the discrete energy associated with the above functional  $E$ ,  $h$  denoting the step of discretization. We say that the problem  $\inf E_h$  is a *variational discrete model* of  $\inf E$  if  $\min E_h$  converges to  $\min E$  when  $h$  goes to zero and if every minimizer  $u_h$  of  $E_h$  tends in  $L^1(0, 1)$  to a minimizer of  $E$ . Provided that  $(u_h)_{h>0}$  be compact, an appropriate convergence for the sequence of functionals  $(E_h)_{h>0}$  leading to this objective, is the so called  $\Gamma$ -convergence of the sequence  $(E_h)_{h>0}$  to the functional  $E$ . Section 4 is devoted to the description of an algorithm giving a solution  $u_h$  of the discrete problem  $\min E_h$ . We conclude the paper by giving some numerical experiments with application in image segmentation. Previous work have all ready applied variational models to image segmentation [8], [9], [10], [2].

## II. EXISTENCE OF A SOLUTION

We equip  $SBV(\Omega)$  with the norm  $\|\cdot\|_{SBV(\Omega)}$  defined by:

$$\|v\|_{BV(\Omega)} = \int_{\Omega} |v| dx + \int_{\Omega} |Dv|,$$

where  $\int_{\Omega} |Dv|$  is the total mass of the total variation  $|Dv|$  of the measure  $Dv$ .

We say that a sequence  $(u_n)_{n \in \mathbf{N}}$  weakly converges to  $u$  in  $BV(\Omega)$  iff  $u_n \rightarrow u$  strongly in  $L^1(\Omega)$  and  $Du_n \rightarrow Du$  weakly in the sense of measures in  $\mathcal{M}(\Omega, \mathbf{R}^N)$ .

Let  $(u_n)_{n \in \mathbf{N}}$  be a minimizing sequence of  $\inf E(v)$ .

From Ambrosio's compactness theorem (see [1], [4]), one can establish the weak lower-semi-continuity of the functional  $E$  together with existence of a weak cluster point  $\bar{u}$  of  $(u_n)_{n \in \mathbf{N}}$ . Combining these two arguments gives  $E(\bar{u}) = \inf E(v)$ . We have established

*Theorem 1:* There exists at least one minimizer of the problem

$$\inf_{v \in A} \left\{ \int_{\Omega} |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v) + \int_{\Omega} |g - v|^2 dx \right\}$$

The following compactness theorem is due to L. Ambrosio (see [1], [4]).

*Theorem 2:* Let  $(u_n)_{n \in \mathbf{N}}$  be a sequence of elements in  $SBV(\Omega)$  satisfying

- (i)  $\sup_{n \in \mathbf{N}} \{ \|u_n\|_{BV(\Omega)} \} < +\infty$ ,
- (ii) the sequence  $(\nabla u_n)_{n \in \mathbf{N}}$  is equi-integrable,
- (iii) the sequence  $(\mathcal{H}^{N-1}(S_{u_n}))_{n \in \mathbf{N}}$  is uniformly bounded.

Then one can extract a subsequence  $(u_{n_k})_{k \in \mathbf{N}}$  which converges to some  $u \in SBV(\Omega)$  such that

$$\begin{aligned} u_{n_k} &\rightarrow u \text{ in } L^1_{loc}(\Omega), \\ \nabla u_{n_k} &\rightharpoonup \nabla u \text{ weakly in } L^1(\Omega, \mathbf{R}^N), \\ J u_{n_k} &\rightharpoonup J u \text{ weakly in } \mathcal{M}(\Omega, \mathbf{R}^N), \\ \mathcal{H}^{N-1}(S_{u_{n_k}}) &\leq \liminf_{k \rightarrow +\infty} \mathcal{H}^{N-1}(S_{u_{n_k}}). \end{aligned} \quad (1)$$

Let  $(u_n)_{n \in \mathbf{N}}$  be a minimizing sequence of  $\inf E(v)$ . From classical argument stemming from measure theory, one can establish the weak lower-semicontinuity of the functional  $E$ . On the other hand, applying Theorem 2, one can easily see that there exists a weak cluster point  $\bar{u}$  of  $(u_n)_{n \in \mathbf{N}}$ . Combining these two arguments gives  $E(\bar{u}) = \inf E(v)$ . We have established

*Theorem 3:* There exists at least one minimizer of the problem

$$\inf_{v \in A} \left\{ \int_{\Omega} |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v) + \int_{\Omega} |g - v|^2 dx \right\}$$

### III. DISCRETE MODEL IN ONE DIMENSIONAL CASE

We assume here that  $\Phi > 0$  on  $[0, 1]$  and  $a > 0$ . For  $h = \frac{1}{N}$  we consider the following density defined for every  $(t, e) \in (0, 1) \times \mathbf{R}$  by:

$$W_h(t, e) := \begin{cases} e^2 & \text{if } e \leq e_h(t) \\ \frac{1}{h} & \text{if } e > e_h(t), \end{cases}$$

where  $e_h(t) = \frac{a}{h\Phi(t)}$  and set  $\tilde{W}_h(t, e) = W_h(ih, e)$  if  $t \in [ih, (i+1)h[$ .

Let  $\mathcal{A}_h(0, 1)$  denote the set of all the continuous functions in  $(0, 1)$ , affine on each interval  $(ih, (i+1)h)$  of  $(0, 1)$  and bounded by  $\|g\|_{\infty}$ . We define the functional:

$$F_h(v) := \begin{cases} \int_0^1 \tilde{W}_h(t, v'(t)) dt & \text{if } v \in \mathcal{A}_h(0, 1), \\ +\infty & \text{if } v \in L^1(0, 1) \setminus \mathcal{A}_h(0, 1). \end{cases}$$

Our objective is to establish the  $\Gamma$ -convergence of the functional  $v \mapsto E_h(v) := F_h(v) + \int_0^1 |g(ih) - v|^2 dx$  to the energy functional  $E$  when  $L^1(0, 1)$  is equipped with its strong

topology. In order to make relevant our variational approximating scheme, we must establish the following compactness result whose proof is a straightforward consequence of the behavior of  $W_h$  and of the compactness of the embedding  $W^{1,1}(0, 1) \subset L^1(0, 1)$ .

*Proposition 1:* Let  $(u_h)_{h>0}$  be a sequence of  $L^1(0, 1)$  satisfying

$$E_h(u_h) = \inf \{ E_h(v) : v \in L^1(0, 1) \}.$$

Then, there exist a subsequence (not relabeled) and  $u \in L^1(0, 1)$  such that  $u_h \rightarrow u$  strongly in  $L^1(0, 1)$ .

Existence of  $u_h$  above is obtained by arguments similar to those of the proof of Theorem 3. We now establish the  $\Gamma$ -convergence of  $E_h$  to  $E$ , a notion of convergence introduced by De Giorgi and Franzoni. For overview, we refer the reader to [3] and references therein.

*Theorem 4:* Let  $u \in L^1(0, 1)$ . Then the sequence  $(E_h)_{h>0}$   $\Gamma$ -converges to  $E$ , i.e., the two following statements hold:

( $\Gamma_1$ ) there exists a sequence  $(u_h)_{h>0}$  strongly converging to  $u$  in  $L^1(0, 1)$  such that:

$$\limsup_{h \rightarrow 0} E_h(u_h) \leq E(u);$$

( $\Gamma_2$ ) for every  $u \in L^1(0, 1)$  and every sequence  $(u_h)_{h>0}$  strongly converging to  $u$ , we have:

$$E(u) \leq \liminf_{h \rightarrow 0} E_h(u_h).$$

We should point out that this convergence is variational, i.e., every cluster point of the sequence  $(u_h)_{h>0}$  given in Proposition 1 is a minimizer of  $E$ , and  $\min\{E_h(v) : v \in L^1(0, 1)\}$  converges to  $\min\{E(v) : v \in L^1(0, 1)\}$ . For a proof of ( $\Gamma_1$ ) and ( $\Gamma_2$ ), we refer the reader to [5].

### IV. DESCRIPTION OF AN ALGORITHM FOR THE DISCRETE MODEL

In this section, we would like to describe an algorithm for the computation of a solution of the discrete variational model stated in the previous section :

$$E_h(u) = \min_{v \in \mathcal{A}_h(0, 1)} \left( F_h(v) + \int_0^1 |g(ih) - v|^2 dx \right).$$

Obviously, one can rewrite the energy  $E_h$  as a functional defined in  $\mathbf{R}^{N+1}$  and the discrete problem becomes:

$$\begin{aligned} E_h(u) &= \min_{v \in \mathbf{R}^{N+1}} \left( \sum_{\{i: v_{i+1} - v_i \leq \frac{a}{\Phi(ih)}\}} \frac{|v_{i+1} - v_i|^2}{h} \right. \\ &\quad \left. + \mathcal{H}^0(\{i : v_{i+1} - v_i > \frac{a}{\Phi(ih)}\}) \right), \end{aligned}$$

where  $v = (v_0, \dots, v_N)$ .

The strategy consists now in conditioning the minimization by fixing:

a) firstly the cardinal of the set  $(\{i : v_{i+1} - v_i > \frac{a}{\Phi(ih)}\})$  that we will call *fracture set*,

b) secondly, for each fixed cardinal, the site of such fracture set. The problem may be written : find  $u$  in  $\mathbf{R}^{N+1}$  solution of:

$$\min_{k=0,\dots,N-1} \min_{F_k} \min_{v \in \mathbf{R}^{N-k}, v_{i+1}-v_i \leq \frac{\alpha}{\Phi(ih)}} \left( \sum_{i \notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + k + \sum_{i \notin F_k} h|g(ih) - v_i|^2 \right).$$

Note that we have replaced the sum  $\sum_{i=0}^N h|g(ih) - v_i|^2$  by the sum  $\sum_{i \notin F_k} h|g(ih) - v_i|^2$  in the definition of the discrete energy. The first minimization problem:

$$v \in \mathbf{R}^{N-k}, v_{i+1}-v_i \leq \frac{\alpha}{\Phi(ih)} \left( \sum_{i \notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + k + \sum_{i \notin F_k} h|g(ih) - v_i|^2 \right)$$

consists in solving the classical quadratic optimization problem

$$\min_{\mathbf{R}^{N-k}} \left( \sum_{i \notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + \sum_{i \notin F_k} h|g(ih) - v_i|^2 \right), \quad (2)$$

and to take into account the solution  $u_{k, F_k}$  if and only if its components satisfy the constraint  $v_{i+1} - v_i \leq \frac{\alpha}{\Phi(ih)}$ .

It is easily seen that (2) is equivalent to the linear problem:

$$\left( A^T A + h^2 I \right) v = h^2 \tilde{g}_h,$$

where  $I$  is the  $(N-k) \times (N-k)$  identity matrix and  $A$  is a  $(N-k) \times (N-k)$  matrix whose entries are  $-1$  or  $1$  on the diagonal  $i = j$  and  $1$  or  $0$  on the diagonal  $j = i + 1$ .

## V. NUMERICAL EXPERIMENTS

The problem is equivalent to the linear problem:

$$\begin{pmatrix} b_0 & a_0 & \cdots & 0 \\ a_0 & b_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}$$

where

$$\begin{cases} b_0 = b_{n-1} & = 1 + \frac{1}{h^2} \\ b_i & = 1 + \frac{2}{h^2} \quad \text{for } 1 \leq i \leq n-1 \\ a_i & = -\frac{1}{h^2} \quad \text{for all } i \end{cases}$$

Set :

$$\begin{cases} w_0 & = \frac{a_0}{b_0} \\ y_0 & = \frac{g_0}{b_0} \end{cases}$$

and replace  $b_1$  by  $b_1 - a_0 w_0$  and  $g_1$  by  $g_1 - a_0 y_0$ .

then, for all  $k > 1$

do :

$$\begin{cases} w_k & = \frac{a_k}{b_k - a_k w_{k-1}} \\ y_k & = \frac{g_k - a_k y_{k-1}}{b_k - a_k w_{k-1}} \end{cases}$$

and replace  $b_k$  by  $b_k - a_{k-1} w_{k-1}$  and  $g_k$  by  $g_k - a_{k-1} y_{k-1}$ .

So, we obtain the solution:

$$\begin{cases} v_n & = y_n \\ v_k & = y_k - w_k v_{k+1} \quad \text{for } 0 \leq k \leq n-1 \end{cases}$$



Fig. 1. Original image of Lena.

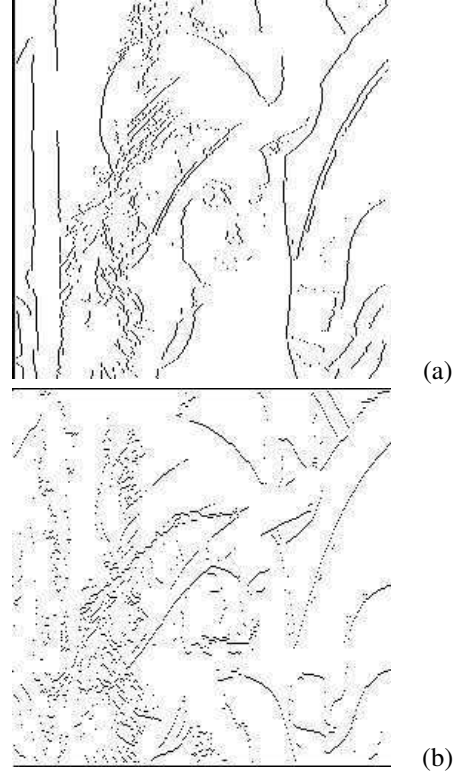


Fig. 2. Without filter, a) On line, b) On column.

To generalize, we introduce coefficients  $\alpha, \beta$  and  $\gamma$  in  $E(v)$  and put:

$$E(v) = \alpha \int_{\Omega} |\nabla v|^2 dx + \beta \int_{\Omega} |g - v|^2 dx + \gamma \int_{S_v} f d\mathcal{H}^{N-1}.$$

We now give some reconstructed pictures obtained with various values of coefficients  $\alpha, \beta$  and  $\gamma$  and various values of the number  $\frac{\alpha}{\Phi(ih)}$  that we call *filter*. For working in one dimension, we scan the picture on line and on column in order to superpose the both. We limit the computation by taking  $\#(F_k) = 1$ .

Without filter, from the original image of Lena, Fig.1, we obtain the detection on line, Fig.2.a, and on column, Fig.2.b.

By superposition of the results of the Fig.2.a and b we have a full detection illustrated Fig.3. For the on line detection, Fig.4 and 5, we have applied several values for the filter and for  $\alpha$ ,  $\beta$  and  $\gamma$ . In particular, for the Fig. 4.c, 4.d and 5, we have separated the pixels in two parts. The values of the parameters  $\alpha$ ,  $\beta$  and  $\gamma$  have been fixed experimentally.

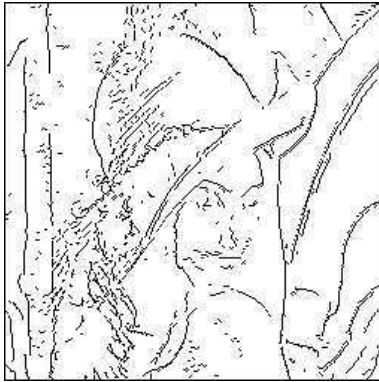


Fig. 3. By superposition of Fig.2.a and 2.b

For the on column detection, Fig.6, we have also applied several values for the filter and for  $\alpha$ ,  $\beta$  and  $\gamma$ . Finally, we illustrate Fig. 7 the superposition of the on line and on column detections.

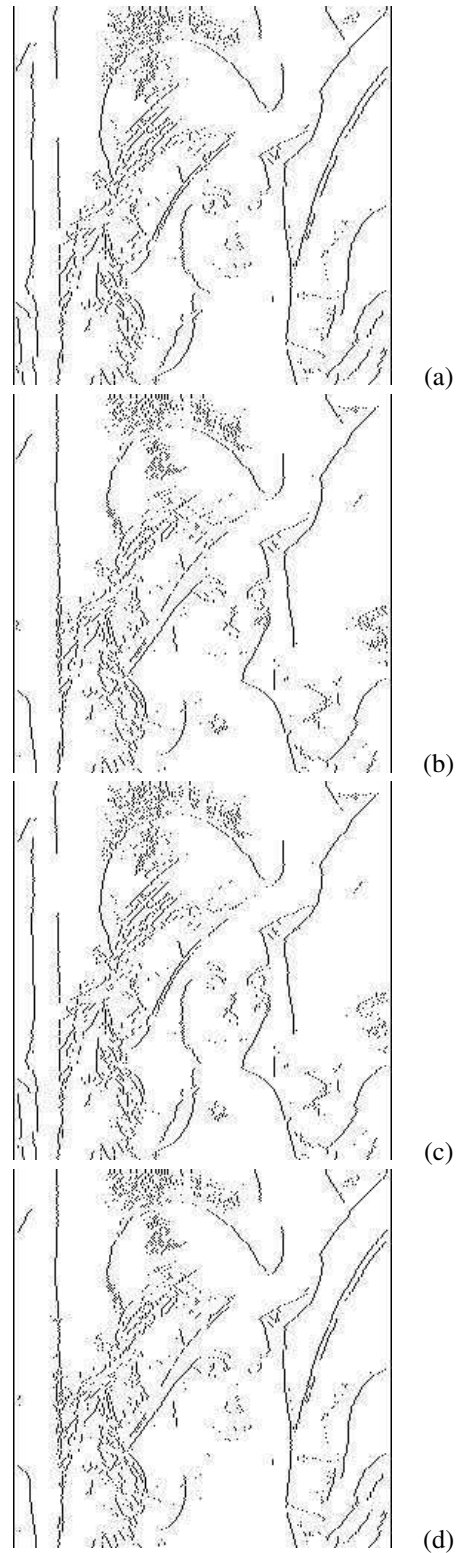


Fig. 4. On line, a) filter = 30 and  $\alpha = \beta = \gamma = 1$ , b) filter = 0.001,  $\alpha = 200, \beta = 0.01$  and  $\gamma = 0.1$ . c) With two filters: filter1 = 0.001 for pixels between 0 and 128 with  $\alpha = 200, \beta = 0.01$  and  $\gamma = 0.1$ , filter2 = 30 for pixels between 129 and 255 with  $\alpha = \beta = \gamma = 1$ , d) With two filters: filter1 = 30 for pixels between 0 and 128 with  $\alpha = \beta = \gamma = 1$ , filter2 = 0.001 for pixels between 129 and 255 with  $\alpha = 200, \beta = 0.01$  and  $\gamma = 0.1$ .

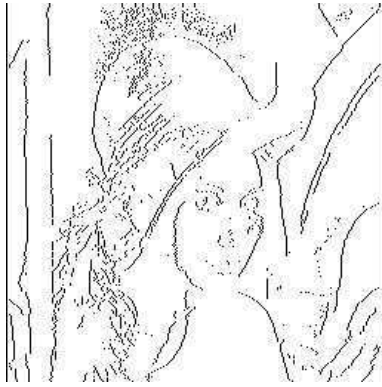


Fig. 5. On line with two filters: filter1 = 0.001 for pixels between 154 and 196 with  $\alpha = 200, \beta = 0.01$  and  $\gamma = 0.1$ , filter2 = 30 for pixels between 0 and 153, and from 197 until 255, with  $\alpha = \beta = \gamma = 1$ .

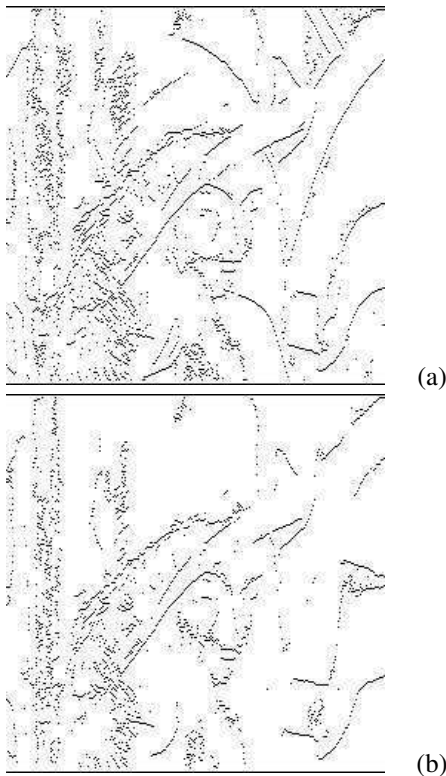


Fig. 6. On column, a) filter = 30,  $\alpha = \beta = \gamma = 1$ , b) filter = 0.001,  $\alpha = 200, \beta = 0.01$  and  $\gamma = 0.1$ .

## VI. CONCLUSION

In this paper we have generalized the well known one dimensional discretization of the Mumford-Shah functional by taking into account a constraint on the outline (i.e. the filter). We have illustrated the method by scanning a picture on line and on column in order to superpose the both. Actually, the used filters in line and column could be considered as the projections of a given two dimensional vector field on the picture. In perspective of this work, we are planning to compare our method with level set methods [7].

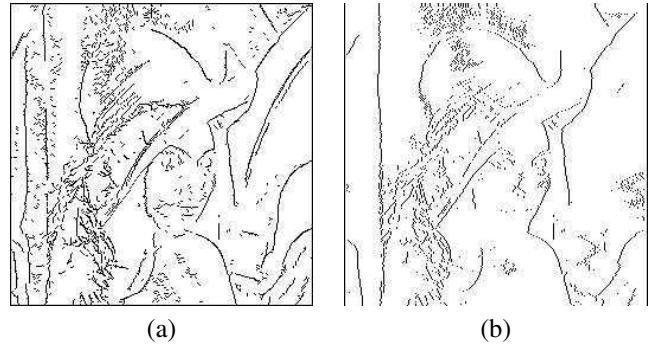


Fig. 7. a) Superposition of Fig.5 and Fig.6.a, b) Superposition of Fig.5 and Fig.6.b.

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