Variational Approximation of a Constraint Signal by a Mumford-Shah Type Energy Functional

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***Abstract*— In this paper, the main objective is to establish an existence result of a variational model in image segmentation constrained by a given vector field. In the one dimensional case, we give a discrete version converging in a variational way to the continuous model. We finally describe the numerical analysis of this model with application in image segmentation.**

***Keywords*— Image processing theory, image segmentation.**

I. Introduction

In this paper, we aim to describe a theoretical and numerical treatment of the following problem stemming from the theory of image segmentation:

$$E(u) = \inf_{v \in SBV(\Omega)} E(v),$$
$$E(v) = \int_\Omega |\nabla v|^2 dx + \int_{S_v} f \, d\mathcal{H}^{N-1} + \int_\Omega |g - v|^2 dx,$$

where Ω is a domain in $\mathbf{R}^N$ and $g : \Omega \to \mathbf{R}$.

The density f is the extended real-valued function $v \mapsto 1 + I_A(v)$ where A is the set of all the functions v in $SBV(\Omega)$ satisfying, for a given $a \in \mathbf{R}$, the condition:

$$[v](x)\nu_v(x).\Phi(x) \geq a,$$

for $\mathcal{H}^{N-1}\lfloor S_v$ a.e. x in Ω. We point out that the energy E is the perturbation of the so called *Mumford-Shah energy* [6]($f \equiv 1$) by the indicator function I_A of the set A (i.e $I_A(v) = 0$ if $v \in A$, $+\infty$ otherwise). The vector valued function Φ belonging to $\mathcal{C}(\overline{\Omega}, \mathbf{R}^N)$, may be considered, when $N = 2$, as a constraint on the outline of an image g in computer vision: the outlines of the image having a jump of grey level in the direction of Φ more than a are selected by the model. In the one dimensional case it leads to detect some selected discontinuities of the signal g.

In the paper $\mathcal{L}$ denotes the Lebesgue measure restricted to Ω, $SBV(\Omega)$ the space of all the functions v of bounded variation whose distributional gradient is of the form $Dv = \nabla v \mathcal{L} + [v]\nu_v \mathcal{H}^{N-1}\lfloor S_v$. We denote by $[v] := u^+ - u^-$ the jump of v through the jump set S_v, ν_v is the unit normal to S_v and $\mathcal{H}^{N-1}\lfloor S_v$ is the restriction to S_v of the $N-1$-dimensional Hausdorff measure. In the one dimensional case, we adopt the following notation for the structure of the distributional derivative of any v in $SBV(0,1)$: $v' = \dot{v} \; dt + [v]\mathcal{H}^0\lfloor S_v$. In this specific case, $[v](t)$ is nothing but the classical jump of v at $t \in S_v$ and $\mathcal{H}^0$ is the counting measure so that $\int [v]\mathcal{H}^0\lfloor S_v = \sum_{t \in S_v} [v]\delta_t$. For more about SBV spaces, we refer the reader to [4].

In Section 2 we establish the existence of a solution u of the problem. In Section 3 we define a variational discrete model in the case $N = 1$. Let E_h be some suitable functional expected to describe the discrete energy associated with the above functional E, h denoting the step of discretization. We say that the problem $\inf E_h$ is a *variational discrete model* of $\inf E$ if $\min E_h$ converges to $\min E$ when h goes to zero and if every minimizer u_h of E_h tends in $L^1(0,1)$ to a minimizer of E. Provided that $(u_h)_{h>0}$ be compact, an appropriate convergence for the sequence of functionals $(E_h)_{h>0}$ leading to this objective, is the so called Γ-*convergence* of the sequence $(E_h)_{h>0}$ to the functional E. Section 4 is devoted to the description of an algorithm giving a solution u_h of the discrete problem $\min E_h$. We conclude the paper by giving some numerical experiments with application in image segmentation. Previous work have all ready applied variational models to image segmentation [8], [9], [10], [2].

II. Existence of a solution

We equip $SBV(\Omega)$ with the norm $\|.\|_{SBV(\Omega)}$ defined by:

$$\|v\|_{BV(\Omega)} = \int_\Omega |v| \; dx + \int_\Omega |Dv|,$$

where $\int_\Omega |Dv|$ is the total mass of the total variation $|Dv|$ of the measure Dv.

We say that a sequence $(u_n)_{n \in \mathbf{N}}$ weakly converges to u in $BV(\Omega)$ iff $u_n \to u$ strongly *in* $L^1(\Omega)$ and $Du_n \rightharpoonup Du$ weakly in the sense of measures in $\mathcal{M}(\Omega, \mathbf{R}^N)$.

Let $(u_n)_{n \in \mathbf{N}}$ be a minimizing sequence of $\inf E(v)$.

From Ambrosio's compactness theorem (see [1], [4]), one can establish the weak lower-semi-continuity of the functional E together with existence of a weak cluster point $\bar{u}$ of $(u_n)_{n\in\mathbf{N}}$. Combining these two arguments gives $E(\bar{u}) = \inf E(v)$. We have established

Theorem 1: There exists at least one minimizer of the problem

$$\inf_{v\in A}\Big\{\int_\Omega |\nabla v|^2\ dx + \mathcal{H}^{N-1}(S_v) + \int_\Omega |g-v|^2\ dx\Big\}$$

The following compactnes theorem is due to L. Ambrosio (see [1], [4]).

Theorem 2: Let $(u_n)_n$ be a sequence of elements in $SBV(\Omega)$ satisfying

(i) $\sup_{n\in\mathbf{N}}\{\|u_n\|_{BV(\Omega)}\} < +\infty$,

(ii) the sequence $(\nabla u_n)_{n\in\mathbf{N}}$ is equi-integrable,

(iii) the sequence $(\mathcal{H}^{N-1}(S_{u_n}))_{n\in\mathbf{N}}$ is uniformly bounded.

Then one can extract a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ which converges to some $u\in SBV(\Omega)$ such that

$$\begin{array}{l} u_{n_k}\to u \text{ in } L^1_{loc}(\Omega),\\ \nabla u_{n_k}\rightharpoonup \nabla u \text{ weakly in } L^1(\Omega,\mathbf{R}^N),\\ Ju_{n_k}\rightharpoonup Ju \text{ weakly in } \mathcal{M}(\Omega,\mathbf{R}^N),\\ \mathcal{H}^{N-1}(S_u)\le \liminf_{k\longrightarrow+\infty}\mathcal{H}^{N-1}(S_{u_{n_k}}). \end{array} \tag{1}$$

Let $(u_n)_{n\in\mathbf{N}}$ be a minimizing sequence of $\inf E(v)$. From classical argument stemming from measure theory, one can establish the weak lower-semicontinuity of the functional E. On the other hand, applying Theorem 2, one can easily see that there exists a weak cluster point $\bar{u}$ of $(u_n)_{n\in\mathbf{N}}$. Combining these two arguments gives $E(\bar{u}) = \inf E(v)$. We have established

Theorem 3: There exists at least one minimizer of the problem

$$\inf_{v\in A}\Big\{\int_\Omega |\nabla v|^2\ dx + \mathcal{H}^{N-1}(S_v) + \int_\Omega |g-v|^2\ dx\Big\}$$

III. Discrete model in one dimensional case

We assume here that $\Phi > 0$ on $[0,1]$ and $a>0$. For $h=\frac{1}{N}$ we consider the following density defined for every $(t,e)\in(0,1)\times\mathbf{R}$ by:

$$W_h(t,e) := \begin{cases} e^2 \text{ if } e\le e_h(t)\\ \frac{1}{h} \text{ if } e> e_h(t),\end{cases}$$

where $e_h(t)=\frac{a}{h\Phi(t)}$ and set $\tilde{W}_h(t,e)=W_h(ih,e)$ if $t\in[ih,(i+1)h[$.

Let $\mathcal{A}_h(0,1)$ denote the set of all the continuous functions in $(0,1)$, affine on each interval $(ih,(i+1)h)$ of $(0,1)$ and bounded by $\|g\|_\infty$. We define the functional:

$$F_h(v) := \begin{cases} \displaystyle\int_0^1 \tilde{W}_h(t,v'(t))\ dt \text{ if } v\in\mathcal{A}_h(0,1),\\ +\infty \text{ if } v\in L^1(0,1)\setminus\mathcal{A}_h(0,1).\end{cases}$$

Our objective is to establish the Γ-convergence of the functional $v\mapsto E_h(v) := F_h(v)+\int_0^1 |g(ih)-v|^2\ dx$ to the energy functional E when $L^1(0,1)$ is equipped with its strong topology. In order to make relevant our variational approximating scheme, we must establish the following compactness result whose proof is a straightforward consequence of the behavior of W_h and of the compactness of the embedding $W^{1,1}(0,1)\subset L^1(0,1)$.

Proposition 1: Let $(u_h)_{h>0}$ be a sequence of $L^1(0,1)$ satisfying

$$E_h(u_h) = \inf\{E_h(v) : v\in L^1(0,1)\}.$$

Then, there exist a subsequence (not relabeled) and $u\in L^1(0,1)$ such that $u_h\to u$ strongly in $L^1(0,1)$.

Existence of u_h above is obtained by arguments similar to those of the proof of Theorem 3. We now establish the Γ-convergence of E_h to E, a notion of convergence introduced by De Giorgi and Franzoni. For overview, we refer the reader to [3] and references therein.

Theorem 4: Let $u\in L^1(0,1)$. Then the sequence $E_h)_{h>0}$ Γ-converges to E, i.e., the two following statements hold:

(Γ_1) there exists a sequence $(u_h)_{h>0}$ strongly converging to u in $L^1(0,1)$ such that:

$$\limsup_{h\to0} E_h(u_h)\le E(u);$$

(Γ_2) for every $u\in L^1(0,1)$ and every sequence $(u_h)_{h>0}$ strongly converging to u, we have:

$$E(u)\le\liminf_{h\to0} E_h(u_h).$$

We should point out that this convergence is variational, i.e., every cluster point of the sequence $(u_h)_{h>0}$ given in Proposition 1 is a minimizer of E, and $\min\{E_h(v) : v\in L^1(0,1)\}$ converges to $\min\{E(v): v\in L^1(0,1)\}$. For a proof of (Γ_1) and (Γ_2), we refer the reader to [5].

IV. Description of an algorithm for the discrete model

In this section, we would like to describe an algorithm for the computation of a solution of the discrete variational model stated in the previous section :

$$E_h(u) = \min_{v\in\mathcal{A}_h(0,1)}\left(F_h(v)+\int_0^1 |g(ih)-v|^2\ dx\right).$$

Obviously, one can rewrite the energy E_h as a functional defined in $\mathbf{R}^{N+1}$ and the discrete problem becomes:

$$\begin{aligned} E_h(u) &= \min_{v\in\mathbf{R}^{N+1}}\Big(\sum_{\{i:v_{i+1}-v_i\le\frac{a}{\Phi(ih)}\}}\frac{|v_{i+1}-v_i|^2}{h}\\ &+ \mathcal{H}^0(\{i : v_{i+1}-v_i>\frac{a}{\Phi(ih)}\})\Big),\end{aligned}$$

where $v=(v_0,\dots,v_N)$.

The strategy consists now in conditioning the minimization by fixing:

a) firstly the cardinal of the set $(\{i : v_{i+1}-v_i>\frac{a}{\Phi(ih)}\}$ that we will call *fracture set*,

b) secondly, for each fixed cardinal, the site of such fracture set. The problem may be written : find u in $\mathbf{R}^{N+1}$ solution of:

$$\min_{k=0,\ldots,N-1} \min_{F_k} \min_{v\in\mathbf{R}^{N-k},\ v_{i+1}-v_i\le\frac{\alpha}{\Phi(ih)}} \Big(\sum_{i\notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + k + \sum_{i\notin F_k} h|g(ih)-v_i|^2\Big).$$

Note that we have replaced the sum $\sum_{i=0}^{N} h|g(ih)-v_i|^2$ by the sum $\sum_{i\notin F_k} h|g(ih)-v_i|^2$ in the definition of the discrete energy. The first minimization problem:

$$\min_{v\in\mathbf{R}^{N-k},\ v_{i+1}-v_i\le\frac{\alpha}{\Phi(ih)}} \Big(\sum_{i\notin F_k} \frac{|v_{i+1}-v_i|^2}{h}+k+\sum_{i\notin F_k} h|g(ih)-v_i|^2\Big)$$

consists in solving the classical quadratic optimization problem

$$\min_{\mathbf{R}^{N-k}} \Big(\sum_{i\notin F_k} \frac{|v_{i+1}-v_i|^2}{h} + \sum_{i\notin F_k} h|g(ih)-v_i|^2\Big), \qquad (2)$$

and to take into account the solution u_{k,F_k} if and only if its components satisfy the constraint $v_{i+1}-v_i \le \frac{\alpha}{\Phi(ih)}$.

It is easily seen that (2) is equivalent to the linear problem:

$$\Big(\mathcal{A}^T\mathcal{A}+h^2I\Big)v = h^2\tilde{g}_h,$$

where I is the $(N-k)\times(N-k)$ identity matrix and $\mathcal{A}$ is a $(N-k)\times(N-k)$ matrix whose entries are -1 or 1 on the diagonal $i=j$ and 1 or 0 on the diagonal $j=i+1$.

V. Numerical experiments

The problem is equivalent to the linear problem:

$$\begin{pmatrix} b_0 & a_0 & \cdots & 0 \\ a_0 & b_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}$$

where

$$\begin{cases} b_0 = b_{n-1} & = & 1+\frac{1}{h^2} & \\ b_i & = & 1+\frac{2}{h^2} & \text{for } 1\le i\le n-1 \\ a_i & = & -\frac{1}{h^2} & \text{for all i} \end{cases}$$

Set :

$$\begin{cases} w_0 & = & \frac{a_0}{b_0} \\ y_0 & = & \frac{g_0}{b_0} \end{cases}$$

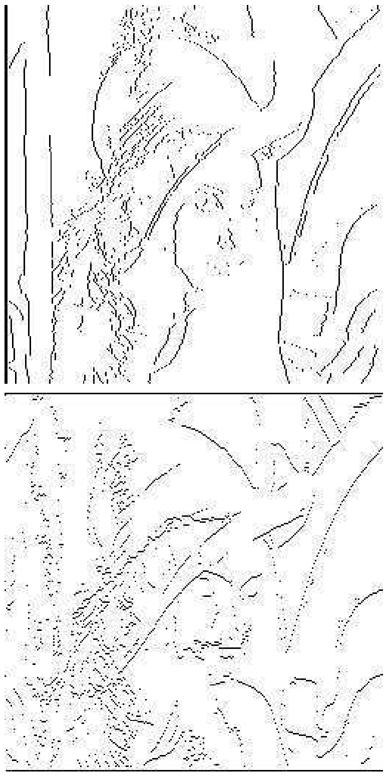
and replace b_1 by $b_1 - a_0w_0$ and g_1 by $g_1 - a_0y_0$.
then, for all $k>1$
do :

$$\begin{cases} w_k & = & \frac{a_k}{b_k-a_kw_{k-1}} \\ y_k & = & \frac{g_k-a_ky_{k-1}}{b_k-a_kw_{k-1}} \end{cases}$$

and replace b_k by $b_k - a_{k-1}w_{k-1}$ and g_k by $g_k - a_{k-1}y_{k-1}$.
So, we obtain the solution:

$$\begin{cases} v_n & = & y_n & \\ v_k & = & y_k - w_kv_k & \text{for } o\le k\le n-1 \end{cases}$$

Fig. 1. Original image of Lena.



(a)

(b)

Fig. 2. Without filter, a) On line, b) On column.

To generalize, we introduce coefficients α,β and γ in $E(v)$ and put:

$$E(v) = \alpha\int_\Omega |\nabla v|^2dx + \beta\int_\Omega |g-v|^2dx + \gamma\int_{S_v} f\ d\mathcal{H}^{N-1}.$$

We now give some reconstructed pictures obtained with various values of coefficients α,β and γ and various values of the number $\frac{a}{\Phi(ih)}$ that we call *filter*. For working in one dimension, we scan the picture on line and on column in order to superpose the both. We limit the computation by taking $\#(F_k)=1$.

Without filter, from the original image of Lena, Fig.1, we obtain the detection on line, Fig.2.a, and on column, Fig.2.b.

By superposition of the results of the Fig.2.a and b we have a full detection illustrated Fig.3. For the on line detection, Fig.4 and 5, we have applied several values for the filter and for α, β and γ. In particular, for the Fig. 4.c, 4.d and 5, we have separated the pixels in two parts. The values of the parameters α, β and γ have been fixed experimentally.

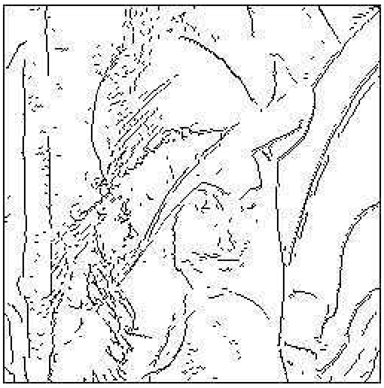
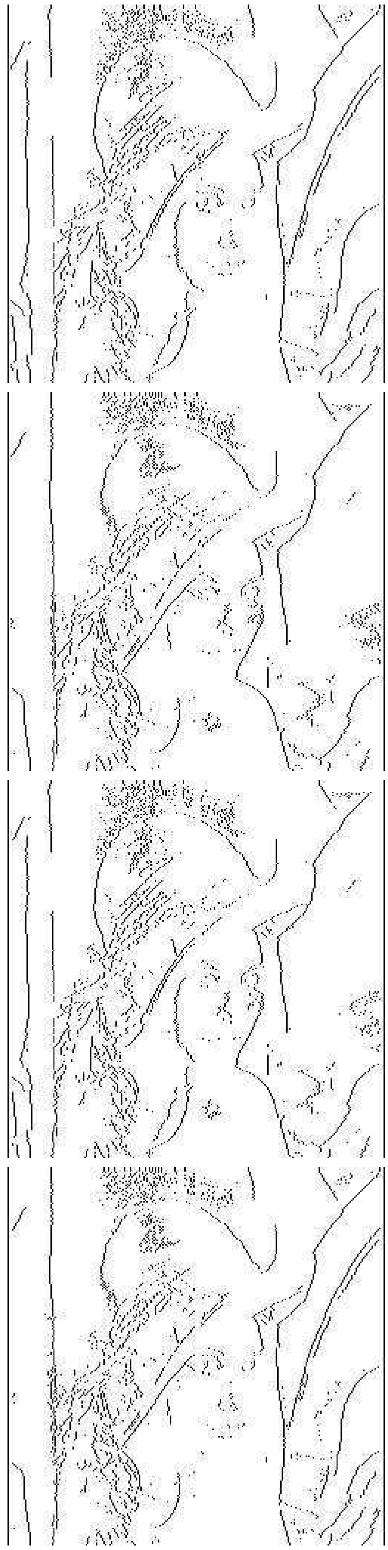


Fig. 3. By superposition of Fig.2.a and 2.b

For the on column detection, Fig.6, we have also applied several values for the filter and for α, β and γ. Finally, we illustrate Fig. 7 the superposition of the on line and on column detections.



(a)

(b)

(c)

(d)

Fig. 4. On line, a) filter = 30 and $\alpha = \beta = \gamma = 1$, b) filter = 0.001, $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$. c) With two filters: filter1 = 0.001 for pixels between 0 and 128 with $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$, filter2 = 30 for pixels between 129 and 255 with $\alpha = \beta = \gamma = 1$, d) With two filters: filter1 = 30 for pixels between 0 and 128 with $\alpha = \beta = \gamma = 1$, filter2 = 0.001 for pixels between 129 and 255 with $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$.

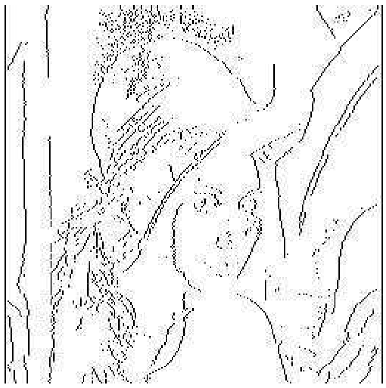
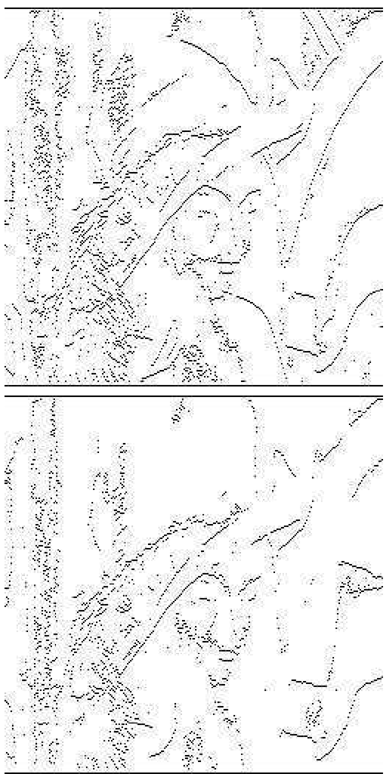


Fig. 5. On line with two filters: filter1 = 0.001 for pixels between 154 and 196 with $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$, filter2 = 30 for pixels between 0 and 153, and from 197 until 255, with $\alpha = \beta = \gamma = 1$.



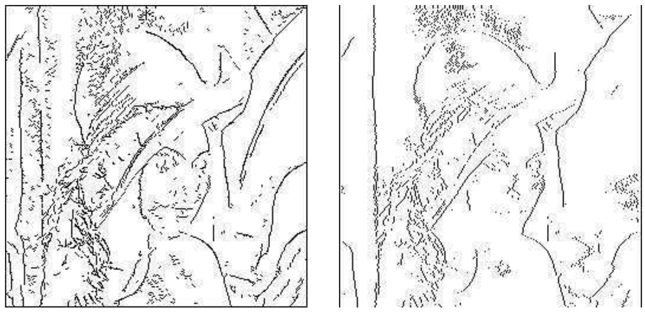
(a)

(b)

Fig. 6. On column, a) filter = 30, $\alpha = \beta = \gamma = 1$, b) filter = 0.001, $\alpha = 200, \beta = 0.01$ and $\gamma = 0.1$.

VI. Conclusion

In this paper we have generalized the well known one dimensional discretization of the Mumford-Shah functional by taking into account a constraint on the outline (i.e. the filter). We have illustrated the method by scanning a picture on line and on column in order to superpose the both. Actually, the used filters in line and column could be considered as the projections of a given two dimensional vector field on the picture. In perspective of this work, we are planning to compare our method with level set methods [7].



(a) (b)

Fig. 7. a) Superposition of Fig.5 and Fig.6.a, b) Superposition of Fig.5 and Fig.6.b.

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