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► **To cite this version:**

Patrice Séébold. k-overlap-free binary words. [Research Report] RR-10031, Lirmm. 2010. <lirmm-00512255>

HAL Id: lirmm-00512255

<https://hal-lirmm.ccsd.cnrs.fr/lirmm-00512255>

Submitted on 29 Aug 2010

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k -overlap-free binary words

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August 29, 2010

Abstract

We study k -overlap-free binary infinite words that are binary infinite words which can contain only overlaps $xyxyx$ with $|x| \leq k - 1$. We prove that no such word can be generated by morphism, except if $k = 1$. On the other hand, for every $k \geq 2$, there exist k -overlap-free binary infinite words which are not $(k - 1)$ -overlap-free. As an application, we prove that, for every integer n , there exists infinitely many k -overlap-free binary infinite partial words with n holes.

1 Introduction

Repetitions, i.e., consecutive occurrences of a given factor within a word, and especially repetition-freeness have been fundamental research subjects in combinatorics on words since the seminal papers of Thue [11, 12] in the beginning of the 20th century (see also [2]). In particular, Thue and Morse independently showed the existence of an overlap-free binary infinite word (the Thue-Morse word [8, 12]), i.e., an infinite word using only two different letters and which does not contain any factor $xyxyx$ with x a non-empty word.

In the present paper we study the case where x must be of length at least k , that is, k -overlap-free binary infinite words which do not contain any factor $xyxyx$ with $|x| \geq k$.

The paper is organized as follows. After general definitions and notations given in Section 2, the notion of k -overlap-freeness is introduced in Section 3 where it is proved that no k -overlap-free binary infinite word can be generated by morphism, except if $k = 1$. In Section 4 we introduce the concept of 0-limited square property (a word has this property if the squares it contains have a particular form) to prove that, for every integer k , there exist k -overlap-free binary infinite words that are not $(k - 1)$ -overlap-free. In Section 5 we consider the particular case of k -overlap-free words which do not contain cubes of some letters. Section 6 is then dedicated to an application to partial words¹.

2 Preliminaries

Generalities on combinatorics on words can be found, e.g., in [7].

Let \mathcal{A} be a finite alphabet. The elements of \mathcal{A} are called *letters*. A word $w = a_1a_2 \cdots a_n$ of length n over the alphabet \mathcal{A} is a mapping $w: \{1, 2, \dots, n\} \rightarrow \mathcal{A}$ such that $w(i) = a_i$. The length of a word w is denoted by $|w|$, and ε is the empty word of length zero. For a word w and a letter a , $|w|_a$ denotes the number of occurrences of the letter a in the word w . By a (right) infinite word $w = a_1a_2a_3 \cdots$ we mean a mapping w from the positive integers \mathbb{N}_+ to the alphabet \mathcal{A} such that $w(i) = a_i$. The set of all finite words is denoted by \mathcal{A}^* , infinite words are denoted by \mathcal{A}^ω and $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$. A finite word v is a *factor*

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¹A preliminary version of this paper was presented at JORCAD'08 [10]. Some results about the case $k = 2$ appeared in [6]. But in these two papers, the 0-limited square property was replaced by the restricted square property, a much more restrictive condition.

34 of $w \in \mathcal{A}^* \cup \mathcal{A}^\omega$ if $w = xvy$, where $x \in \mathcal{A}^*$ and $y \in \mathcal{A}^* \cup \mathcal{A}^\omega$. Words xv and vy are respectively called *a*
 35 *prefix* and *a suffix* of w .

A morphism on \mathcal{A}^* is a mapping $f: \mathcal{A}^* \rightarrow \mathcal{A}^*$ satisfying $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{A}^*$. The morphism f is *erasing* if there exists $a \in \mathcal{A}$ such that $f(a) = \varepsilon$. Note that f is completely defined by the values $f(a)$ for every letter a on \mathcal{A} . A morphism is called *prolongable on a letter a* if $f(a) = aw$ for some word $w \in \mathcal{A}^+$ such that $f^n(w) \neq \varepsilon$ for all integers $n \geq 1$. This implies that $f^n(a)$ is a prefix of $f^{n+1}(a)$ for all integers $n \geq 0$ and a is a *growing letter for f*, that is, $|f^n(a)| < |f^{n+1}(a)|$ for every $n \in \mathbb{N}$. Consequently, the sequence $(f^n(a))_{n \geq 0}$ converges to the unique infinite word generated by f from the letter a ,

$$f^\omega(a) := \lim_{n \rightarrow \infty} f^n(a) = awf(w)f^2(w) \cdots,$$

36 which is a fixed point of f .

37 A morphism $f: \mathcal{A}^* \rightarrow \mathcal{A}^*$ *generates an infinite word w from a letter a* $a \in \mathcal{A}$ if there exists $p \in \mathbb{N}$ such
 38 that the morphism f^p is prolongable on a . We say that the morphism f *generates an infinite word* if it
 39 generates an infinite word from at least one letter.

40 A *kth power* of a word $u \neq \varepsilon$ is the word u^k prefix of length $k \cdot |u|$ of u^ω , where u^ω denotes the infinite
 41 catenation of the word u , and k is a rational number such that $k \cdot |u|$ is an integer. A word w is called
 42 *k-free* if there does not exist a word x such that x^k is a factor of w . If $k = 2$ or $k = 3$, then we talk about
 43 *square-free* or *cube-free* words, respectively. An *overlap* is a word of the form $xyxyx$ where $x \in \mathcal{A}^+$ and
 44 $y \in \mathcal{A}^*$. A word is called *overlap-free* if it does not contain overlaps. Therefore, it can contain squares
 45 but it cannot contain any longer repetitions such as overlaps or cubes. For example, over the alphabet
 46 $\{a, b\}$ the word *abbabaa* is overlap-free but it contains squares *bb*, *aa*, and *baba*. It is easy to verify that
 47 there does not exist a square-free infinite word over a binary alphabet, but as we recall in the next section
 48 there exist overlap-free binary infinite words.

49 In all the paper we will use the two alphabets $\mathcal{A} = \{a, b\}$, $\mathcal{B} = \{0, 1, 2\}$.

50 3 *k*-overlap-free binary words

In [12], Thue introduced the morphism

$$\begin{aligned} \mu: \mathcal{A}^* &\rightarrow \mathcal{A}^* \\ a &\mapsto ab \\ b &\mapsto ba \end{aligned}$$

The Thue-Morse word is the overlap-free binary infinite word

$$t := \lim_{n \rightarrow \infty} \mu^n(a) = abbabaabbaababbaba \cdots$$

51 generated by μ from the letter a (see, e.g., [1] for other definitions and properties, see also [2] for a
 52 translation of the contribution of Thue to the combinatorics on words). Another overlap-free binary
 53 infinite word is t' , the word generated by the morphism μ from the letter b . Note that the word t' can
 54 be obtained from the word t by exchanging all the a 's and b 's.

55 We generalize the notion of overlap with the following definition.

56 **Definition 3.1** *A k-overlap is a word of the form xyxyx where x and y are two words with |x| = k. A*
 57 *word is k-overlap-free² if it does not contain k-overlaps.*

58 For example, the word *baabaab* is not overlap-free but it is 2-overlap-free while the word *baabaaba* is
 59 not.

60 It is important to note that a *k*-overlap-free word can contain a^{3k-1} for each letter a . More generally,
 61 a *k*-overlap-free word can contain ℓ -powers u^ℓ where the value of ℓ , which can be greater than k , depends
 62 on the word u . For example, the word $u = ababababab$, which is a 5-power, is 3-overlap-free. On the
 63 contrary, the word $v = abaabaaba$, which is only a 3-power and which is also such that $|v| < |u|$, is a

²While it is not exactly the same, this notion of *k*-overlap-freeness resembles that of *k*-bounded overlaps introduced by Thue in [12].

64 3-overlap thus not being 3-overlap-free! This peculiarity is one reason for restraining the definition, for
65 example to the case of cube-free words. However, such a restriction seems to be very drastic and it is
66 generally enough to avoid the powers of letters. In the present paper, in Section 5, we study the change
67 in our results when restraining to the case of words without a^3 ... and we will see that the results are
68 not very different.

69 By definition, it is evident that every k -overlap-free word is also k' -overlap-free for $k' \geq k$. Note that
70 a word is 1-overlap-free if and only if it is overlap-free. So, an overlap-free infinite word is a k -overlap-free
71 infinite word for every positive integer k .

72 It is a well-known important property that t and t' are the only overlap-free binary infinite words
73 which are generated by morphism (see, e.g., [5], [9]). Since k -overlap-freeness does not imply ℓ -overlap-
74 freeness for $\ell < k$, k -overlap-freeness is weaker than overlap-freeness when $k \geq 2$. Therefore, we might
75 suppose that there exist binary infinite words, generated by morphism, that are k -overlap-free for some
76 $k \geq 2$ but that are not overlap-free (therefore different from t and t'). In fact, rather surprisingly, for
77 that property, k -overlap-freeness does not give more than only (1)-overlap-freeness.

78 **Theorem 3.2** *Let $k \in \mathbb{N}_+$ and let w be a k -overlap-free binary infinite word. Then w is generated by a*
79 *morphism if and only if $w = t$ or $w = t'$.*

80 *Proof.* The only if part is obvious since $t = \mu^\omega(a)$ and $t' = \mu^\omega(b)$ are k -overlap-free for every positive
81 integer k .

82 Conversely, as we have already seen, if an overlap-free ($k = 1$) binary infinite word is generated by a
83 morphism then $w = t$ or $w = t'$. Thus it remains to prove that an infinite word which contains an overlap
84 but is k -overlap-free for some integer $k \geq 2$ cannot be generated by a morphism.

85 Assume, contrary to what we want to prove, that an infinite word w over \mathcal{A} which contains an overlap
86 but is k -overlap-free for some integer $k \geq 2$, is generated by a morphism f . Then there exists a positive
87 integer r such that f^r is prolongable on the first letter of w . Without loss of generality we may assume
88 that this first letter is the letter a . In particular, w begins with $(f^r)^n(a)$ for every $n \in \mathbb{N}$. Moreover,
89 by definition a is a growing letter for f^r , which implies that there exists a positive integer N_0 such that
90 $|f^{rN_0}(a)| \geq k$.

91 If $f(b) = \varepsilon$ or if $|f(a)|_b = 0$ (which means $f(a) = a^p$, $p \geq 2$, because a is a growing letter for f), then
92 w is the periodic word $(f(a))^\omega$ which contains arbitrarily large powers of $f(a)$.

93 If $f(b) = b^p$, $p \geq 2$, or if $f^r(a)$ ends with b and $f(b) = b$ (which implies that $f^{rn}(a)$ ends with b^n for
94 every integer n), then w contains arbitrarily large powers of b .

95 If $f(a)$ ends with a and begins with $ab^p a$, $p \geq 1$, and $f(b) = b$, then w , which begins with $(f^r)^2(a)$,
96 contains a factor $auaua$ with $u = b^p$.

97 If w contains an overlap $auaua$ as a factor, then $f^{rN_0}(auaua)$ is also a factor of w .

98 The only remaining case is w contains an overlap $bubub$ as a factor and $f(b) = xay$ for some words
99 $x, y \in \mathcal{A}^*$. Then $f^{rN_0}(bubub)$ contains the factor $f^{rN_0}(a)f^{rN_0}(yux)f^{rN_0}(a)f^{rN_0}(yux)f^{rN_0}(a)$.

100 Consequently, since $f^{rN_0}(w) = w$ (because $w = (f^r)^\omega(a)$), in all cases w contains a k -overlap, which
101 contradicts with the hypothesis. ■

102 *Remark.* Although the case of alphabets with more than two letters is out of the scope of the present
103 paper, one can notice that Theorem 3.2 is no more true if we consider larger alphabets. Indeed, over
104 a 3-letter alphabet it is possible, for every integer $k \geq 2$, to find k -overlap-free words that are not
105 $(k - 1)$ -overlap-free and that are generated by morphism.

For example, let us consider the morphism

$$\begin{aligned} \mu_c : (\mathcal{A} \cup \{c\})^* &\rightarrow (\mathcal{A} \cup \{c\})^* \\ a &\mapsto ac^{3(k-1)}b \\ b &\mapsto ba \\ c &\mapsto c \end{aligned}$$

106 This morphism is obtained from the morphism μ by adding $c^{3(k-1)}$ at the middle of $\mu(a)$. Since μ is
107 an overlap-free morphism, the only overlaps in the word $\mu_c^\omega(a)$ are powers of the letter c . Therefore the
108 word $\mu_c^\omega(a)$ is k -overlap-free but not $(k - 1)$ -overlap-free.

109 4 The 0-limited square property

110 We have seen with Theorem 3.2 that the Thue-Morse words t and t' are the only k -overlap-free binary
 111 infinite words generated by morphism, whatever be the value of k . So it is natural to ask about the
 112 existence of k -overlap-free binary infinite words with $k \geq 2$ (then, of course, not generated by morphism),
 113 which are not ℓ -overlap-free for $\ell < k$. The answer is given in the present section where a family of such
 114 words is characterized.

115 Before this, we have to recall some works of Thue.

In order to prove the existence of infinite cube-free words over a two-letter alphabet from square-free words over three letters, Thue used in [11] the application

$$\begin{aligned} \delta : \mathcal{B}^* &\rightarrow \mathcal{A}^* \\ 0 &\mapsto a \\ 1 &\mapsto ab \\ 2 &\mapsto abb \end{aligned}$$

116 Six years later he proved the following result.

117 **Proposition 4.1** [12] [2] *Let $u \in \mathcal{A}^\omega$ and $v \in \mathcal{B}^\omega$ be such that $\delta(v) = u$. The word u is overlap-free if*
 118 *and only if the word v is square-free and does not contain 010 nor 212 as a factor. ■*

119 Thue also remarked that if the word $\delta(w)$ is not overlap-free for a square-free word w (thus containing
 120 010 or 212) then every overlap $xyxyx$ in $\delta(w)$ is such that x is a single letter. Therefore, it suffices to
 121 prove the existence of a square-free ternary infinite word containing either 010 or 212 over \mathcal{B} to obtain a
 122 2-overlap-free binary infinite word that is not overlap-free. Here again such a word is found in [12].

Let τ be the morphism

$$\begin{aligned} \tau : \mathcal{B}^* &\rightarrow \mathcal{B}^* \\ 0 &\mapsto 01201 \\ 1 &\mapsto 020121 \\ 2 &\mapsto 0212021 \end{aligned}$$

123 **Proposition 4.2** [12] *The word $\tau^\omega(0)$ is square-free and it contains 212 as a factor. ■*

124 Now, to prove the existence, for every integer $k \geq 2$, of k -overlap-free binary infinite words that are
 125 not $(k-1)$ -overlap-free, we generalize Thue's idea with the following notion.

126 **Definition 4.3** *An infinite word v over \mathcal{B} has the 0-limited square property if*

- 127 • *the word v does not contain 00 as a factor,*
- 128 • *whenever v contains a non-empty square rr as a factor, then, in v , the factor rr is preceded (if it*
 129 *is not a prefix of v) and followed by the letter 0.*

130 Note that if a word $v \in \mathcal{B}^\omega$ has the 0-limited square property then v is overlap-free and if v contains a
 131 non-empty square rr as a factor, the word r does not begin nor end with the letter 0.

132 The following corollary is straightforward from Proposition 4.2 because each square-free word obvi-
 133 ously has the 0-limited square property.

134 **Corollary 4.4** *The word $\tau^\omega(0)$ has the 0-limited square property. ■*

Now, let k, p be two integers with $k \geq 2$ and $1 \leq p \leq k-1$. We associate to (k, p) the application

$$\begin{aligned} \delta_{k,p} : \mathcal{B}^* &\rightarrow \mathcal{A}^* \\ 0 &\mapsto a^{k-p} \\ 1 &\mapsto a^{k-p}b^p \\ 2 &\mapsto a^{k-p}b^{p+1} \end{aligned}$$

135 Of course, $\delta_{2,1} = \delta$ thus our affirming that this is a generalization of Thue's idea.

136 **Theorem 4.5** Let $u \in \mathcal{A}^\omega$ and $v \in \mathcal{B}^\omega$ be such that $\delta_{k,p}(v) = u$. If the word v has the 0-limited square
 137 property then the word u is k -overlap-free.

138 *Proof.* Suppose that u is not k -overlap-free. Since $u = \delta_{k,p}(v)$, the following cases are possible:

139 • u contains a factor $a^k x a^k x a^k$
 140 If $|x|_b = 0$, or if $x = zx'$ with $|z| \geq k - 2p + 1$ and $|z|_b = 0$ then u contains $a^{k-p} a^{k-p} a$ which means
 141 that v contains 00 .

142 Henceforth, u contains a factor $a^n a^k a^m x' a^k a^m x' a^k$ with $n + m + k = 2(k - p)$, i.e., $k - p = n + m + p$,
 143 and x' begins with the letter b . Therefore, u contains a factor $a^{n+m+p} a^{k-p} x' a^{p+m} a^{k-p} x' a^{p+m} a^{p+n}$,
 144 which implies that v contains a factor $0y y$ where y is such that $\delta_{k,p}(y) = a^{k-p} x' a^{p+m}$ (in particular,
 145 $y \neq \varepsilon$). But in this case, y necessarily ends with 0 because $p + m \geq p \geq 1$. Therefore, either yy is
 146 followed by the letter 0 implying that v contains 00 as a factor, or yy is not followed by the letter 0 .

147 • u contains a factor $a^n b^{p+1} a^m x a^n b^{p+1} a^m x a^n b^{p+1} a^m$ with $n + m = k - p - 1$ (this includes the case
 148 where u contains $b^k x b^k x b^k$ when $p = k - 1$)

149 In this case, v contains a factor $2y2y2$ with $\delta_{k,p}(y2) = a^m x a^n b^{p+1}$.

150 • u contains a factor $a^n b^p a^m x a^n b^p a^m x a^n b^p a^m$ with $n + m = k - p$

151 Here, two cases are possible.

152 1. $m \neq 0$

153 Then u contains a factor $b^p a^m x a^n b^p a^m x a^n b^p a^{k-p}$, which implies that v contains a square yy ,
 154 preceded by 1 or 2 , with $\delta_{k,p}(y) = a^m x a^n b^p$.

155 2. $m = 0$ (then $n = k - p$)

156 Then u contains a factor $a^{k-p} b^p x a^{k-p} b^p x a^{k-p} b^p$, which implies that v contains a square yy ,
 157 followed by 1 or 2 , with $\delta_{k,p}(y) = a^{k-p} b^p x$.

158 • u contains a factor $b^n a^{k-p} b^m x b^n a^{k-p} b^m x b^n a^{k-p} b^m$ with $n + m = p$

159 Here again, two cases are possible.

160 1. $m \neq 0$

161 Then u contains a factor $a^{k-p} b^m x b^n a^{k-p} b^m x b^n a^{k-p} b^p$, which implies that v contains a square
 162 yy , followed by 1 or 2 , with $\delta_{k,p}(y) = a^{k-p} b^m x b^n$.

163 2. $m = 0$ (then $n = p$)

164 Then u contains a factor $b^p a^{k-p} x b^p a^{k-p} x b^p a^{k-p}$, which implies that v contains a square yy ,
 165 preceded by 1 or 2 , with $\delta_{k,p}(y) = a^{k-p} x b^p$.

166 In all the cases v has not the 0-limited square property. ■

167 Conditions given in Definition 4.3 are not sufficient to guarantee that the word v has the 0-limited
 168 square property when $u = \delta_{k,p}(v)$ is k -overlap-free. For example, the word $v = 0\tau^\omega(0)$ contains only one
 169 square, the factor 00 which v begins with. But, since the word $\tau^\omega(0)$ has the 0-limited square property,
 170 the word $\delta_{k,p}(\tau^\omega(0))$ is k -overlap-free which implies that $\delta_{k,p}(v)$ is also k -overlap-free (otherwise, $\delta_{k,p}(v)$
 171 begins with a k -overlap whose prefix is $a^{k-p} a^{k-p} a^{k-p}$, implying that $\delta_{k,p}(\tau^\omega(0))$ contains an occurrence
 172 of this factor $a^{k-p} a^{k-p} a^{k-p}$ from which $\tau^\omega(0)$ contains 00 , a contradiction). However, it is possible to
 173 obtain an equivalence by giving conditions on the words u and v .

174 **Corollary 4.6** Let u , an infinite word over \mathcal{A} , which does not contain the factor $a^{2(k-p)+1}$, and v ,
 175 an infinite word over \mathcal{B} which does not begin with a square, be such that $\delta_{k,p}(v) = u$. The word u is
 176 k -overlap-free if and only if the word v has the 0-limited square property.

177 Remark that here the word u is also $[2(k - p) + 1]$ -free.

178 *Proof.* Let u and v be as in the statement. It is of course equivalent that u does not contain the factor
 179 $a^{2(k-p)+1}$ and v does not contain the factor 00 , thus our assuming that 00 is not a factor of v .

180 From Theorem 4.5, it suffices to prove the necessary condition.

181 Let rr be a factor of v with $r \neq \varepsilon$. According to the hypothesis, rr is not at the beginning of v which
 182 means that in v , rr is preceded (and followed) by at least one letter.

- 183 • If r begins with the letter 0 then, since 00 is not a factor of v , r does not end with 0. Thus
184 $\delta_{k,p}(r) = a^{k-p}sb^p$. For the same reason, rr is preceded by the letter 1 or by the letter 2, so $\delta_{k,p}(rr)$
185 is preceded by b^p . Whatever be the letter following rr , $\delta_{k,p}(rr)$ is followed by a^{k-p} . Consequently,
186 u contains the factor $b^p\delta_{k,p}(rr)a^{k-p} = b^pa^{k-p}sb^pa^{k-p}sb^pa^{k-p}$, a k -overlap. This implies that u is
187 not k -overlap-free.
- 188 • If r ends with the letter 0 then, since v does not contain 00 as a factor, r does not begin with 0,
189 which implies that $\delta_{k,p}(r)$ begins with $a^{k-p}b^p$. Moreover, in v , the factor rr is followed either by
190 1 or by 2. Then u contains the factor $\delta_{k,p}(rr)a^{k-p}b^p = a^{k-p}b^psa^{k-p}b^psa^{k-p}b^p$, a k -overlap. This
191 implies that u is not k -overlap-free.
- 192 • Now if r begins with 1 or 2, and rr is not followed by 0 then $\delta_{k,p}(r)$ begins with $a^{k-p}b^p$ and $\delta_{k,p}(rr)$
193 is followed by $a^{k-p}b^p$, which means that u is not k -overlap-free.
- 194 • Finally, if r ends with 1 or 2, and rr is not preceded by 0 then $\delta_{k,p}(r)$ begins with a^{k-p} and ends
195 with b^p , and $\delta_{k,p}(rr)$ is preceded by b^p . This implies that, since $\delta_{k,p}(rr)$ is followed by a^{k-p} , u is
196 not k -overlap-free.

197 Consequently, if u is k -overlap-free then v has the 0-limited square property. ■

198 Theorem 4.5 gives the first part of the answer to the question we asked at the beginning of this section
199 by showing the existence of k -overlap-free binary infinite words for every integer $k \geq 2$. It remains to
200 prove that some words u satisfying Theorem 4.5 can effectively be constructed containing $(k-1)$ -overlaps.
201 This is done by using again Thue's morphism τ .

202 **Proposition 4.7** *For every integer $k \geq 2$, the word $\delta_{k,k-1}(\tau^\omega(0))$ is k -overlap-free but it contains $(k-1)$ -*
203 *overlaps.*

204 *Proof.* Since from Corollary 4.4 the word $\tau^\omega(0)$ has the 0-limited square property, the word $\delta_{k,k-1}(\tau^\omega(0))$
205 is k -overlap-free from Theorem 4.5.

206 Now, we know from Proposition 4.2 that $\tau^\omega(0)$ contains 212 as a factor. Therefore, $\delta_{k,k-1}(\tau^\omega(0))$
207 contains the factor $\delta_{k,k-1}(212) = ab^kab^{k-1}ab^k$, which implies that the $(k-1)$ -overlap $b^{k-1}ab^{k-1}ab^{k-1}$ is
208 a factor of $\delta_{k,k-1}(\tau^\omega(0))$. ■

209 5 Strongly k -overlap-free binary words

210 A k -overlap-free binary infinite word must contain occurrences of a^2 or b^2 (or both). For if it were not
211 the case the word would be $(ab)^\omega$ or $(ba)^\omega$ which obviously contains k -overlaps for every $k \in \mathbb{N}$.

212 As mentioned after Definition 3.1, the particular case of k -overlap-free binary infinite words without
213 x^3 for some letter $x \in \mathcal{A}$ is of interest. We define such words as follows.

214 **Definition 5.1** *A word over \mathcal{A} is x -strongly k -overlap-free if it is k -overlap-free and if it does not*
215 *contain x^3 , where x is a letter. A word is strongly k -overlap-free if it is x -strongly k -overlap-free for*
216 *every letter $x \in \mathcal{A}$.*

217 For example, the word a^5 is 2-overlap-free; it is b -strongly 2-overlap-free, but it is not strongly 2-overlap-
218 free because it is not a -strongly 2-overlap-free.

219 Notice that there exists effectively strongly k -overlap-free words that are not $(k-1)$ -overlap-free: for
220 example, from Proposition 4.2, the word $\delta(\tau^\omega(0))$ is strongly 2-overlap-free without being overlap-free.

221 Since every strongly k -overlap-free binary infinite word is k -overlap-free and since the Thue-Morse
222 words t and t' are cube-free, Theorem 3.2 remains true in the present case.

223 **Theorem 5.2** *Let $k \in \mathbb{N}_+$ and let w be a strongly k -overlap-free binary infinite word. Then w is generated*
224 *by a morphism if and only if $w = t$ or $w = t'$.*

225 It is obvious that $\mu(u)$ does not contain neither a^3 nor b^3 , whatever be the value of u . This implies
226 that if $\mu(u)$ is a k -overlap-free binary infinite word then it is indeed strongly k -overlap-free. Let us recall
227 the two lemmas used by Thue to prove that the Thue-Morse word t is overlap-free.

228 **Lemma 5.3** Let $\Sigma = \{ab, ba\}$. If $u \in \Sigma^*$ then $aua \notin \Sigma^*$ and $bub \notin \Sigma^*$.

229 **Lemma 5.4** A word $u \in \mathcal{A}^* \cup \mathcal{A}^\omega$ is overlap-free if and only if the word $\mu(u)$ is overlap-free.

230 The following result is an extension of Lemma 5.4.

231 **Proposition 5.5** Let $w \in \mathcal{A}^* \cup \mathcal{A}^\omega$ and let $k \in \mathbb{N}_+$. The word w is k -overlap-free if and only if the word
232 $\mu(w)$ is strongly $(2k - 1)$ -overlap-free.

233 *Proof.* If $k = 1$, the equivalence is true from Lemma 5.4, thus our assuming that $k \geq 2$.

234 If the word w is not k -overlap-free then it contains a factor $XYXYX$ with $|X| = k$. This implies that
235 $\mu(w)$, which contains the factor $\mu(X)\mu(Y)\mu(X)\mu(Y)\mu(X)$ with $|\mu(X)| = 2k$, is not $(2k - 1)$ -overlap-free.

236 Conversely, if the word $\mu(w)$ is not $(2k - 1)$ -overlap-free then it contains a factor $XxYXxYXx$ where
237 $X, Y \in \mathcal{A}^*$, $|X| = 2k - 2$, and $x \in \mathcal{A}$.

238 If $|Y|$ is even then $Y \neq \varepsilon$. For if not $\mu(w)$ would contain $XxXxXx$ which implies that both X and
239 xXx are in Σ^* , a contradiction with Lemma 5.3. So, let $Z \in \mathcal{A}^*$ and $y, z \in \mathcal{A}$ be such that $Y = Zyz$.
240 Then $XxYXxYXx = XxZyzXxZyzXx$ which implies that both X , xZy , yz , zXx , and Z are elements
241 of Σ^* .

242 From Lemma 5.3, $X \in \Sigma^*$ and $zXx \in \Sigma^*$ imply $x \neq z$, and $Z \in \Sigma^*$ and $xZy \in \Sigma^*$ imply $x \neq y$.
243 Therefore, $y = z$ which contradicts with $yz \in \Sigma^*$.

244 Consequently, $|Y|$ is odd so $|XxYXxYXx|$ is odd, and two cases are possible depending on whether,
245 in $\mu(w)$, the factor $XxYXxYXx$ appears at an even index or at an odd index.

246 • $\mu(w) = \mu(w_1)XxYXxYXx\mu(w_2)$ for a letter y .

247 In this case, by definition of μ , the letter y is also the first letter of Y . This implies that
248 $XxYXxYXx\mu(w_2) = \mu(ZY'ZY'Z)$ with $\mu(Z) = XxY$. Since $|XxY| = 2k$, $|Z| = k$ and the word
249 w is not k -overlap-free.

250 • $\mu(w) = \mu(w_1)yXxYXxYXx\mu(w_2)$ for a letter y .

251 In this case, since $k \geq 2$ one has $X \neq \varepsilon$, so let $X = zX'$, $z \in \mathcal{A}$, $X' \in \mathcal{A}^+$. Then $yXxYXxYXx =$
252 $yzX'xYzX'xYzX'x$, and by definition of μ , the letter y is also the last letter of Y . This implies
253 that $yzX'xYzX'xYzX'x = \mu(ZY'ZY'Z)$ with $\mu(Z) = yzX'x$. Since $|yzX'x| = 2k$, $|Z| = k$ and
254 the word w is not k -overlap-free. ■

255 Now, we consider the application $\delta_{k,k-1}$ ($k \geq 2$) already used above. Since $\delta_{k,k-1}$ is defined by
256 $\delta_{k,k-1}(0) = a$, $\delta_{k,k-1}(1) = ab^{k-1}$, $\delta_{k,k-1}(2) = ab^k$, it is straightforward that if $u \in \mathcal{A}^* \cup \mathcal{A}^\omega$ is such
257 that $u = \delta_{k,k-1}(v)$ for some $v \in \mathcal{B}^* \cup \mathcal{B}^\omega$ then u contains a^3 if and only if v contains 00 . Consequently,
258 from Theorem 4.5, if $u \in \mathcal{A}^\omega$ and $v \in \mathcal{B}^\omega$ are such that $u = \delta_{k,k-1}(v)$ then u is a -strongly k -overlap-free
259 whenever v has the 0-limited square property. We have seen above that if $k = 2$, i.e., in the case of Thue's
260 original application δ , the word u is strongly 2-overlap-free.

261 Now we notice that, in the case of $\delta_{k,k-1}$, the statement of Corollary 4.6 can be simplified because
262 $2(k - p) + 1 = 3$ when $p = k - 1$.

263 **Corollary 5.6** Let $u \in \mathcal{A}^\omega$, and v , an infinite word over \mathcal{B} which does not begin with a square, be such
264 that $\delta_{k,k-1}(v) = u$. The word u is a -strongly k -overlap-free if and only if the word v has the 0-limited
265 square property.

266 In this section we have seen that the results given in Section 4 remain the same when adding the
267 condition that words over \mathcal{A} do not contain cubes of some single letter, in particular in using the appli-
268 cation $\delta_{k,k-1}$. In the next section we give another interesting use of this application.

269 6 k -overlap-free binary partial words

270 A *partial word* u of length n over an alphabet \mathcal{A} is a partial function $u: \{1, 2, \dots, n\} \rightarrow \mathcal{A}$. This means
271 that in some positions the word u contains *holes*, i.e., “do not know”-letters. The holes are represented
272 by \diamond , a symbol that does not belong to \mathcal{A} . Classical words (called *full* words) are only partial words
273 without holes. Partial words were first introduced by Berstel and Boasson [3] (see also [4]).

274 Similarly to finite words, we define infinite partial words to be partial functions from \mathbb{N}_+ to \mathcal{A} . We
 275 denote by \mathcal{A}_\diamond^* and $\mathcal{A}_\diamond^\omega$ the sets of finite and infinite partial words, respectively.

276 A partial word $u \in \mathcal{A}_\diamond^*$ is a *factor* of a partial word $v \in \mathcal{A}_\diamond^* \cup \mathcal{A}_\diamond^\omega$ if there exist words $x, u' \in \mathcal{A}_\diamond^*$ and
 277 $y \in \mathcal{A}_\diamond^* \cup \mathcal{A}_\diamond^\omega$ such that $v = xu'y$ with $u'(i) = u(i)$ whenever neither $u(i)$ nor $u'(i)$ is a hole \diamond . Prefixes
 278 and suffixes are defined in the same way.

279 For example, let $u = ab\diamond bba\diamond a$. The length of u is $|u| = 8$, and u contains two holes in positions 3
 280 and 7. Let $v = aa\diamond bb\diamond ba\diamond abbaa\diamond$. The word v contains the word u as a factor in positions 3 and 8. The
 281 word u is a suffix of the word v .

282 Note that a partial word is a factor of all the (full) words of the same length in which each \diamond is replaced
 283 by any letter of \mathcal{A} . We call these (full) words the *completions* of the partial word. In the previous example,
 284 if $\mathcal{A} = \{a, b\}$, the partial word u has four completions: $ababbaaa$, $ababbaba$, $abbbbbaa$, and $abbbbaba$.

285 Let k be a rational number. A partial word u is *k-free* if all its completions are k -free. Overlaps,
 286 k -overlaps, overlap-freeness, and k -overlap-freeness of partial words are defined in the same manner.

287 In [6] it is proved that overlap-free binary infinite partial words cannot contain more than one hole,
 288 when 2-overlap-free binary infinite partial words can contain infinitely many holes. Here we complete
 289 this last result by the following theorem.

290 **Theorem 6.1** *For every integer $k \geq 2$ and for every non-negative integer n , there exist infinitely many*
 291 *k -overlap-free binary infinite partial words containing n holes, and being not $(k - 1)$ -overlap-free.*

292 Proof of Theorem 6.1 is constructive and needs some preliminaries.

The word $\tau^\omega(0)$ contains an infinite number of occurrences of $\tau(01)$:

$$\begin{aligned} \tau^\omega(0) &= \tau(01)u_1\tau(01)u_2 \cdots u_\ell\tau(01)u_{\ell+1} \cdots, u_i \in \mathcal{B}^+ \\ &= \prod_{\ell=1}^{\infty} \tau(01)u_\ell \\ &= \prod_{\ell=1}^{\infty} 01201020121u_\ell. \end{aligned}$$

293 For every integer $n \geq 0$, let Y_n be the word obtained from $\tau^\omega(0)$ by replacing 102 by 22 in n (not
 294 necessarily consecutive) occurrences of $\tau(01)$. Of course $Y_0 = \tau^\omega(0)$.

295 **Proposition 6.2** *For every $n \in \mathbb{N}$, the word Y_n has the 0-limited square property.*

296 *Proof.* In [6], it is proved that the occurrences of 22 are the only squares in the word Y_n . Consequently,
 297 Y_n does not contain 00 as a factor. Moreover, Y_n contains no squares but those 22 obtained from $\tau^\omega(0)$
 298 by replacing the factor 102 by 22 in n occurrences of $\tau(01)$, that is in n factors 01020. This implies
 299 that each of these 22 is preceded and followed by the letter 0. Therefore, since the word Y_n fulfills the
 300 conditions of Definition 4.3 it has the 0-limited square property. ■

301 **Corollary 6.3** *For every integers $k \geq 2$ and p , $1 \leq p \leq k - 1$, and for every integer $n \geq 0$, the word*
 302 *$\delta_{k,p}(Y_n)$ is k -overlap-free.*

303 *Proof.* By Proposition 6.2, the word Y_n has the 0-limited square property which implies, by Theorem 4.5,
 304 that $\delta_{k,p}(Y_n)$ is k -overlap-free. ■

305 In particular, for every integer $n \geq 0$, the words $\delta_{k,k-1}(\tau^\omega(0))$ and $\delta_{k,k-1}(Y_n)$ are k -overlap-free.

306 *Proof of Theorem 6.1.*

$$\begin{aligned} \delta_{k,k-1}(\tau(01)) &= \delta_{k,k-1}(0120)\delta_{k,k-1}(102)\delta_{k,k-1}(0121) \\ &= \delta_{k,k-1}(0120)ab^{k-1}\underline{a}ab^k\delta_{k,k-1}(0121) \end{aligned} \quad (1)$$

307 and

$$\begin{aligned} \delta_{k,k-1}(0120 \ 22 \ 0121) &= \delta_{k,k-1}(0120)\delta_{k,k-1}(22)\delta_{k,k-1}(0121) \\ &= \delta_{k,k-1}(0120)ab^{k-1}\underline{b}ab^k\delta_{k,k-1}(0121) \end{aligned} \quad (2)$$

308 Let us define Z_n to be the word obtained from $\delta_{k,k-1}(\tau^\omega(0))$ by replacing n (not necessarily consec-
 309 utive) occurrences of $\delta_{k,k-1}(\tau(01))$ by $\delta_{k,k-1}(0120)ab^{k-1} \diamond ab^k \delta_{k,k-1}(0121)$.

310 From Corollary 6.3, and equations (1) and (2) above, for every integer $n \geq 0$, the word Z_n is k -
 311 overlap-free. Moreover, from Proposition 4.7, Z_n is not $(k - 1)$ -overlap-free. ■

312 **Corollary 6.4** For every integer $k \geq 2$, there exist infinitely many k -overlap-free binary infinite partial
313 words containing infinitely many holes, and being not $(k - 1)$ -overlap-free.

314 *Proof.* Considering the words Z_n defined in the proof of Theorem 6.1, and making n tend to infinity, we
315 deduce that the word $\prod_{\ell=1}^{\infty} \delta_{k,k-1}(0120)ab^{k-1} \diamond ab^k \delta_{k,k-1}(0121)u_{\ell}$ has the required property.

316 Now, we can choose to leave out a finite number of substitutions of the factor 102 by 22. Since the
317 number of such choices is infinite, the result follows. ■

318 Acknowledgments

319 I am greatly indebted to Professor Gwénaél Richomme whose comments and suggestions on a prelim-
320 inary version of this paper were very useful.

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