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# Submodular partition functions

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## Abstract

Adapting the method introduced in Graph Minors X [6], we propose a new proof of the duality between the bramble-number of a graph and its tree-width. This proof is based on a new definition of submodularity on partition functions which naturally extends the usual one on set functions. The technique simplifies the proof of bramble/tree-width duality since it does not rely on Menger's theorem. One can also derive from it all known dual notions of other classical width-parameters. Finally, it provides a dual for matroid tree-width.

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## 1 Introduction.

In their seminal paper Graph Minors X [6], Robertson and Seymour introduced the notion of branch-width of a graph and its dual notion of tangle.

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Their method is based on bias and tree-labellings. Later on, Seymour and Thomas [7] found a dual notion to tree-width, the *bramble number* (named after Reed [4]). The proof of the *bramble-number/tree-width duality* makes use of Menger's theorem to reconnect partial tree-decompositions, see for instance the textbook of Diestel [1]. Our aim in this paper is to show how the classical dual notions of width-parameters can be deduced from the original method of Graph Minors X.

In this paper,  $E$  will always denote a set with at least two elements. A *partitioning tree* on  $E$  is a tree  $T$  in which the leaves are identified with the elements of  $E$  in a one-to-one way. Therefore, every internal node  $v$  of  $T$ , if any, corresponds to the partition  $T_v$  of  $E$  which parts are the leaves of the subtrees obtained by deleting  $v$ .

An obvious way of forming a partitioning tree is simply to fix a central node which is linked to every element of  $E$  - a *partitioning star*. But what if we are not permitted to do so? Precisely, assume that a restricted set of partitions of  $E$ , called *admissible partitions*, is given. Is it possible to form an *admissible partitioning tree*? (i.e. such that every partition  $T_v$  is *admissible*). An obstruction to the existence of such a tree is the dual notion of *bramble*.

An *admissible bramble* is a nonempty set of pairwise intersecting subsets of  $E$  which contains a part of every *admissible partition* of  $E$ . It is routine to form an *admissible bramble*: just pick an element  $e$  of  $E$ , and collect, for every *admissible partition*, the part which contains  $e$ . Such a bramble is called *principal*. The crucial fact is that if there is a non-principal *admissible bramble*  $\mathcal{B}$ , there is no *admissible partitioning tree*. To see this, assume for contradiction that  $T$  is an *admissible partitioning tree*. For every internal node  $u$  of  $T$ , there is an element  $X$  of  $T_u$  which belongs to  $\mathcal{B}$ . Let  $v$  be the neighbour of  $u$  which belongs to the component of  $T \setminus u$  having set of labels  $X$ . Orient the edge  $uv$  of  $T$  from  $u$  to  $v$ . Note that every internal node becomes the origin of an oriented edge. Observe also that an edge of  $T$  incident to a leaf never gets an orientation since  $\mathcal{B}$  is non-principal. The contradiction follows from the fact that one edge of  $T$  carries two orientations, which is impossible since the elements of  $\mathcal{B}$  are pairwise intersecting. Note that this argument fails when  $T$  has no internal vertex, i.e.  $E$  has two elements. In this case, the unique partitioning tree is by definition *admissible*, and every *admissible bramble* is *principal*.

Unfortunately, if no *principal admissible bramble* exists, there is not necessarily an *admissible partitioning tree*. In the first part of this paper, we prove that for some particular families of *admissible partitions* (e.g. generated by a submodular partition function) we have the following:

- Either there exists an *admissible partitioning tree*.

- Or there exists a non-principal admissible bramble.

The second part of the paper is devoted to the translation of this result into the different notions of width-parameters.

## 2 Submodular partition functions.

The *complement* of a subset  $X$  of  $E$  is the set  $X^c := E \setminus X$ . A *partition* of  $E$  is a set  $\mathcal{X} = \{X_1, \dots, X_n\}$  of subsets of  $E$  satisfying  $X_1 \cup \dots \cup X_n = E$  and  $X_i \cap X_j = \emptyset$  for all  $i \neq j$ . The order in which the  $X_i$ 's appear is irrelevant. We authorise degenerate partitions (i.e. the sets  $X_i$  can be empty). Let  $F$  be a subset of  $E$ . The partition

$$\mathcal{X}_{X_i \rightarrow F} := \{X_1 \cap F, \dots, X_{i-1} \cap F, X_i \cup F^c, X_{i+1} \cap F, \dots, X_n \cap F\}$$

is the partition obtained from  $\mathcal{X}$  by *pushing*  $X_i$  to  $F$ .

A *partition function* is a function  $\Phi$  defined from the set of partitions of  $E$  into the reals. Let  $\mathcal{X}$  be a partition of  $E$ . We call  $\Phi(\mathcal{X})$  the  $\Phi$ -*width*, or simply *width*, of  $\mathcal{X}$ . Let  $k$  be an integer. A  $k$ -*partition* is a partition of width at most  $k$ . A partition function  $\Phi$  is *submodular* if for every pair of partitions  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ , we have:

$$\Phi(\mathcal{X}) + \Phi(\mathcal{Y}) \geq \Phi(\mathcal{X}_{X_1 \rightarrow Y_1}) + \Phi(\mathcal{Y}_{Y_1 \rightarrow X_1})$$

To justify *a posteriori* our terminology, observe that for bipartitions, partition submodularity gives

$$\begin{aligned} \Phi(A, A^c) + \Phi(B, B^c) &= \Phi(A, A^c) + \Phi(B^c, B) \\ &\geq \Phi(A \cup (B^c)^c, A^c \cap B^c) + \Phi(B^c \cup A^c, B \cap A) \\ &\geq \Phi(A \cup B, A^c \cap B^c) + \Phi(A \cap B, A^c \cup B^c) \end{aligned}$$

which corresponds to the usual notion of submodularity when setting  $\Phi(F) := \Phi(F, F^c)$ , for every subset  $F$  of  $E$ .

Unfortunately, since some natural partition functions lack submodularity, we have to define a relaxed version of it. A partition function  $\Phi$  is *weakly submodular* if for every pair of partitions  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_l\}$ , at least one of the following holds:

- (1) There exists  $F$  such that  $X_1 \subseteq F \subseteq (Y_1 \setminus X_1)^c$  and  $\Phi(\mathcal{X}) > \Phi(\mathcal{X}_{X_1 \rightarrow F^c})$
- (2)  $\Phi(\mathcal{Y}) \geq \Phi(\mathcal{Y}_{Y_1 \rightarrow X_1})$

Submodular partition functions are weakly submodular, it suffices to consider  $F = Y_1^c$ . Let us illustrate these notions. In what follows,  $\mathcal{X} = \{X_1, \dots, X_n\}$  is a partition of  $E$ .

- The key-example of a submodular partition function is the function *border* defined on the set of partitions of the edge set  $E$  of a graph  $G = (V, E)$  by letting  $\delta(\mathcal{X}) = |\Delta(\mathcal{X})|$  where

$$\Delta(\mathcal{X}) = \{x \in V \mid \exists xy \in X_i \text{ and } \exists xz \in X_j, i \neq j\}.$$

We will often write, for a subset  $F$  of  $E$ ,  $\Delta(F)$  and  $\delta(F)$  instead of  $\Delta(F, F^c)$  and  $\delta(F, F^c)$ . The proof of submodularity is postponed to Section 5.1. As we will see, the function  $\delta$  gives the tree-width of  $G$ .

- Let  $f$  be a submodular function on  $2^E$ . We form a submodular partition function by letting  $\Sigma_f(\mathcal{X}) = \sum_{i \in I} f(X_i)$ . The proof of submodularity is postponed to Section 5.2. This function gives the tree-width of matroids.
- Let  $f$  be a symmetric submodular function on  $2^E$ , i.e. satisfying moreover  $f(A) = f(A^c)$  for all  $A \subseteq E$ . The function  $\max_{i \in \{1, \dots, n\}} f(X_i)$ , which can be made weakly submodular, gives the notion of branch-width and its relatives like rank-width. It is treated in Section 5.3.
- Let  $\Phi$  be a weakly submodular partition function and  $p \geq 2$  be an integer. We form a weakly submodular partition function by letting  $\Phi_p(\mathcal{X}) = \Phi(\mathcal{X})$  when the number of parts of  $\mathcal{X}$  is at most  $p$ , and  $+\infty$  otherwise (or any large constant integer).
- Let  $\Phi$  be a weakly submodular partition function and  $p \geq 2$  be an integer. By letting  $\Phi'_p(\mathcal{X}) = \Phi(\mathcal{X})$  when the number of  $X_i$  with at least two elements is at most  $p$ , and  $+\infty$  otherwise (or any large constant integer), we obtain a partition function which gives, in particular, the notion of path-width. This is a weakly submodular partition function if we only push subsets which are non-singletons.

*Our choice of the partition submodularity condition is motivated by the analogy with usual submodular functions, when restricted to bipartitions. However, we never use the fact that  $X_1$  and  $Y_1$  may intersect, and could have defined the notion for disjoint  $X_1, Y_1$ . This less constrained definition is perfectly valid for the results presented here.*

### 3 Search-trees.

A *bidirected tree* is a directed graph obtained from an undirected tree by replacing every edge by an oriented circuit of length two. A *search-tree*  $T$  on  $E$  is a bidirected tree on at least two nodes together with a label function  $l$  defined from the arcs of  $T$  into the subsets of  $E$  with the additional requirements:

- If  $u$  is an internal node of  $T$ , the sets  $l(uv)$ , for all outneighbours  $v$  of  $u$ , form a partition of  $E$ . We denote it by  $T_u$ .
- The labels of a 2-circuit do not intersect, i.e.  $l(uv) \cap l(vu) = \emptyset$ .

A 2-circuit  $uv$  is *exact* if  $l(uv) \cup l(vu) = E$ . By extension, a search-tree  $T$  is *exact* if all its 2-circuits are exact. The label of an arc with origin a leaf of  $T$  is called a *leaf-label*. Let  $\mathcal{F}$  be a set of subsets of  $E$ . A search-tree  $T$  is *compatible* with  $\mathcal{F}$  if every leaf-label of  $T$  contains an element of  $\mathcal{F}$ . Let  $uv$  be a 2-circuit of  $T$  where  $u$  is an internal node. Let  $F$  be a subset such that  $l(uv) \subseteq F \subseteq l(vu)^c$ . The key-fact is that replacing the partition  $T_u$  in  $T$  by  $(T_u)_{l(uv) \rightarrow F^c}$  (in the obvious one-to-one way) gives a new search-tree which is still compatible with  $\mathcal{F}$  since the leaf-labels are unchanged.

If  $\Phi$  is a weakly submodular partition function on  $E$ , the  $\Phi$ -*width* of a search-tree  $T$  with at least three nodes is the maximum of  $\Phi(T_u)$ , taken over the internal nodes  $u$ . If no confusion can occur, we just speak of the *width* of  $T$ . A  $k$ -*search-tree* is a search-tree with two nodes or having width at most  $k$ .

**Theorem 1** *If  $\Phi$  is a weakly submodular partition function and  $T$  is a  $k$ -search-tree compatible with  $\mathcal{F}$ , there is a relabelling of  $T$  which is an exact  $k$ -search-tree compatible with  $\mathcal{F}$ .*

**PROOF.** If  $T$  consists of a 2-circuit  $uv$ , we simply set  $l(vu) := l(uv)^c$ . Now, assume that amongst all relabellings of  $T$  which are  $k$ -search-trees compatible with  $\mathcal{F}$ , we minimise the sum of  $\Phi(T_u)$ , taken over all internal nodes  $u$ . Select an internal node  $r$  as the root of  $T$ . If  $T$  is not exact, we select a non exact 2-circuit  $uv$ , with  $u$  chosen closer to  $r$  than  $v$ . If  $v$  is a leaf, we simply replace  $l(vu)$  by  $l(uv)^c$ . If  $v$  is an internal node, by the minimality of  $T$ , there is no  $F$  with  $l(uv) \subseteq F \subseteq l(vu)^c$  for which  $\Phi(T_u) > \Phi((T_u)_{l(uv) \rightarrow F^c})$ . Since  $\Phi$  is weakly submodular, we have  $\Phi(T_v) \geq \Phi((T_v)_{l(vu) \rightarrow l(uv)})$ . We then replace  $T_v$  by  $(T_v)_{l(vu) \rightarrow l(uv)}$ . Observe that both replacements strictly increase the sum of the sizes of the labels of backward arcs of  $T$  (those pointing toward the root). Thus this process stops on an exact  $k$ -search-tree which is still compatible with  $\mathcal{F}$  since the leaf-labels can only increase.  $\square$

In an exact search-tree  $T$ , the set of labels of the arcs entering the leaves forms a partition of  $E$ . Therefore the union of two leaf-labels is equal to  $E$ . When this partition consists of singletons and empty sets,  $T$  is a *partitioning*  $k$ -search-tree. In the full generality of partition functions, empty sets cannot be avoided, however in all the examples given below, we can *prune* partitioning trees to remove them.

#### 4 Tree-bramble duality.

Let  $\Phi$  be a weakly submodular partition function on  $E$ . A *bias* is a nonempty family  $\mathcal{B}$  of subsets of  $E$  such that  $\bigcap \mathcal{B} = \emptyset$ . A *k-bramble*  $\mathcal{B}$  is a nonempty family of subsets of  $E$  such that:

- For all  $X, Y \in \mathcal{B}$ , we have  $X \cap Y \neq \emptyset$ .
- For every  $k$ -partition  $\mathcal{X} = \{X_1, \dots, X_n\}$ , there exists  $i$  such that  $X_i \in \mathcal{B}$ .

A  $k$ -bramble is *principal* if it is not a bias, i.e.  $\bigcap \mathcal{B}$  is nonempty.

**Theorem 2** *Let  $\Phi$  be a weakly submodular partition function on a set  $E$ .*

- Either there exists a non-principal  $k$ -bramble.*
- Or there exists a partitioning  $k$ -search-tree.*

**PROOF.** If there is a partitioning  $k$ -search-tree, every  $k$ -bramble is principal. The proof is given in the introduction in terms of admissible partitions. We now assume that every  $k$ -bramble is principal, and prove the existence of a partitioning  $k$ -search-tree. More generally, we show that every bias has a compatible  $k$ -search-tree. This gives our conclusion when considering the bias  $\{E \setminus e \mid e \in E\}$ . The proof goes by reverse induction on the inclusion order. Let  $\mathcal{B}$  be a bias. We assume that the result holds for every bias  $\mathcal{B}' \neq \mathcal{B}$  such that  $\mathcal{B} \subseteq \mathcal{B}'$ . Two cases can happen:

- For every  $k$ -partition  $\mathcal{X} = \{X_1, \dots, X_n\}$ , there exists  $X_i \in \mathcal{B}$ . Since  $\mathcal{B}$  is not a  $k$ -bramble, it contains two disjoint subsets  $B_i$  and  $B_j$ . Thus the 2-circuit labelled by  $B_i$  and  $B_j$  is a  $k$ -search-tree which is compatible with  $\mathcal{B}$ .
- There exists a  $k$ -partition  $\mathcal{X} = \{X_1, \dots, X_n\}$  such that  $X_i \notin \mathcal{B}$ , for all  $i = 1, \dots, n$ . For each  $X_i$ , we choose a subset  $X'_i \notin \mathcal{B}$  which contains  $X_i$  and which is maximal with respect to inclusion. We form the bias  $\mathcal{B}_i := \mathcal{B} \cup \{X'_i\}$ . By the induction hypothesis and Theorem 1, there exists an exact  $k$ -search-tree  $T_i$  compatible with  $\mathcal{B}_i$ . If  $T_i$  is also compatible with  $\mathcal{B}$ , we are done. If not,  $T_i$  has a leaf-label containing  $X'_i$  and no element of  $\mathcal{B}$ . Hence, by maximality of  $X'_i$ , this leaf-label is exactly  $X'_i$ . Observe that if  $T_i$  has two leaf-labels  $X'_i$ , since their union is  $E$ , we would have  $X'_i = E$  and thus  $T_i$  would also be compatible with  $\mathcal{B}$ . Consequently,  $X'_i$  appears only once as a leaf-label. We form a new tree  $T$  by identifying, for every  $T_i$ , the leaf carrying the leaf-label  $X'_i$ . The tree  $T$  is not a search-tree since the labels of the arcs with origin the identified vertex are  $\{X'_1, X'_2, \dots, X'_n\}$ , which is not a partition. We simply replace these labels by  $X_1, X_2, \dots, X_n$ . Now  $T$  is a  $k$ -search-tree compatible with  $\mathcal{B}$ .  $\square$

## 5 Examples of submodular partition functions.

### 5.1 The submodular partition function $\delta$ .

Let  $G = (V, E)$  be a graph. Let  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_l\}$  be some partitions of  $E$ . We want to prove that:

$$\begin{aligned} \delta(\mathcal{X}) + \delta(\mathcal{Y}) &\geq \delta(\mathcal{X}_{X_1 \rightarrow Y_1}) + \delta(\mathcal{Y}_{Y_1 \rightarrow X_1}) \\ &\geq \delta(X_1 \cup Y_1^c, X_2 \cap Y_1, \dots, X_n \cap Y_1) + \\ &\quad \delta(Y_1 \cup X_1^c, Y_2 \cap X_1, \dots, Y_n \cap X_1) \end{aligned}$$

Let  $x$  be a vertex of  $G$ . Two cases can happen:

- The contribution of  $x$  in the right-hand term of the previous inequality is one, say  $x$  belongs to the border of  $\mathcal{X}_{X_1 \rightarrow Y_1}$ . If  $x$  belongs to the border of  $Y_1$ , it contributes to  $\delta(\mathcal{Y})$ . If not,  $x$  belongs to the border of some  $X_i$  with  $i > 1$ . In both cases, its contribution to the left-hand term is at least one.
- Assume now that  $x$  both belongs to the borders of  $\mathcal{X}_{X_1 \rightarrow Y_1}$  and  $\mathcal{Y}_{Y_1 \rightarrow X_1}$ . Since  $x$  belongs to the border of  $\mathcal{X}_{X_1 \rightarrow Y_1}$  there is an edge  $e_x$  containing  $x$  in some  $X_i \cap Y_1$  with  $i > 1$ . Similarly there is an edge  $f_x$  containing  $x$  in some  $Y_j \cap X_1$  with  $j > 1$ . Since  $e_x \in X_i$  and  $f_x \in X_1$ ,  $x$  is in the border of  $\mathcal{X}$ . Similarly  $x$  is also in the border of  $\mathcal{Y}$ , and thus contributes also for two to the left-hand term.

### 5.2 The submodular partition function $\Sigma_f$ .

Let  $f$  be a submodular function on  $2^E$ .

**Lemma 3** (1) Let  $X$  and  $Y$  be two disjoint subsets of  $E$ . If  $X_1 \subset X$  and  $Y_1 \subset Y$ , we have:

$$f(X) + f(Y) - f(X_1) - f(Y_1) \geq f(X \cup Y) - f(X_1 \cup Y_1)$$

(2) More generally, if  $X_1, \dots, X_r$  are pairwise disjoint subsets of  $E$ , and for all  $i = 1, \dots, r$ ,  $X'_i \subset X_i$ , we have:

$$\sum_{i=1}^r (f(X_i) - f(X'_i)) \geq f\left(\bigcup_{i=1}^r X_i\right) - f\left(\bigcup_{i=1}^r X'_i\right)$$

**PROOF.**



- (1) Apply first the submodularity of  $f$  to the subsets  $A = X \cup Y_1$  and  $B = Y$ . Since  $A \cap B = Y_1$  and  $A \cup B = X \cup Y$ , we obtain:

$$f(X \cup Y_1) + f(Y) \geq f(X \cup Y) + f(Y_1) \quad (1)$$

Apply then the submodularity of  $f$  to the subsets  $A = X_1 \cup Y_1$  and  $B = X$ . Since  $A \cap B = X_1$  and  $A \cup B = X \cup Y_1$ , we obtain:

$$f(X_1 \cup Y_1) + f(X) \geq f(X \cup Y_1) + f(X_1) \quad (2)$$

The conclusion follows from (1)+(2).

- (2) Follows by induction on  $r$ . □

**Proposition 4** *The function  $\Sigma_f$  is a submodular partition function.*

**PROOF.** Let  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_l\}$  be two partitions of  $E$ . We want to prove that  $\Sigma_f(\mathcal{X}) + \Sigma_f(\mathcal{Y}) \geq \Sigma_f(\mathcal{X}_{X_1 \rightarrow Y_1}) + \Sigma_f(\mathcal{Y}_{Y_1 \rightarrow X_1})$ . We must then prove:

$$\begin{aligned} \sum_{i=1}^n f(X_i) + \sum_{j=1}^l f(Y_j) &\geq f(X_1 \cup Y_1^c) + \sum_{i=2}^n f(Y_1 \cap X_i) \\ &\quad + f(Y_1 \cup X_1^c) + \sum_{j=2}^l f(X_1 \cap Y_j). \end{aligned} \quad (3)$$

By applying lemma 3 with  $X'_i = Y_1 \cap X_i$  and since  $X_2 \cup \dots \cup X_n = X_1^c$ , we have:

$$\sum_{i=2}^n f(X_i) - \sum_{i=2}^n f(Y_1 \cap X_i) \geq f(X_1^c) - f(Y_1 \cap X_1^c) \quad (4)$$

Similarly we obtain:

$$\sum_{j=2}^l f(Y_j) - \sum_{j=2}^l f(X_1 \cap Y_j) \geq f(Y_1^c) - f(X_1 \cap Y_1^c) \quad (5)$$

By adding (4) and (5), we obtain

$$\begin{aligned} \sum_{j=2}^l f(Y_j) + \sum_{i=2}^n f(X_i) + f(X_1 \cap Y_1^c) + f(Y_1 \cap X_1^c) &\geq \\ f(Y_1^c) + f(X_1^c) + \sum_{j=2}^l f(X_1 \cap Y_j) + \sum_{i=2}^n f(Y_1 \cap X_i) \end{aligned} \quad (6)$$

By applying submodularity to  $X_1^c$  and  $Y_1$  and to  $X_1$  and  $Y_1^c$ , we obtain:

$$f(X_1) + f(Y_1) - f(X_1 \cap Y_1^c) - f(Y_1 \cap X_1^c) \geq f(X_1 \cup Y_1^c) + f(Y_1 \cup X_1^c) - f(Y_1^c) - f(X_1^c) \quad (7)$$

Adding (6) and (7), we obtain (3). Thus  $\Sigma_f$  is submodular.  $\square$

### 5.3 The weakly submodular partition function $\text{Max}_f$ .

Let  $f$  be a symmetric submodular function on  $2^E$ . Let  $\mathcal{X} = \{X_1, \dots, X_n\}$  be a partition of  $E$ . The function  $\max_{i \in \{1, \dots, n\}} f(X_i)$  is unfortunately not a weakly submodular partition function. We have to shift it a little to break ties. For some arbitrarily small  $\varepsilon > 0$ , we consider instead the function:

$$\text{Max}_f(\mathcal{X}) = \max_{i \in \{1, \dots, n\}} f(X_i) + \varepsilon \Sigma_f(\mathcal{X})$$

**Lemma 5** *The function  $\text{Max}_f$  is a weakly submodular partition function.*

**PROOF.** Let  $\mathcal{X} = \{X_1, \dots, X_n\}$  and  $\mathcal{Y} = \{Y_1, \dots, Y_l\}$  be two partitions of  $E$ . Let  $F$  be a set such that

$$X_1 \subseteq F \subseteq (Y_1 \setminus X_1)^c \quad (8)$$

and chosen minimum with respect to  $f$ . Note that since  $X_1$  satisfies (8), we have  $f(F) \leq f(X_1)$ . Assume that  $f(F) < f(X_1)$ . We claim that  $\text{Max}_f(\mathcal{X}) > \text{Max}_f(\mathcal{X}_{X_1 \rightarrow F^c})$ . Indeed, for every  $i \geq 2$ , we have by submodularity of  $f$ :

$$f(X_i) + f(F^c) \geq f(X_i \cap F^c) + f(X_i \cup F^c) \quad (9)$$

Furthermore, we have  $f(F) \leq f(F \setminus X_i)$  by minimality of  $F$ , and thus by symmetry of  $f$  we get:

$$f(X_i \cup F^c) \geq f(F^c) \quad (10)$$

Adding (9) and (10), we obtain  $f(X_i) \geq f(X_i \cap F^c)$ . Thus the maximum of  $f$  over  $\mathcal{X}$  is at least the maximum of  $f$  over  $\mathcal{X}_{X_1 \rightarrow F^c}$ . We now apply the submodularity of the function  $\Sigma_f$  to the partitions  $\mathcal{X}$  and  $\{F^c, F\}$ . We then obtain  $\Sigma_f(\mathcal{X}) + \Sigma_f(F^c, F) \geq \Sigma_f(\mathcal{X}_{X_1 \rightarrow F^c}) + \Sigma_f(X_1^c, X_1)$ . Since  $f(X_1) > f(F)$ , we have  $\Sigma_f(F^c, F) < \Sigma_f(X_1^c, X_1)$ , hence  $\Sigma_f(\mathcal{X}) > \Sigma_f(\mathcal{X}_{X_1 \rightarrow F^c})$ . Therefore,  $\text{Max}_f(\mathcal{X}) > \text{Max}_f(\mathcal{X}_{X_1 \rightarrow F^c})$ .

Assume now that  $F = X_1$  is a minimum for  $f$ . By the same calculation as above, we obtain  $\text{Max}_f(\mathcal{Y}) \geq \text{Max}_f(\mathcal{Y}_{Y_1 \rightarrow X_1})$ . Thus  $\text{Max}_f$  is a weakly submodular partition function.  $\square$

## 6 Width parameters.

We assume in this section that the reader is somehow familiar with the usual definitions of tree-decompositions (such as tree-width, branch-width, path-width, rank-width,...). Our aim is just to associate a weakly submodular partition function to each of these parameters and show how to translate the exact partitioning  $k$ -search-tree into a tree-decomposition, and the non-principal  $k$ -bramble into the known dual notion (if any). To avoid technicalities, we assume that  $k$  is at least two and that  $G = (V, E)$  is a graph with minimum degree two.

### 6.1 Tree-width of graphs.

The tree-width of  $G$  corresponds to the border  $\delta$  defined on partitions of  $E$ .

Assume first that  $E$  has an exact partitioning  $k$ -search-tree  $T$ . Associate to every internal node  $u$  of  $T$  the bag  $\Delta(T_u)$ . The restriction of  $T$  to its internal nodes is a tree-decomposition of  $G$ . Indeed, for every edge  $xy$  of  $G$ , there is a leaf  $v$  of  $T$  for which  $l(uv) = \{xy\}$ , where  $uv$  is the arc of  $T$  with head  $v$ . Thus  $x$  and  $y$  belong to  $\Delta(T_u)$ , since the minimum degree in  $G$  is two. Furthermore, if a vertex of  $G$  both belongs to  $\Delta(T_u)$  and  $\Delta(T_v)$ , it also belongs to  $\Delta(T_w)$  for every node  $w$  in the  $(u, v)$ -path of  $T$ . Since every bag has size at most  $k$ , the tree-width of  $G$  is at most  $k - 1$ .

Now if  $E$  has a non-principal  $k$ -bramble  $\mathcal{B}$ , we form a bramble  $\mathcal{B}'$  (in the usual sense). Let  $S$  be a subset of  $V$  with  $|S| \leq k$ . We associate to  $S$  the partition  $\{E_1, \dots, E_n\}$  of  $E$  where the sets  $E_i$  are the (nonempty) sets of edges minimal with respect to inclusion for the property  $\Delta(E_i) \subseteq S$ . Observe that this is indeed a partition since  $\Delta(E_i \cap E_j) \subseteq \Delta(E_i) \cup \Delta(E_j) \subseteq S$ . Since  $\mathcal{B}$  is a non-principal  $k$ -bramble, one of the  $E_i$ , with at least two edges, is in  $\mathcal{B}$ . This means that  $X_i = V(E_i) \setminus S$  is a nonempty set of vertices. In other words,  $E_i$  is the set of edges incident to at least one vertex of  $X_i$  (such a set is denoted by  $E(X_i)$ ). We now collect, for all subsets  $S$  with  $|S| \leq k$ , these sets  $X_i$  to form our  $\mathcal{B}'$ . Observe first that, by minimality of  $E_i$ , every element  $X_i$  of  $\mathcal{B}'$  induces a connected subgraph of  $G$ . We have now to prove that for every pair  $X_i, X_j$  of elements of  $\mathcal{B}'$ ,  $X_i \cup X_j$  also induces a connected subgraph of  $G$ . Indeed, let  $E_i = E(X_i)$  and  $E_j = E(X_j)$ . Since the elements of  $\mathcal{B}$  are pairwise intersecting, there is an edge  $xy$  of  $G$  in  $E_i \cap E_j$ . Without loss of generality, we can assume that  $x \in X_i$ . If we also have  $x \in X_j$ , the sets  $X_i$  and  $X_j$  intersect, and thus their union is connected. If  $x \notin X_j$ , we necessarily have  $y \in X_j$ , hence there is an edge of  $G$  connecting  $X_i$  and  $X_j$ . Thus  $\mathcal{B}'$  is a bramble, and the minimum size covering set of  $\mathcal{B}'$  has at least  $k + 1$  elements. In this case

the bramble-number of  $G$  is at least  $k + 1$ .

## 6.2 Branch-width of graphs.

The branch-width of  $G$  corresponds to the weakly submodular partition function  $(\text{Max}_\delta)_3$ , which counts the maximum border of a subset in a partition of  $E$  into two or three subsets. An exact partitioning  $k$ -search tree of  $E$  is precisely a branch-decomposition of  $G$  of width  $k$ . Let us make now the correspondence between a non-principal  $k$ -bramble  $\mathcal{B}$  and a tangle of  $G$ .

First of all,  $\mathcal{B}$  is here a pairwise intersecting family of subsets of  $E$  such that every  $k$ -partition  $\{E_1, E_2\}$  or  $\{E_1, E_2, E_3\}$  contains an element of  $\mathcal{B}$ . The translation into a tangle of order  $k + 1$  is straightforward: when  $(G_1, G_2)$  is a separation of order at most  $k$  of  $G$ , we choose  $(G_1, G_2)$  in the tangle if  $E(G_2) \in \mathcal{B}$ , otherwise we choose  $(G_2, G_1)$ . The second axiom of tangles asserts that if  $(A_1, B_1)$ ,  $(A_2, B_2)$  and  $(A_3, B_3)$  are in the tangle, we have  $G \neq A_1 \cup A_2 \cup A_3$ . It follows from the next proposition:

**Proposition 6** *If  $E_1$ ,  $E_2$  and  $E_3$  are in  $\mathcal{B}$ , we have  $E_1 \cap E_2 \cap E_3 \neq \emptyset$ .*

**PROOF.** Assume for contradiction that  $E_1 \cap E_2 \cap E_3 = \emptyset$ . Observe that

$$\delta(E_1 \cap E_2) + \delta(E_2 \cap E_3) + \delta(E_3 \cap E_1) \leq \delta(E_1) + \delta(E_2) + \delta(E_3).$$

So we can assume without loss of generality that, for instance,  $\delta(E_1 \cap E_2) \leq k$ . We also have that  $\delta(E_1 \setminus E_2) + \delta(E_2 \setminus E_1) \leq \delta(E_1) + \delta(E_2)$ . So we can assume for instance that  $\delta(E_1 \setminus E_2) \leq k$ . Then the partition  $\{E_1^c, E_1 \setminus E_2, E_1 \cap E_2\}$  is a  $k$ -partition. But this is impossible since these three sets are respectively disjoint from  $E_1$ ,  $E_2$  and  $E_3$ , which all belong to  $\mathcal{B}$ .  $\square$

The third axiom of tangles asserts that if  $(G_1, G_2)$  is a separation of  $G$ , we have  $V(G_1) \neq V$ . To see that, assume for contradiction that  $V(G_1) = V$ . We have  $E(G_2) \in \mathcal{B}$ , hence the number of vertices of  $G_2$  is at most  $k$ . Thus every subset  $F$  of edges of  $G_2$  is such that  $\delta(F) \leq k$ . Pick now  $F$  minimum with respect to inclusion such that  $F \subseteq E(G_2)$  and  $F \in \mathcal{B}$ . Since  $\mathcal{B}$  is non principal, we have  $|F| \geq 2$ . Let  $\{F_1, F_2\}$  be a non trivial partition of  $F$ . The contradiction appears when considering the  $k$ -partition  $\{F^c, F_1, F_2\}$  of  $E$  since these three sets are not in  $\mathcal{B}$ .

### 6.3 Rank-width.

The rank-width (see Oum and Seymour [3]) of  $G$  is based on the symmetric submodular function  $cutrk$  defined on subsets of vertices  $X$  where  $cutrk(X)$  is the rank (in  $\mathbb{F}_2$ ) of the adjacency matrix of the bipartite graph  $(X, V \setminus X)$ . The submodular partition function on base set  $V$  is then  $cutrk_3$ . The partitioning exact  $k$ -search tree is precisely a rank-decomposition of  $G$ . A non-principal  $k$ -bramble  $\mathcal{B}$  is here a pairwise intersecting family of subsets of  $V$  such that every  $k$ -partition  $\{V_1, V_2\}$  or  $\{V_1, V_2, V_3\}$  has an element in  $\mathcal{B}$ .

### 6.4 Path-width of graphs.

The path-width of  $G = (V, E)$  corresponds to the partition function  $\delta'_2$ , which is the border of partitions  $\{X_1, \dots, X_n\}$  of  $E$  with at most two parts with more than one element. The following analogue of Theorem 1 holds for partition functions  $\Phi'_p$ , where  $\Phi$  is a weakly submodular partition function, and  $p \geq 2$  is some integer:

**Theorem 7** *If  $T$  is a  $k$ -search-tree (with respect to  $\Phi'_p$ ) compatible with  $\mathcal{F}$ , there is a relabelling of a subtree of  $T$  which is an exact  $k$ -search-tree compatible with  $\mathcal{F}$ .*

**PROOF.** The proof is exactly the same as the one of Theorem 1 except in one case: One cannot always push, for  $u$  and  $v$  internal nodes of  $T$ , the part  $l(uv)$  to  $l(vu)$  in the partition  $T_u$ . Indeed, when  $|l(uv)| \leq 1$ , this could increase the number of parts of  $T_u$  with more than one element. In this case, we simply form a new tree  $T'$  by deleting the nodes of  $T$  which belong to the components of  $T \setminus v$  not containing  $u$ . Now,  $v$  is a leaf of  $T'$ , and we set  $l(vu) = l(uv)^c$ . Observe that  $T'$  is still compatible with  $\mathcal{F}$ . The reason for this is that  $\cap \mathcal{F} = \emptyset$ , hence one of its element is included in  $l(uv)^c$ .  $\square$

It follows that Theorem 2 also holds for  $\Phi'_p$ , and consequently for  $\delta'_2$ . Now assume that  $T$  is a partitioning  $k$ -search-tree. Observe that we can assume that its internal labels have size at least 2, otherwise we just cut the branches as previously. This means that  $T$  is a caterpillar, i.e. a path with some attached leaves. We associate to every internal node  $x$  the bag  $\Delta(T_x)$ . This gives a path-decomposition of  $G$  in the usual sense.

The dual of path-width is the notion of *blockage*, introduced in [5]. Let us assume that  $\mathcal{B}$  is a non-principal  $k$ -bramble in our sense, that is a set of pairwise intersecting subsets of edges, with overall empty intersection, and

containing a part of every partition  $\mathcal{X} = \{X_1, \dots, X_n\}$  with  $\delta'_2(\mathcal{X}) \leq k$ . We form a blockage as follows: A  $k$ -cut  $(V_1, V_2)$  is a pair of subsets of vertices with  $|V_1 \cap V_2| \leq k$ ,  $V_1 \cup V_2 = V$  and such that no edge of  $G$  joins  $V_1 \setminus V_2$  to  $V_2 \setminus V_1$ . In a blockage  $\mathcal{B}'$ , either  $V_1$  or  $V_2$  must be chosen for every  $k$ -cut  $(V_1, V_2)$ , with the additional *inclusion property* that if  $(V_1, V_2)$  and  $(W_1, W_2)$  are some  $k$ -cuts with  $V_1 \subseteq W_1$ , then  $W_1 \in \mathcal{B}'$  implies  $V_1 \in \mathcal{B}'$ . The construction of  $\mathcal{B}'$  is straightforward: if  $(V_1, V_2)$  is a  $k$ -cut, we let  $X_1 := E(V_1) \setminus E(V_2)$ ,  $X_2 := E(V_2) \setminus E(V_1)$ , and we then list all the single edges  $X_3, \dots, X_n$  which belongs to  $E(V_1 \cap V_2)$ . This partition  $\mathcal{X} = \{X_1, \dots, X_n\}$  of  $E$  satisfies  $\delta'_2(\mathcal{X}) \leq k$ . So  $X_1$  or  $X_2$  belongs to  $\mathcal{B}$ . If  $X_1 \in \mathcal{B}$ , we choose  $V_2$  in  $\mathcal{B}'$ , otherwise we choose  $V_1$  in  $\mathcal{B}'$ . The inclusion property follows from the fact that the elements of  $\mathcal{B}$  are pairwise intersecting.

### 6.5 Tree-width of matroids.

Let  $M$  be a matroid on ground set  $E$  with rank function  $r$ . We denote by  $r^c$  the submodular function such that  $r^c(F) := r(F^c)$  for all subsets  $F$  of  $E$  and  $\Phi$  the submodular partition function such that for any partition  $\mathcal{X} = \{X_1, \dots, X_l\}$ ,

$$\Phi(\mathcal{X}) = \Sigma_{r^c}(\mathcal{X}) - (l - 1)r(E)$$

This function is submodular ( $\Sigma_{r^c}$  is submodular by Proposition 4) and gives the tree-width of matroids.

A *tree-decomposition* of  $M$  (see Hlilěný and G. Whittle [2]) is given by a tree  $T$  whose nodes are labelled by the elements of a partition of  $E$ . A node  $u$  of  $T$  labelled  $F_0$  thus corresponds to a partition  $(F_0, \dots, F_d)$  of  $E$ . Its *weight* is  $\Sigma_{i=1}^d r^c(F_i) - (l - 1)r(E)$ . Partitioning search-tree on  $E$  with the weight function  $\Phi$  thus correspond to tree-decompositions but the converse is not true. However, a tree-decomposition can be turned into a partitioning search-tree without increasing its width. Indeed, we can safely prune the empty labelled leaves and suppose  $u$  is either an internal node with a non-empty label or a leaf whose label is not a singleton. Let  $F_0$  be its label. Attach  $|F_0|$  new leaves to  $u$  and move the elements of  $F_0$  to these leaves. The contribution of a new leaf labelled  $e$  to the weight of  $u$  is  $r^c(e) - r(E) \leq 0$  and its weight is  $r(e) = 1$ . Since the width of a leaf labelled  $F$  is  $r(F) \geq 1$ , the width of this new tree-decomposition is at most the width of the previous one.

Non-principal brambles provide a dual notion to matroid tree-width.

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