Diamond-Free Circle Graphs are Helly Circle

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Abstract

The diamond is the graph obtained from K_4 by deleting an edge. Circle graphs are the intersection graphs of chords in a circle. Such a circle model has the Helly property if every three pairwise intersecting chords intersect in a single point, and a graph is Helly circle if it has a circle model with the Helly property. We show that the Helly circle graphs are the diamond-free circle graphs, as conjectured by Durán. This characterization gives an efficient recognition algorithm for Helly circle graphs.

1 Introduction

A *circle graph* is a graph which vertices can be associated to chords of a circle such that two vertices are adjacent if and only if the corresponding chords intersect.

Recently, circle graphs have received renewed attention in relation to the vertex-minor and pivot-minor relations, and rankwidth (see for instance [4] and [11]). Circle graphs indeed play a similar role with respect to vertex minor and rankwidth as planar graphs with respect to minor and treewidth. Circle graphs have been characterized by Bouchet with three excluded graphs as vertex-minor [3], and by Oum and Geelen with a finite list of excluded graphs as pivot-minor in [9]. Another characterization of circle graphs has been given by de Fraysseix [7].

In the following, we prove that a subclass of the circle graphs, namely the Helly circle graphs, are characterized with respect to circle graphs by one excluded

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induced subgraph (the diamond).

A *circle model* of a circle graph G is a function which associates to every $v \in V(G)$ a couple of endpoints of a chord in the unit circle $\mathcal{C}$. For convenience we only consider models where endpoints are pairwise disjoint.

A *sequence model* σ_G of a circle graph G is a circular sequence in which every element of $V(G)$ appears exactly twice according to the order in which we meet the chord endpoints on a clockwise walk around $\mathcal{C}$. Note that many circle models correspond to a given sequence model and that a circle graph G may have several sequence models, for example if G is disconnected.

A *subsequence* σ of σ_G, that we will denote $\sigma \subseteq \sigma_G$, is a circular sequence obtained by deleting from σ_G the two occurrences of every $v \in X$, for some subset $X \subseteq V(G)$.

A family of geometrical objects is said to have the *Helly property* if every pairwise intersecting subfamily shares a common point. Thus a circle model is *Helly* if every three pairwise intersecting chords intersect in a single point. A circle graph G is *Helly* if it has a Helly circle model. Let the *diamond* be the graph obtained from K_4 by deleting an edge. It is clear that every Helly circle graph is diamond-free. Our main result is that the converse holds, as conjectured by Durán [5] (see also [1,6]).

Theorem 1 *Every diamond-free circle graph G is a Helly circle graph.*

This characterization ensures that the complexity of Helly circle graphs recognition is at most that of circle graphs recognition. Using the $O(n^2)$ recognition algorithm of circle graphs by Spinrad [12] yields a $O(n^2)$ recognition algorithm for Helly circle graphs.

2 Proof of Theorem 1

Consider a diamond-free circle graph G and one of its sequence models σ_G. In the following, we make a slight abuse of notation in denoting (G, σ_G) by G. We prove the theorem by showing that G admits a Helly circle model. An induced subgraph H of G is *convex* if for every subsequence (a, b, c, c, b, a) of σ_G, $\{a, c\} \subseteq V(H)$ implies that $b \in V(H)$. A clique K_t is *non-trivial* if and only if $t \geq 2$. An induced subgraph H of G is *clique maximal* if every non-trivial maximal clique of H is a maximal clique of G. An induced subgraph H of G is *almost component maximal* if at most one connected component of H is not a maximal connected component of G. An induced subgraph H of G is *convenient* if it is convex, clique maximal and almost component maximal.

Given an induced subgraph H of G, we denote by σ_H the sequence model of H induced by σ_G. Given an induced subgraph H of G, a *mixed Helly model* of (G, H) is a circle model of G where the induced sequence model σ_H is a Helly circle model of H.

Lemma 1 *Consider a convenient subgraph H of G, and a vertex $u \in V(G) \setminus V(H)$. It is possible to replace the two occurrences of u in σ_G by u^+ and u^- in such a way to avoid $(u^+, x, x, u^-) \subseteq \sigma_G$ or $(u^+, x, y, u^-, x, y) \subseteq \sigma_G$ for x and $y \in V(H)$.*

PROOF. By the clique maximality of H, we do not have $(u, x, y, u, x, y) \subseteq \sigma_G$ for x and $y \in V(H)$. Now, since by convexity of H, we do not have $(u, x, x, u, y, y) \subseteq \sigma_G$ for x and $y \in V(H)$, it is easy to assign u^+ and u^- in σ_G. □

Consider a convenient subgraph H of G, such that there exists a mixed Helly model of (G, H). Such subgraph clearly exists, H could be the empty graph for example. The theorem would follow from the fact that $H = G$. We thus prove that if $H \neq G$ then there exists a convenient subgraph H' of G verifying $H \subsetneq H'$ and such that (G, H') admits a mixed Helly model. To construct such H' we need the following lemma.

Lemma 2 *Given any proper convenient subgraph H of G, there exists a vertex $u \in V(G) \setminus V(H)$ such that $G[V(H) \cup \{u\}]$ remains convex. Furthermore, if H has a component that is a proper subgraph of a component C of G, there exists such a u in C.*

PROOF. Let $\prec$ be the relation on $V(G)\setminus V(H)$ such that $u' \prec u$ if (x, u, u', u', u, x) is a subsequence of σ_G for some $x \in V(H)$. It is easy to see that $\prec$ is anti-symmetric and transitive. Clearly, for any maximal u for $\prec$, $G[V(H) \cup \{u\}]$ is convex. Moreover, if $u' \prec u$ and u' is adjacent to some $y \in V(H)$, then u is adjacent to y. Thus if H has a component that is a proper subgraph of a component C of G, $\prec$ has a maximal element in C. □

We now distinguish the case where there exists a vertex $u \in V(G) \setminus V(H)$ adjacent to some vertex $v \in V(H)$ and the case where no such vertex u exists.

2.1 None of the vertices in $V(G) \setminus V(H)$ is adjacent to some vertex of H.

By Lemma 2, let $u \in V(G) \setminus V(H)$ be such that $G[V(H) \cup \{u\}]$ is convex.

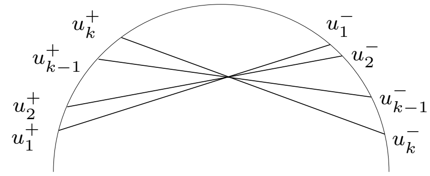

Fig. 1. Drawing the clique $\{u_1, \ldots, u_k\}$.

Lemma 3 *If $|N_G(u)| \neq 0$, there is a vertex $v \in N_G(u)$ such that $G[V(H) \cup \{v\}]$ is convex.*

PROOF. Let $v \in N(u)$ and let $v' \in V(G) \setminus V(H)$, if any, such that $v \prec v'$. Let $x \in V(H)$ such that $(x, v', v, v, v', x) \subseteq \sigma_G$. If $v' \notin N(u)$ then $(x, v', u, u, v', x) \subseteq \sigma_G$, contradicting the convexity of $G[V(H) \cup \{u\}]$. Thus a maximal element v in $N(u)$ is a maximal element in $V(G) \setminus V(H)$, and $G[V(H) \cup \{v\}]$ is convex. □

If $|N_G(u)| \neq 0$, choose a vertex $v \in N_G(u)$ such that $G[V(H) \cup \{v\}]$ is convex. Note that since G is diamond free, G contains a unique maximal clique containing the edge uv, and let us denote it K. If $|N_G(u)| = 0$, let K be the single vertex u.

Lemma 4 *The graph $H' = G[V(H) \cup K]$ is a convenient subgraph of G, and (G, H') admits a mixed Helly model.*

PROOF. Clearly H' is almost component maximal and clique maximal. We show that H' is convex. Suppose that there is a $w \in K$ is such that $G[V(H) \cup \{w\}]$ is not convex. Let $w' \in V(G) \setminus V(H)$ and $x \in V(H)$ be such that $(w, w', x, x, w', w) \subseteq \sigma_G$. By convexity of $G[V(H) \cup \{u\}]$, w' should be adjacent to u. This implies that $|N_G(u)| \neq 0$, and so K contain a vertex $v \neq u$ such that $G[V(H) \cup \{v\}]$ is convex. This implies that w' should be adjacent to v, and thus u, v, w, w' would induce a diamond, contradicting the definition of G.

Now we extend the mixed Helly model of (G, H) to a mixed Helly model of (G, H'). Let $u_1 \ldots u_k$ be the vertices in K such that $(u_1^+, \ldots u_k^+, u_1^-, \ldots, u_k^-) \subseteq \sigma_G$. Since none of the vertices u_i is adjacent to a vertex of H, there exists an open arc $\mathcal{A}$ of $\mathcal{C}$ in the circle model of H such that:

- there are no extremities of chords in $\mathcal{A}$, and such that
- we can draw the chords u_i in $\mathcal{A}$ in order to obtain a circle model of H' that corresponds to $\sigma_{H'}$.

Finally it is clear (see Figure 1) that these chords u_i can be drawn in such a way that this circle model of H' fulfills the Helly property. □

Fig. 2. Left: A vertex $u \in V(G) \setminus V(H)$ and its neighborhood in H. Right: x and y are respectively a predecessor and a successor of u.

2.2 There exists a vertex $u \in V(G) \setminus V(H)$ adjacent to some vertex of H.

By Lemma 2, let $u_0 \in V(G) \setminus V(H)$ be such that $G[V(H) \cup \{u_0\}]$ is convex. Note that by clique maximality of H, for every vertex $u \in V(G) \setminus V(H)$ (like u_0) its neighborhood in H, $N_H(u)$, induces a stable graph.

Lemma 5 *Consider any vertex $u \in V(G) \setminus V(H)$ adjacent to H such that $G[V(H) \cup \{u\}]$ is convex. Let us denote $v_1, \ldots, v_k$, with $k \geq 1$, the neighbors of u in H in such a way that $(u^+, v_1, \ldots, v_k, u^-, v_k, \ldots, v_1) \subseteq \sigma_G$ (see Figure 2). Every common neighbor x of u and v_i, with $1 \leq i \leq k$, is adjacent to exactly one vertex in $N_H(u)$, v_i, and verifies either:*

(P) $i = 1$ and $(u^+, v_1, x, u^-, v_1, x) \subseteq \sigma_G$, or
(S) $i = k$ and $(u^+, x, v_k, u^-, x, v_k) \subseteq \sigma_G$.

In the first case, we call x a predecessor *of u, while in the second case it is a* successor *of u.*

PROOF. If there was a vertex x adjacent to u, v_i and v_j, these four vertices would induce a diamond. We now prove that x could not be such that $(u^+, x, v_i, u^-, x, v_i) \subseteq \sigma_G$, for $1 \leq i < k$. The subgraph H being almost component maximal and u being in the same component of G as v_i and v_{i+1}, the vertices v_i and v_{i+1} are in the same component of H. Thus v_i has at least one neighbor $z \neq v_{i+1}$ in H. Since z is not adjacent to x (by clique maximality of H) we have that $(z, x, v_{i+1}, v_{i+1}, x, z) \subseteq \sigma_G$, contradicting H's convexity. We could similarly prove that x is not such that $(u, v_i, x, u, v_i, x) \subseteq \sigma_G$, for $1 < i \leq k$. This concludes the proof of the lemma. $\square$

Lemma 6 *Consider any vertex $u \in V(G) \setminus V(H)$ adjacent to H such that $G[V(H) \cup \{u\}]$ is convex. For every predecessor (resp. successor) x of u, $G[V(H) \cup \{x\}]$ is convex and u is a successor (resp. a predecessor) of x.*

PROOF. Assume $G[V(H) \cup \{x\}]$ is not convex. There exist $y \in V(G) \setminus V(H)$ and $z \in V(H)$ such that $(x^+, x^-, y, z, z, y) \subseteq \sigma_G$. By convexity of

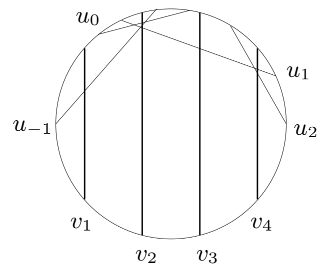

Fig. 3. A sequence $S = (u_{-1}, u_0, u_1, u_2)$ allowing to extend H

$G[V(H) \cup \{u\}]$, the vertex u is adjacent to y (otherwise we would have $(u^+, u^-, y, z, z, y) \subseteq \sigma_G$). Let us denote $v_1, \ldots, v_k$, with $k \geq 1$, the neighbors of u in H in such a way that $(u^+, v_1, \ldots, v_k, u^-, v_k, \ldots, v_1) \subseteq \sigma_G$. Recall that by definition of x, u and x have a common neighbor, say $v_1 \in V(H)$ (resp. $v_k \in V(H)$). Since u, x, y and v_1 (resp. v_k) cannot induce a diamond y and v_1 (resp. v_k) are not adjacent; but this contradicts the convexity of H (we would have (v_1, v_1, y, z, z, y) or $(v_k, v_k, y, z, z, y) \subseteq \sigma_G$). Finally it is clear by Lemma 5 that u is a successor (resp. a predecessor) of x. $\square$

We now define the vertices $u_i \in V(G) \backslash V(H)$ with $-p \leq i \leq q$, in such way that $(u_i)_{-p \leq i \leq q}$ is the longest sequence containing the previously defined element u_0 and such that u_i is a successor of u_{i-1} for every $i \in \{-p+1, \ldots, q\}$. Given the definition of u_0, Lemma 6 implies that all the vertices u_i have a neighbor in H and are such that $G[V(H) \cup \{u_i\}]$ is convex.

Then Lemma 5 allows us to define an increasing sequence $(n_i)_{-p-1 \leq i \leq q}$ and the vertices v_j, for $n_{-p-1} \leq j \leq n_q$, in such way that for every $i \in \{-p, \ldots, q\}$, the neighbors of u_i in H are exactly the vertices v_j with $n_{i-1} \leq j \leq n_i$. The lemma also implies that $(u_i^+, v_{n_{i-1}}, \ldots, v_{n_i}, u_i^-, v_{n_i}, \ldots, v_{n_{i-1}}) \subseteq \sigma_G$, and thus the vertices v_j form a stable of H such that $(v_{n_{-p-1}}, v_{1+n_{-p-1}}, \ldots, v_{n_q}, v_{n_q}, \ldots, v_{1+n_{-p-1}}, v_{n_{-p-1}}) \subseteq \sigma_G$ (see Figure 3). Finally let $H' = G[V(H) \cup \{u_i \mid -p \leq i \leq q\}]$ and let us prove that H' is convenient and admits a Helly circle model.

Lemma 7 *For every $i \in \{-p, \ldots, q\}$, the successors (resp. predecessors) of u_i are the vertices u_k such that $k > i$ and $n_{k-1} = n_i$ (resp. such that $k < i$ and $n_k = n_{i-1}$).*

PROOF. Lemma 5 implies that the relation "successor" on the set of vertices intersecting both u_i and n_{n_i} is a total order. Thus, if a successor of u_i was missing in the sequence $(u_i)_{-p \leq i \leq q}$, one could easily insert it. $\square$

Lemma 8 *H' is convenient.*

PROOF. The graph H' has as many connected components as H, thus it is almost component maximal. Let us show that H' is clique maximal. Assume by contradiction that there exists a vertex $x \in V(G) \setminus V(H')$ adjacent to both ends of an edge $ab \in E(H')$. If $ab \in E(H)$, this would contradict the clique maximality of H. If $a = u_i$ and $b = v_j$, with $-p \leq i \leq q$ and $n_{i-1} \leq j \leq n_i$, the vertex x is a successor or a predecessor of u_i (by Lemma 5) and it thus should belong to H' (by Lemma 7). If $a = u_i$ and $b = u_j$, with $-p \leq i < j \leq q$, then x is either adjacent to v_{n_i} or not. In the first case x should be a vertex u_k, and thus belong to H' (by Lemma 7). In the second case x, a, b and v_{n_i} would induce a diamond.

Finally let us show that H' is convex. By contradiction, assume there exist $x \in V(G) \setminus V(H')$ and $a, b \in V(H')$ such that $(a, a, x, b, b, x) \subseteq \sigma_G$. By convexity of H and $H \cup \{u_i\}$ for every $i \in \{-p, \ldots, q\}$, both a and b belong to $V(H') \setminus V(H)$, say $a = u_i$ and $b = u_j$, with $-p \leq i < j \leq q$. By definition of u_i and u_j, we have that $(v_{n_i}, u_i, v_{n_i}, u_i, u_j, v_{n_j}, u_j, v_{n_j}) \subseteq \sigma_G$. Thus we have either

- $(v_{n_i}, x, u_i, v_{n_i}, u_i, x, u_j, v_{n_j}, u_j, v_{n_j}) \subseteq \sigma_G$ or
- $(v_{n_i}, u_i, v_{n_i}, u_i, x, u_j, v_{n_j}, u_j, v_{n_j}, x) \subseteq \sigma_G$ or
- $(v_{n_i}, u_i, v_{n_i}, u_i, x, u_j, v_{n_j}, u_j, x, v_{n_j}) \subseteq \sigma_G$

which respectively contradicts the convexity of $H \cup \{u_i\}$, H and $H \cup \{u_j\}$. □

Lemma 9 *There is a mixed Helly model of* (G, H')*.*

PROOF. We consider the Helly circle model of H and we extend it to H'. Lemma 5 allows us to distinguish one extremity of v_j, for every j; the distinguished extremity being such that $(u_i^+, v_j^*, u_i^-, v_j) \subseteq \sigma_G$ for every vertex u_i crossing v_j. We extend the Helly circle model of H by processing the chords $u_{-p}, \ldots, u_q$ successively in this order. For any $i \in \{-p-1, \ldots, q\}$ let $H_i = G[V(H) \cup \{u_k \mid -p \leq k \leq i\}]$. At each step we extend an Helly circle model of H_{i-1} to an Helly circle model of H_i. Actually we construct them in such a way that for every i,

(*) the intersection point between u_i and v_{n_i}, lies strictly between the point $v_{n_i}^*$ and the intersection of the chord v_{n_i} and the abstract chord $[v_{n_i-1}^*, v_{n_i+1}^*]$ (see Figure 4).

Assume we have already processed the chords up to u_{i-1} (see Figure 4 Step 1). Since $\sigma_{H_{i-1}} \subseteq \sigma_{H_i}$ it is easy to draw a chord u_i which intersect the desired chords. Now we are going to slightly move this chord in order to fulfill (*) and the Helly property. If $u_i = u_{-p}$, since the neighborhood of u_{-p} in H_{-p} induces a stable, the Helly property follows immediately. So, we just have to move u_{-p}^- close enough to $v_{n_{-p}}^*$ in order to fulfill (*). This is possible since there is no chord extremity in between $v_{n_{-p}}^*$ and u_{-p}^- in $\sigma_{H_{-p}}$. If $u_i \neq u_{-p}$,

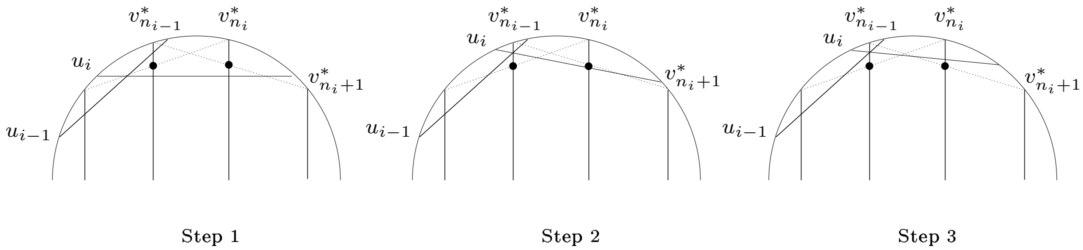

Fig. 4. Processing the chord u_i. Dashed lines are the abstract constraints.

the Helly circle model of H_{i-1} fulfills (*). This ensures that we can move the chord u_i in order to intersect u_{i-1} and $v_{n_{i-1}}$ at their intersection point (see Figure 4 Step 2). At this step, since the neighborhood of u_i in H_i induces a graph with a unique non-trivial maximal clique (the clique with vertex set $\{v_{n_{i-1}}\} \cup \{u_k \mid k < i \textit{ and } n_k = n_{i-1}\}$), the circle model of H_i fulfills the Helly property. Finally if (*) is not satisfied, we just get u_i^- close enough to $v^*_{n_i}$ by rotating u_i around the intersection point of the clique. (see Figure 4 Step 3). This concludes the proof of the theorem. □

3 Concluding Remarks

The first polynomial-time algorithms for circle graphs recognition were independently given by Bouchet [2], Naji [10] and Gabor et al. [8]. The latter was improved by Spinrad who showed that the recognition of circle graphs can be done in $O(n^2)$ time, and that a circle model can be computed in the same time bound [12]. Given a circle model of a circle graph G, the graph induced by the neighborhood of a vertex v in G is a permutation graph. Moreover a permutation model of $G[N(v)]$ can be computed in $O(n)$ if a circle model is known. One can easily check in $O(n)$ time if a permutation graph is P_3-free (*i.e.* is a disjoint union of complete graphs) using its permutation model. Thus one can check if a circle graph G given with a circle model is diamond-free in $O(n^2)$ (check for every vertex v if the permutation graph $G[N(v)]$ is P_3-free). In consequence we have:

Proposition 1 *Helly circle graphs can be recognized in time* $O(n^2)$.

Actually, the test for an induced diamond can even be carried out in linear time $O(n + m)$ with an adequate data structure. This means that the complexity of Helly circle graphs recognition is at most the complexity of circle graphs recognition.

As a byproduct of the proof, we can also treat the case of 1-string Helly circle graphs. A graph is 1-*string circle* if it is the intersection graph of strings in a disk with both endpoints on its border, such that two strings intersect at most once, and no two strings touch. Note that a graph is 1-string circle if and only if it is a circle graph.

A graph is 1-*string Helly circle* if it has a 1-string circle model where the strings have the Helly property. The proof of Theorem 1 can be reformulated in these terms: the 1-string Helly circle graphs are the Helly circle graphs.

Another aspect is the generalization in greater dimensions. A graph is a *d-dimensional sphere graph* if it is an intersection graph of $(d-1)$-dimensional disks with border on a d-dimensional sphere. We wonder whether Theorem 1 could be generalized in the following way:

Question 1 *Does every d-dimensional sphere graph that is $(K_{d+2} \setminus e)$-free admit a Helly embedding ?*

The reverse is not true unless we forbid that d disks intersect in a segment.

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