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Submodular partition functions

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Abstract

Adapting the method introduced in Graph Minors X [6], we propose a new proof of the duality between the bramble-number of a graph and its tree-width. This proof is based on a new definition of submodularity on partition functions which naturally extends the usual one on set functions. The technique simplifies the proof of bramble/tree-width duality since it does not rely on Menger's theorem. One can also derive from it all known dual notions of other classical width-parameters. Finally, it provides a dual for matroid tree-width.

1 Introduction.

In their seminal paper Graph Minors X [6], Robertson and Seymour introduced the notion of branch-width of a graph and its dual notion of tangle.

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Their method is based on bias and tree-labellings. Later on, Seymour and Thomas [7] found a dual notion to tree-width, the *bramble number* (named after Reed [4]). The proof of the *bramble-number/tree-width duality* makes use of Menger's theorem to reconnect partial tree-decompositions, see for instance the textbook of Diestel [1]. Our aim in this paper is to show how the classical dual notions of width-parameters can be deduced from the original method of Graph Minors X.

In this paper, E will always denote a set with at least two elements. A *partitioning tree* on E is a tree T in which the leaves are identified with the elements of E in a one-to-one way. Therefore, every internal node v of T , if any, corresponds to the partition T_v of E which parts are the leaves of the subtrees obtained by deleting v .

An obvious way of forming a partitioning tree is simply to fix a central node which is linked to every element of E - a *partitioning star*. But what if we are not permitted to do so? Precisely, assume that a restricted set of partitions of E , called *admissible partitions*, is given. Is it possible to form an *admissible partitioning tree*? (i.e. such that every partition T_v is admissible). An obstruction to the existence of such a tree is the dual notion of *bramble*.

An *admissible bramble* is a nonempty set of pairwise intersecting subsets of E which contains a part of every admissible partition of E . It is routine to form an *admissible bramble*: just pick an element e of E , and collect, for every admissible partition, the part which contains e . Such a bramble is called *principal*. The crucial fact is that if there is a *non-principal admissible bramble* \mathcal{B} , there is no *admissible partitioning tree*. To see this, assume for contradiction that T is an *admissible partitioning tree*. For every internal node u of T , there is an element X of T_u which belongs to \mathcal{B} . Let v be the neighbour of u which belongs to the component of $T \setminus u$ having set of labels X . Orient the edge uv of T from u to v . Note that every internal node becomes the origin of an oriented edge. Observe also that an edge of T incident to a leaf never gets an orientation since \mathcal{B} is *non-principal*. The contradiction follows from the fact that one edge of T carries two orientations, which is impossible since the elements of \mathcal{B} are pairwise intersecting. Note that this argument fails when T has no internal vertex, i.e. E has two elements. In this case, the unique partitioning tree is by definition *admissible*, and every *admissible bramble* is *principal*.

Unfortunately, if no *principal admissible bramble* exists, there is not necessarily an *admissible partitioning tree*. In the first part of this paper, we prove that for some particular families of *admissible partitions* (e.g. generated by a *submodular partition function*) we have the following:

- Either there exists an *admissible partitioning tree*.

- Or there exists a non-principal admissible bramble.

The second part of the paper is devoted to the translation of this result into the different notions of width-parameters.

2 Submodular partition functions.

The *complement* of a subset X of E is the set $X^c := E \setminus X$. A *partition* of E is a set $\mathcal{X} = \{X_1, \dots, X_n\}$ of subsets of E satisfying $X_1 \cup \dots \cup X_n = E$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$. The order in which the X_i 's appear is irrelevant. We authorise degenerate partitions (i.e. the sets X_i can be empty). Let F be a subset of E . The partition

$$\mathcal{X}_{X_i \rightarrow F} := \{X_1 \cap F, \dots, X_{i-1} \cap F, X_i \cup F^c, X_{i+1} \cap F, \dots, X_n \cap F\}$$

is the partition obtained from \mathcal{X} by *pushing* X_i to F .

A *partition function* is a function Φ defined from the set of partitions of E into the reals. Let \mathcal{X} be a partition of E . We call $\Phi(\mathcal{X})$ the Φ -*width*, or simply *width*, of \mathcal{X} . Let k be an integer. A k -*partition* is a partition of width at most k . A partition function Φ is *submodular* if for every pair of partitions $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$, we have:

$$\Phi(\mathcal{X}) + \Phi(\mathcal{Y}) \geq \Phi(\mathcal{X}_{X_1 \rightarrow Y_1}) + \Phi(\mathcal{Y}_{Y_1 \rightarrow X_1})$$

To justify *a posteriori* our terminology, observe that for bipartitions, partition submodularity gives

$$\begin{aligned} \Phi(A, A^c) + \Phi(B, B^c) &= \Phi(A, A^c) + \Phi(B^c, B) \\ &\geq \Phi(A \cup (B^c)^c, A^c \cap B^c) + \Phi(B^c \cup A^c, B \cap A) \\ &\geq \Phi(A \cup B, A^c \cap B^c) + \Phi(A \cap B, A^c \cup B^c) \end{aligned}$$

which corresponds to the usual notion of submodularity when setting $\Phi(F) := \Phi(F, F^c)$, for every subset F of E .

Unfortunately, since some natural partition functions lack submodularity, we have to define a relaxed version of it. A partition function Φ is *weakly submodular* if for every pair of partitions $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$, at least one of the following holds:

- (1) There exists F such that $X_1 \subseteq F \subseteq (Y_1 \setminus X_1)^c$ and $\Phi(\mathcal{X}) > \Phi(\mathcal{X}_{X_1 \rightarrow F^c})$
- (2) $\Phi(\mathcal{Y}) \geq \Phi(\mathcal{Y}_{Y_1 \rightarrow X_1})$

Submodular partition functions are weakly submodular, it suffices to consider $F = Y_1^c$. Let us illustrate these notions. In what follows, $\mathcal{X} = \{X_1, \dots, X_n\}$ is a partition of E .

- The key-example of a submodular partition function is the function *border* defined on the set of partitions of the edge set E of a graph $G = (V, E)$ by letting $\delta(\mathcal{X}) = |\Delta(\mathcal{X})|$ where

$$\Delta(\mathcal{X}) = \{x \in V \mid \exists xy \in X_i \text{ and } \exists xz \in X_j, i \neq j\}.$$

We will often write, for a subset F of E , $\Delta(F)$ and $\delta(F)$ instead of $\Delta(F, F^c)$ and $\delta(F, F^c)$. The proof of submodularity is postponed to Section 5.1. As we will see, the function δ gives the tree-width of G .

- Let f be a submodular function on 2^E . We form a submodular partition function by letting $\Sigma_f(\mathcal{X}) = \sum_{i \in I} f(X_i)$. The proof of submodularity is postponed to Section 5.2. This function gives the tree-width of matroids.
- Let f be a symmetric submodular function on 2^E , i.e. satisfying moreover $f(A) = f(A^c)$ for all $A \subseteq E$. The function $\max_{i \in \{1, \dots, n\}} f(X_i)$, which can be made weakly submodular, gives the notion of branch-width and its relatives like rank-width. It is treated in Section 5.3.
- Let Φ be a weakly submodular partition function and $p \geq 2$ be an integer. We form a weakly submodular partition function by letting $\Phi_p(\mathcal{X}) = \Phi(\mathcal{X})$ when the number of parts of \mathcal{X} is at most p , and $+\infty$ otherwise (or any large constant integer).
- Let Φ be a weakly submodular partition function and $p \geq 2$ be an integer. By letting $\Phi'_p(\mathcal{X}) = \Phi(\mathcal{X})$ when the number of X_i with at least two elements is at most p , and $+\infty$ otherwise (or any large constant integer), we obtain a partition function which gives, in particular, the notion of path-width. This is a weakly submodular partition function if we only push subsets which are non-singletons.

Our choice of the partition submodularity condition is motivated by the analogy with usual submodular functions, when restricted to bipartitions. However, we never use the fact that X_1 and Y_1 may intersect, and could have defined the notion for disjoint X_1, Y_1 . This less constrained definition is perfectly valid for the results presented here.

3 Search-trees.

A *bidirected tree* is a directed graph obtained from an undirected tree by replacing every edge by an oriented circuit of length two. A *search-tree* T on E is a bidirected tree on at least two nodes together with a label function l defined from the arcs of T into the subsets of E with the additional requirements:

- If u is an internal node of T , the sets $l(uv)$, for all outneighbours v of u , form a partition of E . We denote it by T_u .
- The labels of a 2-circuit do not intersect, i.e. $l(uv) \cap l(vu) = \emptyset$.

A 2-circuit uv is *exact* if $l(uv) \cup l(vu) = E$. By extension, a search-tree T is *exact* if all its 2-circuits are exact. The label of an arc with origin a leaf of T is called a *leaf-label*. Let \mathcal{F} be a set of subsets of E . A search-tree T is *compatible* with \mathcal{F} if every leaf-label of T contains an element of \mathcal{F} . Let uv be a 2-circuit of T where u is an internal node. Let F be a subset such that $l(uv) \subseteq F \subseteq l(vu)^c$. The key-fact is that replacing the partition T_u in T by $(T_u)_{l(uv) \rightarrow F^c}$ (in the obvious one-to-one way) gives a new search-tree which is still compatible with \mathcal{F} since the leaf-labels are unchanged.

If Φ is a weakly submodular partition function on E , the Φ -*width* of a search-tree T with at least three nodes is the maximum of $\Phi(T_u)$, taken over the internal nodes u . If no confusion can occur, we just speak of the *width* of T . A k -*search-tree* is a search-tree with two nodes or having width at most k .

Theorem 1 *If Φ is a weakly submodular partition function and T is a k -search-tree compatible with \mathcal{F} , there is a relabelling of T which is an exact k -search-tree compatible with \mathcal{F} .*

PROOF. If T consists of a 2-circuit uv , we simply set $l(vu) := l(uv)^c$. Now, assume that amongst all relabellings of T which are k -search-trees compatible with \mathcal{F} , we minimise the sum of $\Phi(T_u)$, taken over all internal nodes u . Select an internal node r as the root of T . If T is not exact, we select a non exact 2-circuit uv , with u chosen closer to r than v . If v is a leaf, we simply replace $l(vu)$ by $l(uv)^c$. If v is an internal node, by the minimality of T , there is no F with $l(uv) \subseteq F \subseteq l(vu)^c$ for which $\Phi(T_u) > \Phi((T_u)_{l(uv) \rightarrow F^c})$. Since Φ is weakly submodular, we have $\Phi(T_v) \geq \Phi((T_v)_{l(vu) \rightarrow l(uv)})$. We then replace T_v by $(T_v)_{l(vu) \rightarrow l(uv)}$. Observe that both replacements strictly increase the sum of the sizes of the labels of backward arcs of T (those pointing toward the root). Thus this process stops on an exact k -search-tree which is still compatible with \mathcal{F} since the leaf-labels can only increase. \square

In an exact search-tree T , the set of labels of the arcs entering the leaves forms a partition of E . Therefore the union of two leaf-labels is equal to E . When this partition consists of singletons and empty sets, T is a *partitioning* k -search-tree. In the full generality of partition functions, empty sets cannot be avoided, however in all the examples given below, we can *prune* partitioning trees to remove them.

4 Tree-bramble duality.

Let Φ be a weakly submodular partition function on E . A *bias* is a nonempty family \mathcal{B} of subsets of E such that $\bigcap \mathcal{B} = \emptyset$. A *k-bramble* \mathcal{B} is a nonempty family of subsets of E such that:

- For all $X, Y \in \mathcal{B}$, we have $X \cap Y \neq \emptyset$.
- For every k -partition $\mathcal{X} = \{X_1, \dots, X_n\}$, there exists i such that $X_i \in \mathcal{B}$.

A k -bramble is *principal* if it is not a bias, i.e. $\bigcap \mathcal{B}$ is nonempty.

Theorem 2 *Let Φ be a weakly submodular partition function on a set E .*

- i. Either there exists a non-principal k -bramble.*
- ii. Or there exists a partitioning k -search-tree.*

PROOF. If there is a partitioning k -search-tree, every k -bramble is principal. The proof is given in the introduction in terms of admissible partitions. We now assume that every k -bramble is principal, and prove the existence of a partitioning k -search-tree. More generally, we show that every bias has a compatible k -search-tree. This gives our conclusion when considering the bias $\{E \setminus e \mid e \in E\}$. The proof goes by reverse induction on the inclusion order. Let \mathcal{B} be a bias. We assume that the result holds for every bias $\mathcal{B}' \neq \mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{B}'$. Two cases can happen:

- For every k -partition $\mathcal{X} = \{X_1, \dots, X_n\}$, there exists $X_i \in \mathcal{B}$. Since \mathcal{B} is not a k -bramble, it contains two disjoint subsets B_i and B_j . Thus the 2-circuit labelled by B_i and B_j is a k -search-tree which is compatible with \mathcal{B} .
- There exists a k -partition $\mathcal{X} = \{X_1, \dots, X_n\}$ such that $X_i \notin \mathcal{B}$, for all $i = 1, \dots, n$. For each X_i , we choose a subset $X'_i \notin \mathcal{B}$ which contains X_i and which is maximal with respect to inclusion. We form the bias $\mathcal{B}_i := \mathcal{B} \cup \{X'_i\}$. By the induction hypothesis and Theorem 1, there exists an exact k -search-tree T_i compatible with \mathcal{B}_i . If T_i is also compatible with \mathcal{B} , we are done. If not, T_i has a leaf-label containing X'_i and no element of \mathcal{B} . Hence, by maximality of X'_i , this leaf-label is exactly X'_i . Observe that if T_i has two leaf-labels X'_i , since their union is E , we would have $X'_i = E$ and thus T_i would also be compatible with \mathcal{B} . Consequently, X'_i appears only once as a leaf-label. We form a new tree T by identifying, for every T_i , the leaf carrying the leaf-label X'_i . The tree T is not a search-tree since the labels of the arcs with origin the identified vertex are $\{X'_1, X'_2, \dots, X'_n\}$, which is not a partition. We simply replace these labels by X_1, X_2, \dots, X_n . Now T is a k -search-tree compatible with \mathcal{B} . \square

5 Examples of submodular partition functions.

5.1 The submodular partition function δ .

Let $G = (V, E)$ be a graph. Let $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ be some partitions of E . We want to prove that:

$$\begin{aligned} \delta(\mathcal{X}) + \delta(\mathcal{Y}) &\geq \delta(\mathcal{X}_{X_1 \rightarrow Y_1}) + \delta(\mathcal{Y}_{Y_1 \rightarrow X_1}) \\ &\geq \delta(X_1 \cup Y_1^c, X_2 \cap Y_1, \dots, X_n \cap Y_1) + \\ &\quad \delta(Y_1 \cup X_1^c, Y_2 \cap X_1, \dots, Y_n \cap X_1) \end{aligned}$$

Let x be a vertex of G . Two cases can happen:

- The contribution of x in the right-hand term of the previous inequality is one, say x belongs to the border of $\mathcal{X}_{X_1 \rightarrow Y_1}$. If x belongs to the border of Y_1 , it contributes to $\delta(\mathcal{Y})$. If not, x belongs to the border of some X_i with $i > 1$. In both cases, its contribution to the left-hand term is at least one.
- Assume now that x both belongs to the borders of $\mathcal{X}_{X_1 \rightarrow Y_1}$ and $\mathcal{Y}_{Y_1 \rightarrow X_1}$. Since x belongs to the border of $\mathcal{X}_{X_1 \rightarrow Y_1}$ there is an edge e_x containing x in some $X_i \cap Y_1$ with $i > 1$. Similarly there is an edge f_x containing x in some $Y_j \cap X_1$ with $j > 1$. Since $e_x \in X_i$ and $f_x \in X_1$, x is in the border of \mathcal{X} . Similarly x is also in the border of \mathcal{Y} , and thus contributes also for two to the left-hand term.

5.2 The submodular partition function Σ_f .

Let f be a submodular function on 2^E .

Lemma 3 (1) *Let X and Y be two disjoint subsets of E . If $X_1 \subset X$ and $Y_1 \subset Y$, we have:*

$$f(X) + f(Y) - f(X_1) - f(Y_1) \geq f(X \cup Y) - f(X_1 \cup Y_1)$$

(2) *More generally, if X_1, \dots, X_r are pairwise disjoint subsets of E , and for all $i = 1, \dots, r$, $X'_i \subset X_i$, we have:*

$$\sum_{i=1}^r (f(X_i) - f(X'_i)) \geq f\left(\bigcup_{i=1}^r X_i\right) - f\left(\bigcup_{i=1}^r X'_i\right)$$

PROOF.

- (1) Apply first the submodularity of f to the subsets $A = X \cup Y_1$ and $B = Y$. Since $A \cap B = Y_1$ and $A \cup B = X \cup Y$, we obtain:

$$f(X \cup Y_1) + f(Y) \geq f(X \cup Y) + f(Y_1) \quad (1)$$

Apply then the submodularity of f to the subsets $A = X_1 \cup Y_1$ and $B = X$. Since $A \cap B = X_1$ and $A \cup B = X \cup Y_1$, we obtain:

$$f(X_1 \cup Y_1) + f(X) \geq f(X \cup Y_1) + f(X_1) \quad (2)$$

The conclusion follows from (1)+(2).

- (2) Follows by induction on r . □

Proposition 4 *The function Σ_f is a submodular partition function.*

PROOF. Let $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ be two partitions of E . We want to prove that $\Sigma_f(\mathcal{X}) + \Sigma_f(\mathcal{Y}) \geq \Sigma_f(\mathcal{X}_{X_1 \rightarrow Y_1}) + \Sigma_f(\mathcal{Y}_{Y_1 \rightarrow X_1})$. We must then prove:

$$\begin{aligned} \sum_{i=1}^n f(X_i) + \sum_{j=1}^l f(Y_j) &\geq f(X_1 \cup Y_1^c) + \sum_{i=2}^n f(Y_1 \cap X_i) \\ &\quad + f(Y_1 \cup X_1^c) + \sum_{j=2}^l f(X_1 \cap Y_j). \end{aligned} \quad (3)$$

By applying lemma 3 with $X'_i = Y_1 \cap X_i$ and since $X_2 \cup \dots \cup X_n = X_1^c$, we have:

$$\sum_{i=2}^n f(X_i) - \sum_{i=2}^n f(Y_1 \cap X_i) \geq f(X_1^c) - f(Y_1 \cap X_1^c) \quad (4)$$

Similarly we obtain:

$$\sum_{j=2}^l f(Y_j) - \sum_{j=2}^l f(X_1 \cap Y_j) \geq f(Y_1^c) - f(X_1 \cap Y_1^c) \quad (5)$$

By adding (4) and (5), we obtain

$$\begin{aligned} \sum_{j=2}^l f(Y_j) + \sum_{i=2}^n f(X_i) + f(X_1 \cap Y_1^c) + f(Y_1 \cap X_1^c) &\geq \\ f(Y_1^c) + f(X_1^c) + \sum_{j=2}^l f(X_1 \cap Y_j) + \sum_{i=2}^n f(Y_1 \cap X_i) &\end{aligned} \quad (6)$$

By applying submodularity to X_1^c and Y_1 and to X_1 and Y_1^c , we obtain:

$$f(X_1) + f(Y_1) - f(X_1 \cap Y_1^c) - f(Y_1 \cap X_1^c) \geq f(X_1 \cup Y_1^c) + f(Y_1 \cup X_1^c) - f(Y_1^c) - f(X_1^c) \quad (7)$$

Adding (6) and (7), we obtain (3). Thus Σ_f is submodular. \square

5.3 The weakly submodular partition function Max_f .

Let f be a symmetric submodular function on 2^E . Let $\mathcal{X} = \{X_1, \dots, X_n\}$ be a partition of E . The function $\max_{i \in \{1, \dots, n\}} f(X_i)$ is unfortunately not a weakly submodular partition function. We have to shift it a little to break ties. For some arbitrarily small $\varepsilon > 0$, we consider instead the function:

$$\text{Max}_f(\mathcal{X}) = \max_{i \in \{1, \dots, n\}} f(X_i) + \varepsilon \Sigma_f(\mathcal{X})$$

Lemma 5 *The function Max_f is a weakly submodular partition function.*

PROOF. Let $\mathcal{X} = \{X_1, \dots, X_n\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ be two partitions of E . Let F be a set such that

$$X_1 \subseteq F \subseteq (Y_1 \setminus X_1)^c \quad (8)$$

and chosen minimum with respect to f . Note that since X_1 satisfies (8), we have $f(F) \leq f(X_1)$. Assume that $f(F) < f(X_1)$. We claim that $\text{Max}_f(\mathcal{X}) > \text{Max}_f(\mathcal{X}_{X_1 \rightarrow F^c})$. Indeed, for every $i \geq 2$, we have by submodularity of f :

$$f(X_i) + f(F^c) \geq f(X_i \cap F^c) + f(X_i \cup F^c) \quad (9)$$

Furthermore, we have $f(F) \leq f(F \setminus X_i)$ by minimality of F , and thus by symmetry of f we get:

$$f(X_i \cup F^c) \geq f(F^c) \quad (10)$$

Adding (9) and (10), we obtain $f(X_i) \geq f(X_i \cap F^c)$. Thus the maximum of f over \mathcal{X} is at least the maximum of f over $\mathcal{X}_{X_1 \rightarrow F^c}$. We now apply the submodularity of the function Σ_f to the partitions \mathcal{X} and $\{F^c, F\}$. We then obtain $\Sigma_f(\mathcal{X}) + \Sigma_f(F^c, F) \geq \Sigma_f(\mathcal{X}_{X_1 \rightarrow F^c}) + \Sigma_f(X_1^c, X_1)$. Since $f(X_1) > f(F)$, we have $\Sigma_f(F^c, F) < \Sigma_f(X_1^c, X_1)$, hence $\Sigma_f(\mathcal{X}) > \Sigma_f(\mathcal{X}_{X_1 \rightarrow F^c})$. Therefore, $\text{Max}_f(\mathcal{X}) > \text{Max}_f(\mathcal{X}_{X_1 \rightarrow F^c})$.

Assume now that $F = X_1$ is a minimum for f . By the same calculation as above, we obtain $\text{Max}_f(\mathcal{Y}) \geq \text{Max}_f(\mathcal{Y}_{Y_1 \rightarrow X_1})$. Thus Max_f is a weakly submodular partition function. \square

6 Width parameters.

We assume in this section that the reader is somehow familiar with the usual definitions of tree-decompositions (such as tree-width, branch-width, path-width, rank-width,...). Our aim is just to associate a weakly submodular partition function to each of these parameters and show how to translate the exact partitioning k -search-tree into a tree-decomposition, and the non-principal k -bramble into the known dual notion (if any). To avoid technicalities, we assume that k is at least two and that $G = (V, E)$ is a graph with minimum degree two.

6.1 Tree-width of graphs.

The tree-width of G corresponds to the border δ defined on partitions of E .

Assume first that E has an exact partitioning k -search-tree T . Associate to every internal node u of T the bag $\Delta(T_u)$. The restriction of T to its internal nodes is a tree-decomposition of G . Indeed, for every edge xy of G , there is a leaf v of T for which $l(uv) = \{xy\}$, where uv is the arc of T with head v . Thus x and y belong to $\Delta(T_u)$, since the minimum degree in G is two. Furthermore, if a vertex of G both belongs to $\Delta(T_u)$ and $\Delta(T_v)$, it also belongs to $\Delta(T_w)$ for every node w in the (u, v) -path of T . Since every bag has size at most k , the tree-width of G is at most $k - 1$.

Now if E has a non-principal k -bramble \mathcal{B} , we form a bramble \mathcal{B}' (in the usual sense). Let S be a subset of V with $|S| \leq k$. We associate to S the partition $\{E_1, \dots, E_n\}$ of E where the sets E_i are the (nonempty) sets of edges minimal with respect to inclusion for the property $\Delta(E_i) \subseteq S$. Observe that this is indeed a partition since $\Delta(E_i \cap E_j) \subseteq \Delta(E_i) \cup \Delta(E_j) \subseteq S$. Since \mathcal{B} is a non-principal k -bramble, one of the E_i , with at least two edges, is in \mathcal{B} . This means that $X_i = V(E_i) \setminus S$ is a nonempty set of vertices. In other words, E_i is the set of edges incident to at least one vertex of X_i (such a set is denoted by $E(X_i)$). We now collect, for all subsets S with $|S| \leq k$, these sets X_i to form our \mathcal{B}' . Observe first that, by minimality of E_i , every element X_i of \mathcal{B}' induces a connected subgraph of G . We have now to prove that for every pair X_i, X_j of elements of \mathcal{B}' , $X_i \cup X_j$ also induces a connected subgraph of G . Indeed, let $E_i = E(X_i)$ and $E_j = E(X_j)$. Since the elements of \mathcal{B} are pairwise intersecting, there is an edge xy of G in $E_i \cap E_j$. Without loss of generality, we can assume that $x \in X_i$. If we also have $x \in X_j$, the sets X_i and X_j intersect, and thus their union is connected. If $x \notin X_j$, we necessarily have $y \in X_j$, hence there is an edge of G connecting X_i and X_j . Thus \mathcal{B}' is a bramble, and the minimum size covering set of \mathcal{B}' has at least $k + 1$ elements. In this case

the bramble-number of G is at least $k + 1$.

6.2 Branch-width of graphs.

The branch-width of G corresponds to the weakly submodular partition function $(\text{Max}_\delta)_3$, which counts the maximum border of a subset in a partition of E into two or three subsets. An exact partitioning k -search tree of E is precisely a branch-decomposition of G of width k . Let us make now the correspondence between a non-principal k -bramble \mathcal{B} and a tangle of G .

First of all, \mathcal{B} is here a pairwise intersecting family of subsets of E such that every k -partition $\{E_1, E_2\}$ or $\{E_1, E_2, E_3\}$ contains an element of \mathcal{B} . The translation into a tangle of order $k + 1$ is straightforward: when (G_1, G_2) is a separation of order at most k of G , we choose (G_1, G_2) in the tangle if $E(G_2) \in \mathcal{B}$, otherwise we choose (G_2, G_1) . The second axiom of tangles asserts that if (A_1, B_1) , (A_2, B_2) and (A_3, B_3) are in the tangle, we have $G \neq A_1 \cup A_2 \cup A_3$. It follows from the next proposition:

Proposition 6 *If E_1 , E_2 and E_3 are in \mathcal{B} , we have $E_1 \cap E_2 \cap E_3 \neq \emptyset$.*

PROOF. Assume for contradiction that $E_1 \cap E_2 \cap E_3 = \emptyset$. Observe that

$$\delta(E_1 \cap E_2) + \delta(E_2 \cap E_3) + \delta(E_3 \cap E_1) \leq \delta(E_1) + \delta(E_2) + \delta(E_3).$$

So we can assume without loss of generality that, for instance, $\delta(E_1 \cap E_2) \leq k$. We also have that $\delta(E_1 \setminus E_2) + \delta(E_2 \setminus E_1) \leq \delta(E_1) + \delta(E_2)$. So we can assume for instance that $\delta(E_1 \setminus E_2) \leq k$. Then the partition $\{E_1^c, E_1 \setminus E_2, E_1 \cap E_2\}$ is a k -partition. But this is impossible since these three sets are respectively disjoint from E_1, E_2 and E_3 , which all belong to \mathcal{B} . \square

The third axiom of tangles asserts that if (G_1, G_2) is a separation of G , we have $V(G_1) \neq V$. To see that, assume for contradiction that $V(G_1) = V$. We have $E(G_2) \in \mathcal{B}$, hence the number of vertices of G_2 is at most k . Thus every subset F of edges of G_2 is such that $\delta(F) \leq k$. Pick now F minimum with respect to inclusion such that $F \subseteq E(G_2)$ and $F \in \mathcal{B}$. Since \mathcal{B} is non principal, we have $|F| \geq 2$. Let $\{F_1, F_2\}$ be a non trivial partition of F . The contradiction appears when considering the k -partition $\{F^c, F_1, F_2\}$ of E since these three sets are not in \mathcal{B} .

6.3 Rank-width.

The rank-width (see Oum and Seymour [3]) of G is based on the symmetric submodular function $cutrk$ defined on subsets of vertices X where $cutrk(X)$ is the rank (in \mathbb{F}_2) of the adjacency matrix of the bipartite graph $(X, V \setminus X)$. The submodular partition function on base set V is then $cutrk_3$. The partitioning exact k -search tree is precisely a rank-decomposition of G . A non-principal k -bramble \mathcal{B} is here a pairwise intersecting family of subsets of V such that every k -partition $\{V_1, V_2\}$ or $\{V_1, V_2, V_3\}$ has an element in \mathcal{B} .

6.4 Path-width of graphs.

The path-width of $G = (V, E)$ corresponds to the partition function δ'_2 , which is the border of partitions $\{X_1, \dots, X_n\}$ of E with at most two parts with more than one element. The following analogue of Theorem 1 holds for partition functions Φ'_p , where Φ is a weakly submodular partition function, and $p \geq 2$ is some integer:

Theorem 7 *If T is a k -search-tree (with respect to Φ'_p) compatible with \mathcal{F} , there is a relabelling of a subtree of T which is an exact k -search-tree compatible with \mathcal{F} .*

PROOF. The proof is exactly the same as the one of Theorem 1 except in one case: One cannot always push, for u and v internal nodes of T , the part $l(uv)$ to $l(vu)$ in the partition T_u . Indeed, when $|l(uv)| \leq 1$, this could increase the number of parts of T_u with more than one element. In this case, we simply form a new tree T' by deleting the nodes of T which belong to the components of $T \setminus v$ not containing u . Now, v is a leaf of T' , and we set $l(vu) = l(uv)^c$. Observe that T' is still compatible with \mathcal{F} . The reason for this is that $\cap \mathcal{F} = \emptyset$, hence one of its element is included in $l(uv)^c$. \square

It follows that Theorem 2 also holds for Φ'_p , and consequently for δ'_2 . Now assume that T is a partitioning k -search-tree. Observe that we can assume that its internal labels have size at least 2, otherwise we just cut the branches as previously. This means that T is a caterpillar, i.e. a path with some attached leaves. We associate to every internal node x the bag $\Delta(T_x)$. This gives a path-decomposition of G in the usual sense.

The dual of path-width is the notion of *blockage*, introduced in [5]. Let us assume that \mathcal{B} is a non-principal k -bramble in our sense, that is a set of pairwise intersecting subsets of edges, with overall empty intersection, and

containing a part of every partition $\mathcal{X} = \{X_1, \dots, X_n\}$ with $\delta'_2(\mathcal{X}) \leq k$. We form a blockage as follows: A k -cut (V_1, V_2) is a pair of subsets of vertices with $|V_1 \cap V_2| \leq k$, $V_1 \cup V_2 = V$ and such that no edge of G joins $V_1 \setminus V_2$ to $V_2 \setminus V_1$. In a blockage \mathcal{B}' , either V_1 or V_2 must be chosen for every k -cut (V_1, V_2) , with the additional *inclusion property* that if (V_1, V_2) and (W_1, W_2) are some k -cuts with $V_1 \subseteq W_1$, then $W_1 \in \mathcal{B}'$ implies $V_1 \in \mathcal{B}'$. The construction of \mathcal{B}' is straightforward: if (V_1, V_2) is a k -cut, we let $X_1 := E(V_1) \setminus E(V_2)$, $X_2 := E(V_2) \setminus E(V_1)$, and we then list all the single edges X_3, \dots, X_n which belongs to $E(V_1 \cap V_2)$. This partition $\mathcal{X} = \{X_1, \dots, X_n\}$ of E satisfies $\delta'_2(\mathcal{X}) \leq k$. So X_1 or X_2 belongs to \mathcal{B} . If $X_1 \in \mathcal{B}$, we choose V_2 in \mathcal{B}' , otherwise we choose V_1 in \mathcal{B}' . The inclusion property follows from the fact that the elements of \mathcal{B} are pairwise intersecting.

6.5 Tree-width of matroids.

Let M be a matroid on ground set E with rank function r . We denote by r^c the submodular function such that $r^c(F) := r(F^c)$ for all subsets F of E and Φ the submodular partition function such that for any partition $\mathcal{X} = \{X_1, \dots, X_l\}$,

$$\Phi(\mathcal{X}) = \sum_{r^c}(\mathcal{X}) - (l - 1)r(E)$$

This function is submodular (\sum_{r^c} is submodular by Proposition 4) and gives the tree-width of matroids.

A *tree-decomposition* of M (see Hlířný and G. Whittle [2]) is given by a tree T whose nodes are labelled by the elements of a partition of E . A node u of T labelled F_0 thus corresponds to a partition (F_0, \dots, F_d) of E . Its *weight* is $\sum_{i=1}^d r^c(F_i) - (l - 1)r(E)$. Partitioning search-tree on E with the weight function Φ thus correspond to tree-decompositions but the converse is not true. However, a tree-decomposition can be turned into a partitioning search-tree without increasing its width. Indeed, we can safely prune the empty labelled leaves and suppose u is either an internal node with a non-empty label or a leaf whose label is not a singleton. Let F_0 be its label. Attach $|F_0|$ new leaves to u and move the elements of F_0 to these leaves. The contribution of a new leaf labelled e to the weight of u is $r^c(e) - r(E) \leq 0$ and its weight is $r(e) = 1$. Since the width of a leaf labelled F is $r(F) \geq 1$, the width of this new tree-decomposition is at most the width of the previous one.

Non-principal brambles provide a dual notion to matroid tree-width.

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