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Propagating Conjunctions of \textsc{AllDifferent} Constraints

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Abstract

We study propagation algorithms for the conjunction of two \textsc{AllDifferent} constraints. Solutions of an \textsc{AllDifferent} constraint can be seen as perfect matchings on the variable/value bipartite graph. Therefore, we investigate the problem of finding simultaneous bipartite matchings. We present an extension of the famous Hall theorem which characterizes when simultaneous bipartite matchings exist. Unfortunately, finding such matchings is \textsc{NP}-hard in general. However, we prove a surprising result that finding a simultaneous matching on a convex bipartite graph takes just polynomial time. Based on this theoretical result, we provide the first polynomial time bound consistency algorithm for the conjunction of two \textsc{AllDifferent} constraints. We identify a pathological problem on which this propagator is exponentially faster compared to existing propagators. Our experiments show that this new propagator can offer significant benefits over existing methods.

Introduction

Global constraints are a critical factor in the success of constraint programming. They capture patterns that often occur in practice (e.g. “these courses must occur at different times”). In addition, fast propagation algorithms are associated with each global constraint to reason about potential solutions (e.g. “these 4 courses have only 3 time slots between them so, by a pigeonhole argument, the problem is infeasible”). One of the oldest and most useful global constraints is the \textsc{AllDifferent} constraint (Laurière 1978). This specifies that a set of variables takes all different values. Many different algorithms have been proposed for propagating the \textsc{AllDifferent} constraint (Régis 1994; Leconte 1996; Puget 1998). Such propagators can have a significant impact on our ability to solve problems (Sergiou & Walsh 1999).

Problems often contain multiple \textsc{AllDifferent} constraints (e.g. “The CS courses must occur at different times, as must the IT courses. In addition, CS and IT have several courses in common”). Currently, constraint solvers ignore information about the overlap between multiple constraints (except for the limited communication provided by the domains of common variables). Here, we show the benefits of reasoning about such overlap. This is a challenging problem as finding a solution to just two \textsc{AllDifferent} constraints is \textsc{NP}-hard (Kutz \textit{et al.} 2008) and existing approaches to deal with such overlaps require exponential space (Lardeux \textit{et al.} 2008). Our approach is to focus on domains that are ordered, as often occurs in practice. For example, in our time-tableing problem, values might represent times (which are naturally ordered). In such cases, domains can be compactly represented by intervals. Propagation algorithms can narrow such intervals using the notion of bound consistency. Our main result is to prove we can enforce bound consistency on two \textsc{AllDifferent} constraints in polynomial time. Our algorithm exploits a connection with matching on bipartite graphs. In particular, we consider \textit{simultaneous} matchings. By generalizing Hall’s theorem, we identify a necessary and sufficient condition for the existence of such a matching and show that this problem is polynomial for convex graphs.

Formal background

\textbf{Constraint programming.} We use capitals for variables and lower case for values. Values range over 1 to \textsc{d}. We write $D(X)$ for the domain of values for $X$, $lb(X) \ (ub(X))$ for the smallest (greatest) value in $D(X)$. A \textit{global constraint} is one in which the number of variables $n$ is a parameter. For instance, \textsc{AllDifferent}($\{X_1, \ldots, X_n\}$) ensures that $X_i \neq X_j$ for any $i < j$. Constraint solvers prune search by enforcing properties like domain consistency. A constraint is \textit{domain consistent} (DC) iff when a variable is assigned any value in its domain, there are compatible values in the domains of all other variables. Such an assignment is a \textit{support}. A constraint is \textit{bound consistent} (BC) iff when a variable is assigned the minimum or maximum value in its domain, there are compatible values between the minimum and maximum domain value for all other variables. Such an assignment is a \textit{bound support}. A constraint is \textit{bound disentailed} iff no possible assignment is a bound support.

\textbf{Graph Theory.} Solutions of \textsc{AllDifferent} correspond to matchings in a bipartite variable/value graph (Régis 1994).

\textbf{Definition 1.} The graph $G = (V, E)$ is bipartite if $V$ partitions into 2 classes, $V = A \cup B$ and $A \cap B = \emptyset$, such that every edge has ends in different classes.

\textbf{Definition 2.} Let $G = (A \cup B, E)$ be a bipartite graph. A matching that covers $A$ is a set of pairwise non-adjacent edges $M \subseteq E$ such that every vertex from $A$ is incident to
exactly one edge from $\mathcal{M}$.

We will consider simultaneous matchings on bipartite graphs ($\text{SIM-BM}$) (Kutz et al. 2008).

**Definition 3.** An overlapping bipartite graph is a bipartite graph $G = (A \cup B, E)$ and two sets $S$ and $T$ such that $A = S \cup T$, $A \cap B = \emptyset$, and $S \cap T \neq \emptyset$.

**Definition 4.** Let $(A \cup B, E)$ and $S$, $T$ be an overlapping bipartite graph. A simultaneous matching is a set of edges $M$ such that $S \cap M \neq \emptyset$ and $T \cap M \neq \emptyset$. Matches that cover $S$ and $T$, respectively.

In the following, we use the convention that a set of vertices $P$ is a subset of the partition $A$. We write $N(P)$ for the neighborhood of $P$. $P^S = P \cap (S \setminus T)$, $P^T = P \cap (T \setminus S)$ and $P^{ST} = P \cap S \cap T$. $\text{SIM-BM}$-problems frequently occur in real-world applications like production scheduling and timetabling. We introduce here a simple example timetabling problem that will serve as a running example.

**Running example.** We have 7 exams offered over 5 days and 2 students. The first student has to take the first 5 exams and the second student has to take the last 5 exams. Due to the availability of examiners, not every exam is offered each day. For example, the first exam cannot be on the last day of the week. Only one exam can be sat each day. This problem can be encoded as a $\text{SIM-BM}$ problem. A represents the exams and contains 7 vertices $X_1$ to $X_7$. $B$ represents the days and contains the vertices 1 to 5. $S = \{X_1, X_2, X_3, X_4, X_5\}$ and $T = \{X_3, X_4, X_5, X_6, X_7\}$. We connect vertices between $A$ and $B$ to encode the availability restrictions of the examiners. The adjacency matrix of the graph is as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^S \setminus T$</td>
<td>$X_1$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$X_2$</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A^{ST} \setminus S \cap T$</td>
<td>$X_3$</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$X_4$</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$X_5$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$A^T \setminus S$</td>
<td>$X_6$</td>
<td></td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$X_7$</td>
<td>*</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Finding a solution for this $\text{SIM-BM}$ problem is equivalent to solving the timetabling problem.

**Simultaneous Bipartite Matching**

We now consider how to find a simultaneous matching. Unfortunately, this problem is NP-complete in general (Kutz et al. 2008). Our contribution here is to identify a necessary and sufficient condition for the existence of a simultaneous matching based on an extension of Hall’s theorem (Hall 1935). We use this to show that a simultaneous matching on a convex bipartite graph can be found in polynomial time.

In the following, let $G'_{(u,v)}$ be the subgraph of the overlapping bipartite graph $G$ that is induced by choosing an edge $(u, v)$ to be in the simultaneous matching. If $u \in A^{ST}$ then $G'_{(u,v)} = G - \{u, v\}$. If $u \in A^S$ (and symmetrically if $u \in A^T$) then $G'_{(u,v)} = \{V - \{u, v\}, E \setminus \{(u', v) | u' \in S\}$. If $M$ is a $\text{SIM-BM}$ in $G'_{(u,v)}$, then $M \cup \{(u, v)\}$ is a $\text{SIM-BM}$ in $G$. Since the edge $(u, v)$ is implied throughout, we write $G' = G'_{(u,v)}$. In addition, we write $N'(P) = N_{G'}(P)$.

**Extension of Hall’s Theorem**

Hall’s theorem provides a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.

**Theorem 1** (Hall Condition (Hall 1935)). Let $G = (A \cup B, E)$ such that $A \cap B = \emptyset$. There exists a perfect matching iff $|N(P)| \geq |P|$ for $P \subseteq A$.

Interestingly we only need a small adjustment for simultaneous matching.

**Theorem 2** (Simultaneous Hall Condition (SIM-HC)). Let $G = (A \cup B, E)$ and sets $S$, $T$ be an overlapping bipartite graph. There exists a $\text{SIM-BM}$, iff $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P|$ for $P \subseteq A$.

**Proof.** We prove $\text{SIM-HC}$ by induction on $|A|$. When $|A| = 1$, the statement holds. Let $|A| = k > 1$. If $A^S = \emptyset$ or $A^T = \emptyset$ then $\text{SIM-HC}$ reduces to the condition of Hall’s theorem and the statement is true for that reason. Hence, we assume $A^S \neq \emptyset$ and $A^T \neq \emptyset$. We show that there is an edge $(u, v)$ that can be chosen for a simultaneous matching and the graph $G'_{(u,v)}$ will satisfy $\text{SIM-HC}$. Following (Diestel 2006), page 37, we consider two cases. The first case when all subsets of $A$ satisfy the strict $\text{SIM-HC}$, namely, $|N(P)| + |N(P^S) \cap N(P^T)| > |P|$ and the second case when we have an equality.

**Case 1.** Suppose $|N(P)| + |N(P^S) \cap N(P^T)| > |P|$ for all sets $P \subseteq A$. As $A^S \neq \emptyset$ we select any edge $(u, v), u \in A^S$ and construct the graph $G'_{(u,v)}$ (the case $u \in A^T$ is symmetric). For any set $P \subseteq A \setminus \{u\}$ we consider two cases: either $v \notin N(P)$ or $v \in N(P)$. In the first case, the neighborhood of $P$ is the same in $G$ and $G'$, so the $\text{SIM-HC}$ holds for $P$. In the case that $v \notin N(P)$, then either $v$ is a shared neighbor of $P^S$ and $P^T$, which means that $N'(P^S) \cap N'(P^T) = N'(P^S) \cap N'(P^T) - 1$ but $N'(P) = N(P)$ by construction, or $v$ is a neighbor of $P^S$ but not of $P^T$. Therefore $|N'(P)| \geq |N(P)| - 1$. But $N'(P^S) \cap N'(P^T) = N'(P^S) \cap N'(P^T)$ by construction. In either case, $N'(P) + |N'(P^S) \cap N'(P^T)| \geq |N(P)| + |N(P^S) \cap N(P^T)| - 1 \geq |P|$ for any set $P$ in $G'$. By the inductive hypothesis there exists a simultaneous matching in $G'$.

**Case 2.** Suppose that there exists a set $P \subseteq A$ such that $|N(P)| + |N(P^S) \cap N(P^T)| = |P|$. Let $Q = (A' \cup B', E')$ such that $A' = A \setminus P$, $B' = B \setminus (N(P^S) \cap N(P^T))$ and $E' = \{(u, v) \in E \setminus (A' \times B') |$

$$\begin{align*}
(u \in A^S' &\implies v \notin N(P) \setminus N(P^T)) \quad \land \\
(u \in A^T' &\implies v \notin N(P) \setminus N(P^S)) \quad \land \\
(u \in A^{ST'} &\implies v \notin N(P)) \quad \} 
\end{align*}$$

There exists a simultaneous matching in $G - Q$ by the inductive hypothesis. We claim that the $\text{SIM-HC}$ holds also for $Q$. This implies that, by the inductive hypothesis, there exists a simultaneous matching in $Q$. Suppose there exists a set $P' \subseteq A'$ that violates the $\text{SIM-HC}$ in $Q$.

We denote as $N(P)$ the neighborhood of $P$ in $G$ and $N_Q(P^S)$ as the neighborhood of $P^S$ in $Q$. We know that the sets $P'$ and $P$ are disjoint. We observe that $N(P \cup$
Removing edges

To build a propagator, we consider how to detect edges that cannot appear in any simultaneous matching.

**Definition 5.** Let $G = (A \cup B, E)$ and sets $S, T$ be an overlapping bipartite graph. A set $P, P \subseteq A$, is

- a **simultaneous Hall set** if $|N(P)| + |N(P^S) \cap N(P^T)| = |P|.$
- an **almost simultaneous Hall set** if $|N(P)| + |N(P^S) \cap N(P^T)| = |P| + 1.$
- a **loose set** if $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P| + 2.$

**Theorem 3.** $G = (A \cup B, E)$ and sets $S, T$ be an overlapping bipartite graph. Each edge $(u, v), u \in A$ and $v \in B$ can be extended to a matching that covers $S$ and $T$ iff

1. for each set $P$: (a) $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P|$
2. for each simultaneous Hall set $P$: (a) if $u \notin P$ then $v \notin (N(P^S) \cap N(P^T))$
   (b) if $u \notin S \setminus (T \cup P)$ then $v \notin (N(P^T))$
   (c) if $u \notin T \setminus (S \cup P)$ then $v \notin (N(P^S))$
   (d) if $u \notin (S \cap T) \setminus P$ then $v \notin (N(P))$
3. for each almost simultaneous Hall set $P$: (a) if $u \notin (S \cap T) \setminus P$ then $v \notin (N(P^S) \cap N(P^T))$

**Proof. Soundness.** The soundness of Rule 1a follows from Theorem 2. Let $(u, v)$ be an edge that we want to extend to a matching. Suppose that $(u, v)$ violates one of the rules for a SIM-HALL-SET or an A-SIM-HALL-SET $P$ in $G$. We show that if $(u, v)$ is selected to be in a matching, then $P$ fails SIM-HC in $G'(u,v)$.

- **Rule 2a:** If $(u, v)$ violates Rule 2a for a SIM-HALL-SET $P$ then $|N(P^S) \cap N(P^T)| = |N(P^S) \cap N(P^T)| − 1$ and $N'(P) = N(P)$, so the SIM-HC is violated for $P$ in $G'$.
- **Rule 2b:** If $(u, v)$ violates Rule 2b for a SIM-HALL-SET $P$ then $|N(P)| = |N(P)| − 1$ and $|N'(P^S) \cap N'(P^T)| = |N'(P^S) \cap N'(P^T)|$ so the SIM-HC is violated for $P$ in $G'$.
- **Rule 2c:** Symmetric to Rule 2b.
- **Rule 2d:** If $(u, v)$ violates Rule 2d for a SIM-HALL-SET $P$ then $|N'(P)| = |N(P)| − 1$ so the SIM-HC is violated for $P$ in $G'$.

- **Rule 3a:** If $(u, v)$ violates Rule 3a for an A-SIM-HALL-SET $P$ then $|N'(P)| = |N(P)| − 1$ and $|N'(P^S) \cap N'(P^T)| = |N'(P^S) \cap N'(P^T)| − 1$, so $N'(P)| + |N'(P^S) \cap N'(P^T)| = |P| − 1$ and the SIM-HC is violated for $P$ in $G'$.

**Completeness.** Second, we show that Rules 2a-3a are complete. We will show that we can use any edge $(u, v)$ in a matching by showing that the graph $G'(u,v)$ satisfies the SIM-HC, thus has a SIM-BM.

Suppose there is a set $P$ that violates the SIM-HC in $G'$ but not in $G$ so that

\[
|N'(P)| + |N'(P^S) \cap N'(P^T)| < |P| \tag{1}
\]

and

\[
|N(P)| + |N(P^S) \cap N(P^T)| \geq |P| \tag{2}
\]
Note that $N' = N \setminus \{v\}$ and $N'(P) \cap N'(P) = N(P) \cap N(P) = 1$. Hence, $|N'(P)| + |N'(P)| \geq |N(P)| + |N(P)| - 1 \geq -1$.

There are three cases to consider for $P$, if $P$ is a loose set, a SIM-HALLSET and an A-SIM-HALLSET in $G$. These cases are similar, so we consider only the most difficult case. Let $P$ be an A-SIM-HALLSET in $G$. We show that Rules 2a–3a remove every edge that can be pruned.

Definition 6. We now use these results to build a propagator.

Running example. Consider again our running example. By Rule 3a, we prune $X_i$.

In both cases, $|N'(P)| + |N'(P)| \geq |N(P)| + |N(P)| - 1 \geq 1$ by (1) and (2) cannot both be true.

If $u \in A^S$ ($u \in A_T$ is symmetric), then $v \in N(P) \cap N(P)$ or its complement. In the first case $N'(P) = N(P)$, while in the second $N'(P) \cap N(P) = N(P)$.

The overlapping ALLDIFFERENT constraint

We now use these results to build a propagator.

Definition 6. OVERLAPPINGALLDIFF({X}, S, T) where $S \subseteq X$, $T \subseteq X$, $S \cup T = X$ holds iff ALLDIFFERENT(S) and ALLDIFFERENT(T) hold simultaneously.

Enforcing BC on the OVERLAPPINGALLDIFF constraint is $NP$-hard (Bessiere et al. 2007). We consider instead enforcing just BC. This relaxation is equivalent to the simultaneous matching problem on a bipartite convex variable-value graph. Our main result is an algorithm that enforces BC on the OVERLAPPINGALLDIFF constraint in $O(n^2)$ time.

The algorithm is based on the decomposition of the OVERLAPPINGALLDIFF constraint into a set of arithmetic constraints derived from Rules 2b–3a. It is inspired by a decomposition of ALLDIFFERENT (Bessiere et al. 2009).

The overlapping ALLDIFFERENT constraint

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The algorithm is based on the decomposition of the OVERLAPPINGALLDIFF constraint into a set of arithmetic constraints derived from Rules 2b–3a. It is inspired by a decomposition of ALLDIFFERENT (Bessiere et al. 2009). As there, we introduce Boolean variables $a_{ilu}$, $b_{il}$ to represent whether $X_i$ takes a value in the interval $[l, u]$ and the variables $C^S$, $C^T$ to represent bounds on the number of variables from $S \setminus T$, $T \setminus S$ and $S \cap T$ that may take values in the interval $[l, u]$. We introduce the following set of constraints for $1 \leq i \leq n$, $1 \leq l \leq u \leq d$ and $u - l < n$:

\[
\begin{align*}
  b_{il} &= 1 \iff X_i \leq l \\
  a_{ilu} &= 1 \iff (b_{il-1} = 0 \land b_{il} = 1) \\
  C^S_{ilu} &= \sum_{i \in T} a_{ilu} \\
  C^T_{ilu} &= \sum_{i \in T} a_{ilu} \\
  C^S_{il} &= \sum_{i \in T} a_{ilu} \\
  C^T_{il} &= \sum_{i \in T} a_{ilu} \\
  C^S_{il} &= C^S_{il} + C^S_{il+1} \\
  C^T_{il} &= C^T_{il} + C^T_{il+1} \\
  C^S_{il} + C^T_{il} &\leq u - l + 1 \\
  C^S_{il} + C^T_{il} &\leq u - l + 1
\end{align*}
\]

We also introduce a dummy variable $C^S_{10} = 0$ to simplify the following lemma and theorems.

**Lemma 1.** Consider a sequence of values $v_1, v_2, \ldots, v_k$. Enforcing BC on (8) ensures $ub(C^S_{v_i, v_{i+1}-1}) \leq \sum_{v_i < v_{i+1}} ub(C^S_{v_i, v_{i+1}-1}) - \sum_{v_i > v_{i+1}} lb(C^S_{v_i, v_{i+1}-1})$.

**Proof.** For every $i$ such that $v_i < v_{i+1}$, constraint (8) ensures $ub(C^S_{v_i, v_{i+1}-1}) \leq ub(C^S_{v_i, v_{i+1}}) + ub(C^S_{v_{i+1}, v_{i+1}-1})$.

**Theorem 4.** Enforcing BC on (3)-(10) detects bound entailment of OVERLAPPINGALLDIFF in $O(n^2)$ time but does not enforce BC on OVERLAPPINGALLDIFF.

**Proof.** First we derive useful upper bounds for the variables $C^S_{il}$. Consider a set $P$ and an interval $[a, b]$ such that $N(P) = [a, b]$. Let $c_1, d_1, \ldots, c_n, d_n$ be a set of intervals that tightly contain variables from $P$ so that $\forall i, [c_i, d_i] \in N(P); [e_i, f_i] \in N(P), \ldots, [e_m, f_m] \in N(P)$. For any $i$ such that $v_i > v_{i+1}$, constraint (8) ensures $ub(C^S_{v_i, v_{i+1}-1}) \leq ub(C^S_{v_i, v_{i+1}}) - lb(C^S_{v_i, v_{i+1}}) - lb(C^S_{v_{i+1}, v_{i+1}-1})$.

**Lemma 1.** Consider a sequence of values $v_1, v_2, \ldots, v_k$. Enforcing BC on (8) ensures $ub(C^S_{v_i, v_{i+1}-1}) \leq \sum_{v_i < v_{i+1}} ub(C^S_{v_i, v_{i+1}-1}) - \sum_{v_i > v_{i+1}} lb(C^S_{v_i, v_{i+1}-1})$.

**Proof.** For every $i$ such that $v_i < v_{i+1}$, constraint (8) ensures $ub(C^S_{v_i, v_{i+1}-1}) \leq ub(C^S_{v_i, v_{i+1}}) + ub(C^S_{v_{i+1}, v_{i+1}-1})$.

**Theorem 4.** Enforcing BC on (3)-(10) detects bound entailment of OVERLAPPINGALLDIFF in $O(n^2)$ time but does not enforce BC on OVERLAPPINGALLDIFF.
We sort the union of remaining intervals \([c_i, d_i]\) and \([e_i, f_i]\) by their lower bounds and list them as semi-open intervals \([g_1, g_2), (g_3, g_4), \ldots, (g_{k'+m'}, g_{k'+m'} + 1, a)^x\).

Using the sequence \(a, g_1, g_2, \ldots, g_{k'+m'} + b + 1, a)^x\) where \((.)^x\) indicates a repetition of \(x\) times the same sequence, Lemma 1 provides the inequality
\[
ub(C_{ST}^{ST} \cap a) \leq ub(C_{ST}^{ST} \cap a) + x(ub(C_{ST}^{ST} \cap a) + ub(C_{ST}^{ST} \cap a) + b) + \sum_{i=1}^{k'+m'} ub(C_{ST}^{ST} \cap a) + \sum_{i=1}^{m'} ub(C_{ST}^{ST} \cap a).
\]

Substituting the inequalities that we already defined, we obtain
\[
ub(C_{ST}^{ST} \cap a) \leq a - 1 + x(b - a + 1 + \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a) - \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a) + \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a))
\]

For any removed interval \(I_j \in RI\) we have \(|I_j| - lb(C_{ST}^{ST} \cap a)) \geq 0\) or \(|I_j| - lb(C_{ST}^{ST} \cap a)) \geq 0\). We reaggregate all removed intervals into the inequality to get
\[
ub(C_{ST}^{ST} \cap a) \leq a - 1 + x(b - a + 1 - \sum_{i=1}^{k'+m'} lb(C_{ST}^{ST} \cap a) - \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a)) + \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a).
\]

Note that \(\sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a)) - \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a)) = \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a).
\]

Hence
\[
ub(C_{ST}^{ST} \cap a) \leq a - 1 + x(b - a + 1 - lb(C_{ST}^{ST} \cap a) - lb(C_{ST}^{ST} \cap a)) + \sum_{i=1}^{m'} lb(C_{ST}^{ST} \cap a)
\]

**Bound disentailment.** Suppose, for the purpose of contradiction, that OVERLAPPINGALLDIFF is bound disentailed and that constraints (3)-(10) are bound consistent. Then, there exists a set \(P\), such that \(N(P)\) is an interval and \(N(P) \cap \cap(P) < |P|\). As \(P\) fails SIM-HC, it holds that \(lb(C_{ST}^{ST} \cap a) + lb(C_{ST}^{ST} \cap a) + lb(C_{ST}^{ST} \cap a) > |P|\). Substituting the last inequality in (*) gives \(ub(C_{ST}^{ST} \cap a) < a - 1 - x\). Choosing a large enough value for \(x\) (say \(a\)) gives the contradiction \(ub(C_{ST}^{ST} \cap a) < 0\).

**Bound consistency.** To show that this decomposition does not enforce BC, consider the conjunction of ALLDIFFERENT \((X_1, X_2, X_3)\) and ALLDIFFERENT \((X_2, X_3, X_4)\) with \(D(X_1) = [2, 3], D(X_2) = [2, 4], D(X_3) = [1, 3], D(X_4) = [1, 2]\). Enforcing BC on (3)-(10) does not remove the bound inconsistent value \(X_2 = 2\).

**Complexity.** There are \(O(nd)\) constraints (3) that can be invoked \(O(d)\) times at most. There are \(O(nd^2)\) constraints (4) that can be invoked \(O(1)\) times. There are \(O(d^2)\) constraints (8) that can be propagated \(O(n)\) times. There are \(O(d^2)\) constraints (5)-(7) that can be propagated in \(O(n)\). The remaining constraints take \(O(nd^2)\) to propagate. The total time complexity is \(O(nd^2)\).

It follows immediately that the simultaneous matching problem is polynomial on bipartite graph convex groups.

**Theorem 5.** A simultaneous matching can be found in polynomial time on an overlapping convex bipartite graph.

Next, we present an algorithm to enforce BC. We show that constraints (3)-(10) together with the following two constraints enforce all but one of the rules from Theorem 3.
\[
C_{ST}^{ST} = C_{ST}^{ST} + C_{ST}^{ST} \quad 1 \leq l \leq u \leq d
\]
\[
C_{ST}^{ST} = C_{ST}^{ST} + C_{ST}^{ST} \quad 1 \leq l \leq u \leq d
\]

**Theorem 6.** Constraints (3)-(12) enforce Rules 2a—2d.

**Proof Sketch.** Based on Lemma 1, similar to the proof of Theorem 4, we show that all intervals that contain variables from a SIM-HALLSET \(P\) become saturated intervals, so that the lower bounds of the corresponding variables \(C_{ST}^{ST}, C_{ST}^{ST}\) and \(C_{ST}^{ST}\) equal to their upper bounds. Hence, these values are pruned from domains of variables outside the set \(P\).

**Theorem 7.** Suppose constraints (3)-(12) together with \(C_{ST}^{ST} = C_{ST}^{ST} + C_{ST}^{ST}\) \(2 \leq l \leq k \leq u \leq d\) have their fixpoint. Rule 3a can now be enforced in \(O(nd^3)\) time.

**Proof Sketch.** Let \(P\) is an A-SIM-HALLSET, \(N(P) = [a, b]\). Similar to the proof of Theorem 4, we can obtain that \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \geq 1 \geq ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST})\). Hence, we can identify intervals, that might contain an A-SIM-HALLSET \(P\). Next, we observe that if we add a dummy variable \(Z, D(Z) = [a, b]\) to the set \(P\), then \(P' = P \cup \{Z\}, Z \in P\) is a SIM-HALLSET. This allows us to identify the set \(N(P') \cap N(P')\) by simulating constraints (3)-(12) inside the interval \([a, b]\) taking into account the variable \(Z\). There are \(O(d^2)\) intervals. Finding \(N(P') \cap N(P')\) takes \(O(nd^2)\) time inside an interval. Enforcing the rule takes \(O(nd)\) time. Hence, the total time complexity is \(O(nd^3)\).

From Theorems 6 and 7 it follows that

**Theorem 8.** BC on OVERLAPPINGALLDIFF can be enforced in \(O(nd^3)\) time.

**Running example.** We demonstrate the action of constraints (3)-(12). The interval \([1, 4]\) contains a SIM-HALLSET \(P = \{X_2, X_3, X_4, X_6\}\). Rule 2d, \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \leq ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (8) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 2\). The interval \([1, 3]\) is saturated, as \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (9) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1\). The interval \([1, 2]\) is saturated, as \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (10) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1\). The interval \([1, 2]\) is saturated, as \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (10) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1\). The interval \([1, 2]\) is saturated, as \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (10) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1\). The interval \([1, 2]\) is saturated, as \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (10) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1\). The interval \([1, 2]\) is saturated, as \(lb(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1 \Rightarrow (10) \Rightarrow ub(C_{ST}^{ST} + C_{ST}^{ST} + C_{ST}^{ST}) \leq 1\).

**Rule 2a.** This is satisfied as 2 is removed from all variables outside \(P\).

**Exponential separation.** We now give a pathological problem on which our new propagator does exponentially less work than existing methods.

**Theorem 9.** There exists a class of problems such that enforcing BC on OVERLAPPINGALLDIFF immediately detects unsatisfiability while a search method that enforces BC on the decomposition into ALLDIFFERENT constraints explores an exponential search tree regardless of branching.

**Proof.** The instance \(I_n\) is defined as follows \(I_n = ALLDIFFERENT([X \cup Y]) \land ALLDIFFERENT([Y \cup Z])\), \(D(X_i) = [1, 2n - 1], i = 1, \ldots, n, D(Y_j) = [1, 4n - 1], i = 1, \ldots, n, D(Z_k) = [2n, 4n - 1], i = 1, \ldots, n\).

OVERLAPPINGALLDIFF. Consider the interval \([1, 4n - 1], [P] = 4n, |N(P)| = 4n - 1 and |N(P) \cap N(P)| = 0\). By Theorem 2, we detect unsatisfiability.
**Decomposition.** Consider any ALLDIFFERENT constraint. A subset of \( n \) or fewer variables has at least \( 2n - 1 \) values in their domains and a subset of \( n + 1 \) to \( 3n \) variables has \( 4n - 1 \) values in their domains. Thus, to obtain a Hall set and prune, we must instantiate at least \( n - 1 \) variables.

**Experimental results**

To evaluate the performance of our decomposition we carried out an experiment on random problems. We used Ilog 6.2 on an Intel Xeon 4 CPU, 2.0 GHz, 4GB RAM. We compare the performance of the \( DC \), \( BC \) (Lopez-Ortiz et al. 2003) propagators and our decomposition into constraints (3)-(12) for the OVERLAPPING ALLDIFFERENT constraint (oBC). We use randomly generated problems with three global constraints: ALLDIFFERENT(\( X \cup W \)), ALLDIFFERENT(\( Y \cup W \)) and ALLDIFFERENT(\( Z \cup W \)), and a linear number of binary ordering relations between variables in \( X \), \( Y \) and \( Z \). We use a random variable ordering and run each instance with 50 different seeds. As Table 1 shows, our decomposition reduces the search space significantly, is much faster and solves more instances overall.

Table 1: Random problems. \( n \) is the size of \( X \), \( Y \) and \( Z \);
\( o \) is the size of \( W \); \( d \) is the size of variable domains. Number of instances solved in 300 sec out of 50 runs / average backtracks/average time to solve.

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<th>( n \times d \times o )</th>
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<th>( DC )</th>
<th>oBC</th>
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**TOTALS**

sol/total | 54/450 | 181/450 | 285/450 |

avg time for sol | 78.072 | 70.551 | 24.689 |

avg bt for sol | 2818926 | 1666568 | 9561 |

as NVALUE (Bessiere et al. 2006). It may be possible to apply similar insights to develop propagators for conjunctions of other global constraints, or to improve existing propagators for global constraints that decompose into overlapping constraints like SEQUENCE (Brand et al. 2005). Finally, we may be able to develop polynomial time propagators for otherwise intractable cases if certain parameters are fixed (Bessiere et al. 2008).

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**References**


