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Diamond-free Circle Graphs are Helly Circle

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\begin{abstract}
The diamond is the graph obtained from $K_4$ by deleting an edge. Circle graphs are the intersection graphs of chords in a circle. Such a circle model has the Helly property if every three pairwise intersecting chords intersect in a single point, and a graph is Helly circle if it has a circle model with the Helly property. We show that the Helly circle graphs are the diamond-free circle graphs, as conjectured by Durán. This characterization gives an efficient recognition algorithm for Helly circle graphs.
\end{abstract}

\section{Introduction}

A circle graph is a graph which vertices can be associated to chords of a circle such that two vertices are adjacent if and only if the corresponding chords intersect.

Recently, circle graphs have received renewed attention in relation to the vertex-minor and pivot-minor relations, and rankwidth (see for instance \cite{4} and \cite{11}). Circle graphs indeed play a similar role with respect to vertex minor and rankwidth as planar graphs with respect to minor and treewidth. Circle graphs have been characterized by Bouchet with three excluded graphs as vertex-minor \cite{3}, and by Oum and Geelen with a finite list of excluded graphs as pivot-minor in \cite{9}. Another characterization of circle graphs has been given by de Fraysseix \cite{7}.

In the following, we prove that a subclass of the circle graphs, namely the Helly circle graphs, are characterized with respect to circle graphs by one excluded

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induced subgraph (the diamond).

A circle model of a circle graph $G$ is a function which associates to every $v \in V(G)$ a couple of endpoints of a chord in the unit circle $C$. For convenience we only consider models where endpoints are pairwise disjoint.

A sequence model $\sigma_G$ of a circle graph $G$ is a circular sequence in which every element of $V(G)$ appears exactly twice according to the order in which we meet the chord endpoints on a clockwise walk around $C$. Note that many circle models correspond to a given sequence model and that a circle graph $G$ may have several sequence models, for example if $G$ is disconnected.

A subsequence $\sigma$ of $\sigma_G$, that we will denote $\sigma \subseteq \sigma_G$, is a circular sequence obtained by deleting from $\sigma_G$ the two occurrences of every $v \in X$, for some subset $X \subseteq V(G)$.

A family of geometrical objects is said to have the Helly property if every pairwise intersecting subfamily shares a common point. Thus a circle model is Helly if every three pairwise intersecting chords intersect in a single point. A circle graph $G$ is Helly if it has a Helly circle model. Let the diamond be the graph obtained from $K_4$ by deleting an edge. It is clear that every Helly circle graph is diamond-free. Our main result is that the converse holds, as conjectured by Durán [5] (see also [1,6]).

**Theorem 1** Every diamond-free circle graph $G$ is a Helly circle graph.

This characterization ensures that the complexity of Helly circle graphs recognition is at most that of circle graphs recognition. Using the $O(n^2)$ recognition algorithm of circle graphs by Spinrad [12] yields a $O(n^2)$ recognition algorithm for Helly circle graphs.

## 2 Proof of Theorem 1

Consider a diamond-free circle graph $G$ and one of its sequence models $\sigma_G$. In the following, we make a slight abuse of notation in denoting $(G; \sigma_G)$ by $G$. We prove the theorem by showing that $G$ admits a Helly circle model. An induced subgraph $H$ of $G$ is convex if for every subsequence $(a, b, c, c, b, a)$ of $\sigma_G$, $\{a, c\} \subseteq V(H)$ implies that $b \in V(H)$. A clique $K_t$ is non-trivial if and only if $t \geq 2$. An induced subgraph $H$ of $G$ is clique maximal if every non-trivial maximal clique of $H$ is a maximal clique of $G$. An induced subgraph $H$ of $G$ is almost component maximal if at most one connected component of $H$ is not a maximal connected component of $G$. An induced subgraph $H$ of $G$ is convenient if it is convex, clique maximal and almost component maximal.
Given an induced subgraph $H$ of $G$, we denote by $\sigma_H$ the sequence model of $H$ induced by $\sigma_G$. Given an induced subgraph $H$ of $G$, a mixed Helly model of $(G, H)$ is a circle model of $G$ where the induced sequence model $\sigma_H$ is a Helly circle model of $H$.

**Lemma 1** Consider a convenient subgraph $H$ of $G$, and a vertex $u \in V(G) \setminus V(H)$. It is possible to replace the two occurrences of $u$ in $\sigma_G$ by $u^+$ and $u^-$ in such a way to avoid $(u^+, x, x, u^-) \subseteq \sigma_G$ or $(u^+, x, y, u^-, x, y) \subseteq \sigma_G$ for $x$ and $y \in V(H)$.

**PROOF.** By the clique maximality of $H$, we do not have $(u, x, y, u, x, y) \subseteq \sigma_G$ for $x$ and $y \in V(H)$. Now, since by convexity of $H$, we do not have $(u, x, x, u, y, y) \subseteq \sigma_G$ for $x$ and $y \in V(H)$, it is easy to assign $u^+$ and $u^-$ in $\sigma_G$. □

Consider a convenient subgraph $H$ of $G$, such that there exists a mixed Helly model of $(G, H)$. Such subgraph clearly exists, $H$ could be the empty graph for example. The theorem would follow from the fact that $H = G$. We thus prove that if $H \neq G$ then there exists a convenient subgraph $H'$ of $G$ verifying $H \subseteq H'$ and such that $(G, H')$ admits a mixed Helly model. To construct such $H'$ we need the following lemma.

**Lemma 2** Given any proper convenient subgraph $H$ of $G$, there exists a vertex $u \in V(G) \setminus V(H)$ such that $G[V(H) \cup \{u\}]$ remains convex. Furthermore, if $H$ has a component that is a proper subgraph of a component $C$ of $G$, there exists such a $u$ in $C$.

**PROOF.** Let $\prec$ be the relation on $V(G) \setminus V(H)$ such that $u' \prec u$ if $(x, u, u', u', u, x)$ is a subsequence of $\sigma_G$ for some $x \in V(H)$. It is easy to see that $\prec$ is anti-symmetric and transitive. Clearly, for any maximal $u$ for $\prec$, $G[V(H) \cup \{u\}]$ is convex. Moreover, if $u' \prec u$ and $u'$ is adjacent to some $y \in V(H)$, then $u$ is adjacent to $y$. Thus if $H$ has a component that is a proper subgraph of a component $C$ of $G$, $\prec$ has a maximal element in $C$. □

We now distinguish the case where there exists a vertex $u \in V(G) \setminus V(H)$ adjacent to some vertex $v \in V(H)$ and the case where no such vertex $u$ exists.

2.1 None of the vertices in $V(G) \setminus V(H)$ is adjacent to some vertex of $H$.

By Lemma 2, let $u \in V(G) \setminus V(H)$ be such that $G[V(H) \cup \{u\}]$ is convex.
Fig. 1. Drawing the clique \( \{u_1, \ldots, u_k\} \).

**Lemma 3** If \(|N_G(u)| \neq 0\), there is a vertex \( v \in N_G(u) \) such that \( G[V(H) \cup \{v\}] \) is convex.

**Proof.** Let \( v \in N(u) \) and let \( v' \in V(G) \setminus V(H) \), if any, such that \( v < v' \). Let \( x \in V(H) \) such that \( (x, v', v, v', x) \subseteq \sigma_G \). If \( v' \notin N(u) \) then \( (x, v', u, u, v', x) \subseteq \sigma_G \), contradicting the convexity of \( G[V(H) \cup \{u\}] \). Thus a maximal element \( v \) in \( N(u) \) is a maximal element in \( V(G) \setminus V(H) \), and \( G[V(H) \cup \{v\}] \) is convex. \( \square \)

If \(|N_G(u)| \neq 0\), choose a vertex \( v \in N_G(u) \) such that \( G[V(H) \cup \{v\}] \) is convex. Note that since \( G \) is diamond free, \( G \) contains a unique maximal clique containing the edge \( uv \), and let us denote it \( K \). If \(|N_G(u)| = 0\), let \( K \) be the single vertex \( u \).

**Lemma 4** The graph \( H' = G[V(H) \cup K] \) is a convenient subgraph of \( G \), and \((G, H')\) admits a mixed Helly model.

**Proof.** Clearly \( H' \) is almost component maximal and clique maximal. We show that \( H' \) is convex. Suppose that there is a \( w \in K \) is such that \( G[V(H) \cup \{w\}] \) is not convex. Let \( w' \in V(G) \setminus V(H) \) and \( x \in V(H) \) be such that \( (w, w', x, x, w', w) \subseteq \sigma_G \). By convexity of \( G[V(H) \cup \{u\}] \), \( w' \) should be adjacent to \( u \). This implies that \(|N_G(u)| \neq 0\), and so \( K \) contain a vertex \( v \neq u \) such that \( G[V(H) \cup \{v\}] \) is convex. This implies that \( w' \) should be adjacent to \( v \), and thus \( u, v, w, w' \) would induce a diamond, contradicting the definition of \( G \).

Now we extend the mixed Helly model of \((G, H)\) to a mixed Helly model of \((G, H')\). Let \( u_1 \ldots u_k \) be the vertices in \( K \) such that \((u_1^+, \ldots, u_k^+, u_1^-, \ldots, u_k^-) \subseteq \sigma_G \). Since none of the vertices \( u_i \) is adjacent to a vertex of \( H \), there exists an open arc \( A \) of \( C \) in the circle model of \( H \) such that:

- there are no extremities of chords in \( A \), and such that
- we can draw the chords \( u_i \) in \( A \) in order to obtain a circle model of \( H' \) that corresponds to \( \sigma_{H'} \).

Finally it is clear (see Figure 1) that these chords \( u_i \) can be drawn in such a way that this circle model of \( H' \) fulfills the Helly property. \( \square \)
Fig. 2. Left: A vertex \( u \in V(G) \setminus V(H) \) and its neighborhood in \( H \). Right: \( x \) and \( y \) are respectively a predecessor and a successor of \( u \).

2.2 There exists a vertex \( u \in V(G) \setminus V(H) \) adjacent to some vertex of \( H \).

By Lemma 2, let \( u_0 \in V(G) \setminus V(H) \) be such that \( G[V(H) \cup \{u_0\}] \) is convex. Note that by clique maximality of \( H \), for every vertex \( u \in V(G) \setminus V(H) \) (like \( u_0 \)) its neighborhood in \( H \), \( N_H(u) \), induces a stable graph.

**Lemma 5** Consider any vertex \( u \in V(G) \setminus V(H) \) adjacent to \( H \) such that \( G[V(H) \cup \{u\}] \) is convex. Let us denote \( v_1, \ldots, v_k \), with \( k \geq 1 \), the neighbors of \( u \) in \( H \) in such a way that \( (u^+, v_1, \ldots, v_k, u^-, v_k, \ldots, v_1) \subseteq \sigma_G \) (see Figure 2). Every common neighbor \( x \) of \( u \) and \( v_i \), with \( 1 \leq i \leq k \), is adjacent to exactly one vertex in \( N_H(u) \), \( v_i \), and verifies either:

\begin{align*}
(P) & \quad i = 1 \text{ and } (u^+, v_1, x, u^-, v_1, x) \subseteq \sigma_G, \text{ or} \\
(S) & \quad i = k \text{ and } (u^+, x, v_k, u^-, x, v_k) \subseteq \sigma_G.
\end{align*}

In the first case, we call \( x \) a predecessor of \( u \), while in the second case it is a successor of \( u \).

**PROOF.** If there was a vertex \( x \) adjacent to \( u \), \( v_i \), and \( v_j \), these four vertices would induce a diamond. We now prove that \( x \) could not be such that \( (u^+, x, v_i, u^-, x, v_i) \subseteq \sigma_G \), for \( 1 \leq i < k \). The subgraph \( H \) being almost component maximal and \( u \) being in the same component of \( G \) as \( v_i \) and \( v_{i+1} \), the vertices \( v_i \) and \( v_{i+1} \) are in the same component of \( H \). Thus \( v_i \) has at least one neighbor \( z \neq v_{i+1} \) in \( H \). Since \( z \) is not adjacent to \( x \) (by clique maximality of \( H \)) we have that \( (z, x, v_{i+1}, v_{i+1}, x, z) \subseteq \sigma_G \), contradicting \( H \)'s convexity. We could similarly prove that \( x \) is not such that \( (u, v_i, x, u, v_i, x) \subseteq \sigma_G \), for \( 1 < i \leq k \). This concludes the proof of the lemma. \( \square \)

**Lemma 6** Consider any vertex \( u \in V(G) \setminus V(H) \) adjacent to \( H \) such that \( G[V(H) \cup \{u\}] \) is convex. For every predecessor (resp. successor) \( x \) of \( u \), \( G[V(H) \cup \{x\}] \) is convex and \( u \) is a successor (resp. a predecessor) of \( x \).

**PROOF.** Assume \( G[V(H) \cup \{x\}] \) is not convex. There exist \( y \in V(G) \setminus V(H) \) and \( z \in V(H) \) such that \( (x^+, x^-, y, z, z, y) \subseteq \sigma_G \). By convexity of
Fig. 3. A sequence \( S = (u_{-1}, u_0, u_1, u_2) \) allowing to extend \( H \)

\( G[V(H) \cup \{u\}] \), the vertex \( u \) is adjacent to \( y \) (otherwise we would have \((u^+, u^-, y, z, z, y) \subseteq \sigma_G\)). Let us denote \( v_1, \ldots, v_k \), with \( k \geq 1 \), the neighbors of \( u \) in \( H \) in such a way that \((u^+, v_1, \ldots, v_k, u^-, v_k, \ldots, v_1) \subseteq \sigma_G\). Recall that by definition of \( x \), \( u \) and \( x \) have a common neighbor, say \( v_1 \in V(H) \) (resp. \( v_k \in V(H) \)). Since \( u \), \( x \), \( y \) and \( v_1 \) (resp. \( v_k \)) cannot induce a diamond \( y \) and \( v_1 \) (resp. \( v_k \)) are not adjacent; but this contradicts the convexity of \( H \) (we would have \((v_1, v_1, y, z, z, y) \) or \((v_k, v_k, y, z, z, y) \subseteq \sigma_G\)). Finally it is clear by Lemma 5 that \( u \) is a successor (resp. a predecessor) of \( x \). \( \square \)

We now define the vertices \( u_i \in V(G) \setminus V(H) \) with \(-p \leq i \leq q\), in such way that \((u_i)_{p \leq i \leq q} \) is the longest sequence containing the previously defined element \( u_0 \) and such that \( u_i \) is a successor of \( u_{i-1} \) for every \( i \in \{-p + 1, \ldots, q\} \). Given the definition of \( u_0 \), Lemma 6 implies that all the vertices \( u_i \) have a neighbor in \( H \) and are such that \( G[V(H) \cup \{u_i\}] \) is convex.

Then Lemma 5 allows us to define an increasing sequence \((n_i)_{-p-1 \leq i \leq q} \) and the vertices \( v_j \), for \( n_{-p-1} \leq j \leq n_q \), in such way that for every \( i \in \{-p, \ldots, q\} \), the neighbors of \( u_i \) in \( H \) are exactly the vertices \( v_j \) with \( n_{i-1} \leq j \leq n_i \). The lemma also implies that \((u_i^+, v_{n_{i-1}}, \ldots, v_{n_i}, u_i^-, v_{n_i}, \ldots, v_{n_{i-1}}) \subseteq \sigma_G\), and thus the vertices \( v_j \) form a stable of \( H \) such that \((v_{n_{-p-1}}, v_{1+n_{-p-1}}, \ldots, v_{n_q}, v_{n_q}, \ldots, v_{1+n_{-p-1}}, v_{n_{-p-1}}) \subseteq \sigma_G\) (see Figure 3). Finally let \( H' = G[V(H) \cup \{u_i | -p \leq i \leq q\}] \) and let us prove that \( H' \) is convenient and admits a Helly circle model.

**Lemma 7** For every \( i \in \{-p, \ldots, q\} \), the successors (resp. predecessors) of \( u_i \) are the vertices \( u_k \) such that \( k > i \) and \( n_{k-1} = n_i \) (resp. such that \( k < i \) and \( n_k = n_{i-1} \)).

**Proof.** Lemma 5 implies that the relation “successor” on the set of vertices intersecting both \( u_i \) and \( n_{ni} \) is a total order. Thus, if a successor of \( u_i \) was missing in the sequence \((u_i)_{p \leq i \leq q}\), one could easily insert it. \( \square \)

**Lemma 8** \( H' \) is convenient.
PROOF. The graph $H'$ has as many connected components as $H$, thus it is almost component maximal. Let us show that $H'$ is clique maximal. Assume by contradiction that there exists a vertex $x \in V(G) \setminus V(H')$ adjacent to both ends of an edge $ab \in E(H')$. If $ab \in E(H)$, this would contradict the clique maximality of $H$. If $a = u_i$ and $b = v_j$, with $-p \leq i \leq q$ and $n_i - 1 \leq j \leq n_i$, the vertex $x$ is a successor or a predecessor of $u_i$ (by Lemma 5) and it thus should belong to $H'$ (by Lemma 7). If $a = u_i$ and $b = u_j$, with $-p \leq i < j \leq q$, then $x$ is either adjacent to $v_n_i$ or not. In the first case $x$ should be a vertex $u_k$, and thus belong to $H'$ (by Lemma 7). In the second case, $a, b$ and $v_n_i$ would induce a diamond.

Finally let us show that $H'$ is convex. By contradiction, assume there exist $x \in V(G) \setminus V(H')$ and $a, b \in V(H')$ such that $(a, a, x, b, b, x) \subseteq \sigma_G$. By convexity of $H$ and $H \cup \{u_i\}$ for every $i \in \{-p, \ldots, q\}$, both $a$ and $b$ belong to $(V(H') \setminus V(H)$, say $a = u_i$ and $b = u_j$, with $-p \leq i < j \leq q$. By definition of $u_i$ and $u_j$, we have that $(v_n_i, u_i, v_n_i, u_i, x, u_j, v_n_j, u_j, v_n_j) \subseteq \sigma_G$. Thus we have either

- $(v_n_i, x, u_i, v_n_i, u_i, x, u_j, v_n_j, u_j, v_n_j) \subseteq \sigma_G$ or
- $(v_n_i, u_i, v_n_i, u_i, x, u_j, v_n_j, u_j, v_n_j) \subseteq \sigma_G$ or
- $(v_n_i, u_i, v_n_i, u_i, x, u_j, v_n_j, u_j, x, v_n_j) \subseteq \sigma_G$

which respectively contradicts the convexity of $H \cup \{u_i\}$, $H$ and $H \cup \{u_j\}$. □

Lemma 9 There is a mixed Helly model of $(G, H')$.

PROOF. We consider the Helly circle model of $H$ and we extend it to $H'$. Lemma 5 allows us to distinguish one extremity of $v_j$, for every $j$; the distinguished extremity being such that $(u_i^+, v_j^+, u_i^-, v_j) \subseteq \sigma_G$ for every vertex $u_i$ crossing $v_j$. We extend the Helly circle model of $H$ by processing the chords $u_{-p}, \ldots, u_q$ successively in this order. For any $i \in \{-p - 1, \ldots, q\}$ let $H_i = G[V(H) \cup \{u_k\} - p \leq k \leq i]$. At each step we extend an Helly circle model of $H_{i-1}$ to an Helly circle model of $H_i$. Actually we construct them in such a way that for every $i$,

(*) the intersection point between $u_i$ and $v_n_i$, lies strictly between the point $v_n_i^*$ and the intersection of the chord $v_n_i$ and the abstract chord $[v_n_{i-1}^*, v_{n+1}^*]$ (see Figure 4).

Assume we have already processed the chords up to $u_{i-1}$ (see Figure 4 Step 1). Since $\sigma_{H_{i-1}} \subseteq \sigma_{H_i}$ it is easy to draw a chord $u_i$ which intersect the desired chords. Now we are going to slightly move this chord in order to fulfill (*) and the Helly property. If $u_i = u_{-p}$ since the neighborhood of $u_{-p}$ in $H_{-p}$ induces a stable, the Helly property follows immediately. So, we just have to move $u_{-p}$ close enough to $v_{n_{-p}}^*$ in order to fulfill (*). This is possible since there is no chord extremity in between $v_{n_{-p}}^*$ and $u_{-p}$ in $\sigma_{H_{-p}}$. If $u_i \neq u_{-p}$,
Fig. 4. Processing the chord $u_i$. Dashed lines are the abstract constraints.

the Helly circle model of $H_{i-1}$ fulfills (*). This ensures that we can move the chord $u_i$ in order to intersect $u_{i-1}$ and $v_{n_{i-1}}$ at their intersection point (see Figure 4 Step 2). At this step, since the neighborhood of $u_i$ in $H_i$ induces a graph with a unique non-trivial maximal clique (the clique with vertex set $\{v_{n_{i-1}}\} \cup \{u_k \mid k < i \text{ and } n_k = n_{i-1}\}$), the circle model of $H_i$ fulfills the Helly property. Finally if (*) is not satisfied, we just get $u_i^-$ close enough to $v_{n_i}^*$ by rotating $u_i$ around the intersection point of the clique. (see Figure 4 Step 3).

This concludes the proof of the theorem. □

3 Concluding Remarks

The first polynomial-time algorithms for circle graphs recognition were independently given by Bouchet [2], Naji [10] and Gabor et al. [8]. The latter was improved by Spinrad who showed that the recognition of circle graphs can be done in $O(n^2)$ time, and that a circle model can be computed in the same time bound [12]. Given a circle model of a circle graph $G$, the graph induced by the neighborhood of a vertex $v$ in $G$ is a permutation graph. Moreover a permutation model of $G[N(v)]$ can be computed in $O(n)$ if a circle model is known. One can easily check in $O(n)$ time if a permutation graph is $P_3$-free (i.e. is a disjoint union of complete graphs) using its permutation model. Thus one can check if a circle graph $G$ given with a circle model is diamond-free in $O(n^2)$ (check for every vertex $v$ if the permutation graph $G[N(v)]$ is $P_3$-free).

In consequence we have:

**Proposition 1** Helly circle graphs can be recognized in time $O(n^2)$.

Actually, the test for an induced diamond can even be carried out in linear time $O(n + m)$ with an adequate data structure. This means that the complexity of Helly circle graphs recognition is at most the complexity of circle graphs recognition.

As a byproduct of the proof, we can also treat the case of 1-string Helly circle graphs. A graph is 1-string circle if it is the intersection graph of strings in a disk with both endpoints on its border, such that two strings intersect at most once, and no two strings touch. Note that a graph is 1-string circle if
and only if it is a circle graph.

A graph is 1-string Helly circle if it has a 1-string circle model where the strings have the Helly property. The proof of Theorem 1 can be reformulated in these terms: the 1-string Helly circle graphs are the Helly circle graphs.

Another aspect is the generalization in greater dimensions. A graph is a $d$-dimensional sphere graph if it is an intersection graph of $(d - 1)$-dimensional disks with border on a $d$-dimensional sphere. We wonder whether Theorem 1 could be generalized in the following way:

**Question 1** Does every $d$-dimensional sphere graph that is $(K_{d+2} \setminus e)$-free admit a Helly embedding?

The reverse is not true unless we forbid that $d$ disks intersect in a segment.

**References**


