Propagating Conjuctions of ALLDIFFERENT Constraints

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Abstract

We study propagation algorithms for the conjunction of two ALLDIFFERENT constraints. Solutions of an ALLDIFFERENT constraint can be seen as perfect matchings on the variable/value bipartite graph. Therefore, we investigate the problem of finding simultaneous bipartite matchings. We present an extension of the famous Hall theorem which characterizes when simultaneous bipartite matchings exists. Unfortunately, finding such matchings is NP-hard in general. However, we prove a surprising result that finding a simultaneous matching on a convex bipartite graph takes just polynomial time. Based on this theoretical result, we provide the first polynomial time bound consistency algorithm for the conjunction of two ALLDIFFERENT constraints. We identify a pathological problem on which this propagator is exponentially faster compared to existing propagators. Our experiments show that this new propagator can offer significant benefits over existing methods.

Introduction

Global constraints are a critical factor in the success of constraint programming. They capture patterns that often occur in practice (e.g. "these courses must occur at different times"). In addition, fast propagation algorithms are associated with each global constraint to reason about potential solutions (e.g. "these 4 courses have only 3 time slots between them so, by a pigeonhole argument, the problem is infeasible"). One of the oldest and most useful global constraints is the ALLDIFFERENT constraint (Laurière 1978). This specifies that a set of variables takes all different values. Many different algorithms have been proposed for propagating the ALLDIFFERENT constraint (Régin 1994; Leconte 1996; Puget 1998). Such propagators can have a significant impact on our ability to solve problems (Stergiou & Walsh 1999).

Problems often contain multiple ALLDIFFERENT constraints (e.g. "The CS courses must occur at different times, as must the IT courses. In addition, CS and IT have several courses in common"). Currently, constraint solvers ignore information about the overlap between multiple constraints (except for the limited communication provided by the domains of common variables). Here, we show the benefits of reasoning about such overlap. This is a challenging problem as finding a solution to just two ALLDIFFERENT constraints is NP-hard (Kutz *et al.* 2008) and existing approaches to deal with such overlaps require exponential space (Lardeux *et al.* 2008). Our approach is to focus on domains that are ordered, as often occurs in practice. For example in our time-tabling problem, values might represent times (which are naturally ordered). In such cases, domains can be compactly represented by intervals. Propagation algorithms can narrow such intervals using the notion of bound consistency. Our main result is to prove we can enforce bound consistency on two ALLDIFFERENT constraints in polynomial time. Our algorithm exploits a connection with matching on bipartite graphs. In particular, we consider *simultaneous* matchings. By generalizing Hall's theorem, we identify a necessary and sufficient condition for the existence of such a matching and show that the this problem is polynomial for convex graphs.

Formal background

Constraint programming. We use capitals for variables and lower case for values. Values range over 1 to d. We write $D(X)$ for the domain of values for X, $lb(X)$ ($ub(X)$) for the smallest (greatest) value in $D(X)$. A *global constraint* is one in which the number of variables n is a parameter. For instance, ALLDIFFERENT$([X_1, \ldots, X_n])$ ensures that $X_i \neq X_j$ for any $i < j$. Constraint solvers prune search by enforcing properties like domain consistency. A constraint is *domain consistent* (*DC*) iff when a variable is assigned any value in its domain, there are compatible values in the domains of all other variables. Such an assignment is a *support*. A constraint is *bound consistent* (*BC*) iff when a variable is assigned the minimum or maximum value in its domain, there are compatible values between the minimum and maximum domain value for all other variables. Such an assignment is a *bound support*. A constraint is *bound disentailed* iff no possible assignment is a bound support.

Graph Theory. Solutions of ALLDIFFERENT correspond to matchings in a bipartite variable/value graph (Régin 1994).

Definition 1. *The graph $G = \langle V, E\rangle$ is bipartite if V partitions into 2 classes, $V = A \cup B$ and $A \cap B = \emptyset$, such that every edge has ends in different classes.*

Definition 2. *Let $G = \langle A \cup B, E\rangle$ be a bipartite graph. A matching that covers A is a set of pairwise non-adjacent edges $M \subseteq E$ such that every vertex from A is incident to* *exactly one edge from M.*

We will consider simultaneous matchings on bipartite graphs(SIM-BM) (Kutz *et al.* 2008).

Definition 3. *An* overlapping bipartite graph *is a bipartite graph $G = \langle A \cup B, E\rangle$ and two sets S and T such that $A = S \cup T$, $A \cap B = \emptyset$, and $S \cap T \neq \emptyset$.*

Definition 4. *Let $\langle A \cup B, E\rangle$ and S, T be an overlapping bipartite graph. A simultaneous matching is a set of edges $M \subseteq E$ such that $M \cap (S \times B)$ and $M \cap (T \times B)$ are matchings that cover S and T, respectively.*

In the following, we use the convention that a set of vertices P is a subset of the partition A. We write $N(P)$ for the neighborhood of P, $P^S = P \cap (S \setminus T)$, $P^T = P \cap (T \setminus S)$ and $P^{ST} = P \cap S \cap T$. SIM-BM problems frequently occur in real world applications like production scheduling and timetabling. We introduce here a simple exam timetabling problem that will serve as a running example.

Running example. *We have 7 exams offered over 5 days and 2 students. The first student has to take the first 5 exams and the second student has to take the last 5 exams. Due to the availability of examiners, not every exam is offered each day. For example, the first exam cannot be on the last day of the week. Only one exam can be sat each day. This problem can be encoded as a* SIM-BM *problem. A represents the exams and contains 7 vertices X_1 to X_7. B represents the days and contains the vertices 1 to 5. $S = [X_1, X_2, X_3, X_4, X_5]$ and $T = [X_3, X_4, X_5, X_6, X_7]$. We connect vertices between A and B to encode the availability restrictions of the examiners. The adjacency matrix of the graph is as follows:*

		1	2	3	4	5
$A^S = S \setminus T$	X_1	*	*	*	*	
	X_2	*	*			
$A^{ST} = S \cap T$	X_3	*	*	*		
	X_4	*	*	*		
	X_5	*	*	*	*	*
$A^T = T \setminus S$	X_6		*	*		
	X_7		*	*	*	*

Finding a solution for this SIM-BM *problem is equivalent to solving the timetabling problem.* □

Simultaneous Bipartite Matching

We now consider how to find a simultaneous matching. Unfortunately, this problem is NP-complete in general (Kutz *et al.* 2008). Our contribution here is to identify a necessary and sufficient condition for the existence of a simultaneous matching based on an extension of Hall's theorem (Hall 1935). We use this to show that a simultaneous matching on a convex bipartite graph can be found in polynomial time.

In the following, let $G'_{(u,v)}$ be the subgraph of the overlapping bipartite graph G that is induced by choosing an edge (u, v) to be in the simultaneous matching. If $u \in A^{ST}$ then $G'_{(u,v)} = G - \{u, v\}$. If $u \in A^S$ (and symmetrically if $u \in A^T$) then $G'_{(u,v)} = \langle V - \{u\}, E \setminus \{(u', v)|u' \in S\}\rangle$. If M is a SIM-BM in $G'_{(u,v)}$, then $M \cup \{(u, v)\}$ is a SIM-BM in G. Since the edge (u, v) is implied throughout, we write $G' = G'_{(u,v)}$. In addition, we write $N'(P) = N_{G'}(P)$.

Extension of Hall's Theorem

Hall's theorem provides a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.

Theorem 1 (Hall Condition (Hall 1935))**.** *Let $G = \langle A \cup B, E\rangle$ such that $A \cap B = \emptyset$. There exists a perfect matching iff $|N(P)| \geq |P|$ for $P \subseteq A$.*

Interestingly we only need a small adjustment for simultaneous matching.

Theorem 2 (Simultaneous Hall Condition (SIM-HC))**.** *Let $G = \langle A \cup B, E\rangle$ and sets S, T be an overlapping bipartite graph. There exists a* SIM-BM, *iff $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P|$ for $P \subseteq A$.*

Proof. We prove SIM-HC by induction on $|A|$. When $|A| = 1$, the statement holds. Let $|A| = k > 1$.

If $A^S = \emptyset$ or $A^T = \emptyset$ then SIM-HC reduces to the condition of Hall's theorem and the statement is true for that reason. Hence, we assume $A^S \neq \emptyset$ and $A^T \neq \emptyset$. We show that there is an edge (u, v) that can be chosen for a simultaneous matching and the graph $G'_{(u,v)}$ will satisfy SIM-HC. Following (Diestel 2006), page 37, we consider two cases. The first case when all subsets of A satisfy the strict SIM-HC, namely, $|N(P)| + |N(P^S) \cap N(P^T)| > |P|$ and the second case when we have an equality.

Case 1. Suppose $|N(P)| + |N(P^S) \cap N(P^T)| > |P|$ for all sets $P \subset A$. As $A^S \neq \emptyset$ we select any edge $(u, v), u \in A^S$ and construct the graph $G'_{(u,v)}$ (the case $u \in A^T$ is symmetric). For any set $P \subset A \setminus \{u\}$ we consider two cases: either $v \notin N(P)$ or $v \in N(P)$. In the first case, the neighborhood of P is the same in G and G', so the SIM-HC holds for P. In the case that $v \in N(P)$, then either v is a shared neighbor of P^S and P^T, which means that $N'(P^S) \cap N'(P^T) = N(P^S) \cap N(P^T) - 1$ but $N'(P) = N(P)$ by construction, or v is a neighbor of P^S but not of P^T. Therefore $N'(P) \geq N(P) - 1$. But $N'(P^S) \cap N'(P^T) = N(P^S) \cap N(P^T)$ by construction. In either case, $|N'(P)| + |N'(P^S) \cap N'(P^T)| \geq |N(P)| + |N(P^S) \cap N(P^T)| - 1 \geq |P|$ for any set P in G'. By the inductive hypothesis there exists a simultaneous matching in it.

Case 2. Suppose that there exists a set $P \subsetneq A$ such that $|N(P)| + |N(P^S) \cap N(P^T)| = |P|$. Let $Q = \langle A' \cup B', E'\rangle$ such that $A' = A \setminus P$, $B' = B \setminus (N(P^S) \cap N(P^T))$ and

$$
\begin{aligned}
E' = \quad & \{(u, v) \in E \cap (A' \times B') \mid \\
& (u \in A^{S'} \implies v \notin N(P) \setminus N(P^T)) \quad \wedge \\
& (u \in A^{T'} \implies v \notin N(P) \setminus N(P^S)) \quad \wedge \\
& (u \in A^{ST'} \implies v \notin N(P)) \quad \}
\end{aligned}
$$

There exists a simultaneous matching in $G - Q$ by the inductive hypothesis. We claim that the SIM-HC holds also for Q. This implies that, by the inductive hypothesis, there exists a simultaneous matching in Q. Suppose there exists a set $P' \subseteq A'$ that violates the SIM-HC in Q.

We denote as $N(P)$ the neighborhood of P in G and $N_Q(P')$ as the neighborhood of P' in Q. We know that the sets P' and P are disjoint. We observe that $N(P \cup$

$P') = N(P) \cup N(P') = N(P) \cup N_Q(P')$, because $(N(P') \setminus N_Q(P')) \subseteq N(P)$ by construction of Q. Moreover $|N(P) \cup N_Q(P')| = |N(P)| + |N_Q(P')| - |N(P) \cap N_Q(P')| = |N(P)| + |N_Q(P')| - |N(P) \cap (N_Q(P^{S'}) \cup N_Q(P^{T'}) \cup N_Q(P^{ST'}))|$. By construction of Q, we have that $N_Q(P^{ST'}) \cap N(P) = \emptyset$, $N_Q(P^{S'}) \cap N(P) = N_Q(P^{S'}) \cap N(P^T)$ and $N_Q(P^{T'}) \cap N(P) = N_Q(P^{T'}) \cap N(P^S)$. Hence, $|N(P) \cup N_Q(P')| = |N(P)| + |N_Q(P')| - |D|$, where $D = (N(P^S) \cap N_Q(P^{T'})) \cup (N(P^T) \cap N_Q(P^{S'}))$. Similarly, $N(P^S \cup P^{S'}) = N(P^S) \cup N_Q(P^{S'})$ and $N(P^T \cup P^{T'}) = N(P^T) \cup N_Q(P^{T'})$. Therefore, $|N(P^S \cup P^{S'}) \cap N(P^T \cup P^{T'})| = |(N(P^S) \cup N_Q(P^{S'})) \cap (N(P^T) \cup N_Q(P^{T'}))| \leq |N(P^S) \cap N(P^T)| + |N_Q(P^{S'}) \cap N_Q(P^{T'})| + |D|$. Finally, we have that

$$
\begin{aligned}
|N(P \cup P')| + |N(P^S \cup P^{S'}) \cap N(P^T \cup P^{T'})| &\leq \\
|N(P)| + |N_Q(P')| - |D| &+ \\
|N(P^S) \cap N(P^T)| + |N_Q(P^{S'}) \cap N_Q(P^{T'})| + |D| &= \\
|N(P)| + |N(P^S) \cap N(P^T)| &+ \\
|N_Q(P')| + |N_Q(P^{S'}) \cap N_Q(P^{T'}| &< \\
|P| + |P'| = |P \cup P'|. &
\end{aligned}
$$

Hence $P \cup P'$ violates the SIM-HC in G, a contradiction. Therefore, there exists a simultaneous matching in Q.

Let M be a simultaneous matching in Q. For any edge $(u,v) \in M$, we construct the graph $R = (G - Q)_{(u,v)}$ and show that $N_R(P^*) = N_G(P^*)$ and $N_R(P^{S^*}) \cap N_R(P^{T^*}) = N(P^{S^*}) \cap N(P^{T^*})$ for any $P^* \subseteq P$.

Let (u,v) be an edge in M. By construction of Q, we have that $v \notin N(P^{S^*}) \cap N(P^{T^*})$. Hence, the construction of R leaves the size of $N_R(P^{S^*}) \cap N_R(P^{T^*})$ the same as in G. Moreover, $u \in A^{ST'} \Rightarrow v \notin N(P^*)$, $u \in A^{S'} \Rightarrow v \notin N(P^{S^*})$, $u \in A^{T'} \Rightarrow v \notin N(P^{T^*})$.

Consider the remaining options for the edge (u,v). If $u \in A^{S'}$ ($u \in A^{T'}$ is similar) then v can be in $N(P^{T^*})$, so v is a shared vertex with $N(P^*)$. There are two cases to consider: $v \in N(P^{T^*}) \setminus N(P^{ST^*})$ and $v \in N(P^{T^*}) \cap N(P^{ST^*})$. In the first case, the construction of R leaves the size of $N_R(P^{T^*})$ the same as $N(P^{T^*})$ because $u \in A^{S'}$ and can share vertices with P^{T^*}. In the latter case, $N_R(P^{ST^*}) = N(P^{ST^*}) \setminus \{v\}$, but $N_R(P^{T^*}) = N(P^{T^*})$. Hence, $N_R(P^*) = N(P^*)$ in both cases. Therefore, the SIM-HC holds for any $P^* \subseteq P$. □

Running example. *In our running example $A = [X_1, X_2, X_3, X_4, X_5, X_6, X_7]$, $A^S = [X_1, X_2]$, $A^{ST} = [X_3, X_4, X_5]$ and $A^T = [X_6, X_7]$. It is easy to check that the simultaneous Hall condition holds for all subsets of the partition A.*

Note that Theorem 2 does not give a polynomial time method to decide if a simultaneous matching exists. Verifying that SIM-HC holds requires checking the exponential number of subsets of A.

Removing edges

To build a propagator, we consider how to detect edges that cannot appear in any simultaneous matching.

Definition 5. *Let $G = \langle A \cup B, E\rangle$ and sets S, T be an overlapping bipartite graph. A set P, $P \subseteq A$, is*

***a simultaneous Hall set** iff* $|N(P)| + |N(P^S) \cap N(P^T)| = |P|$.

***an almost simultaneous Hall set** iff* $|N(P)| + |N(P^S) \cap N(P^T)| = |P| + 1$.

***a loose set** iff* $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P| + 2$.

Theorem 3. *$G = \langle A \cup B, E\rangle$ and sets S, T be an overlapping bipartite graph. Each edge (u,v), $u \in A$ and $v \in B$ can be extended to a matching that covers S and T iff*

1. ***for each set** P:*
   (a) $|N(P)| + |N(P^S) \cap N(P^T)| \geq |P|$
2. ***for each simultaneous Hall set** P:*
   (a) *if $u \notin P$ then $v \notin (N(P^S) \cap N(P^T))$*
   (b) *if $u \in S \setminus (T \cup P)$ then $v \notin \big(N(P) \setminus N(P^T)\big)$*
   (c) *if $u \in T \setminus (S \cup P)$ then $v \notin \big(N(P) \setminus N(P^S)\big)$*
   (d) *if $u \in (S \cap T) \setminus P$ then $v \notin N(P)$*
3. ***for each almost simultaneous Hall set** P:*
   (a) *if $u \in (S \cap T) \setminus P$ then $v \notin N(P^S) \cap N(P^T)$*

Proof. **Soundness**. The soundness of Rule 1a follows from Theorem 2. Let (u,v) be an edge that we want to extend to a matching. Suppose that (u,v) violates one of the rules for a SIM-HALLSET or an A-SIM-HALLSET P in G. We show that if (u,v) is selected to be in a matching, then P fails SIM-HC in $G'_{(u,v)}$.

Rule 2a: If (u,v) violates Rule 2a for a SIM-HALLSET P then $|N'(P^S) \cap N'(P^T)| = |N(P^S) \cap N(P^T)| - 1$ and $N'(P) = N(P)$, so the SIM-HC is violated for P in G'.

Rule 2b: If (u,v) violates Rule 2b for a SIM-HALLSET P then $|N'(P)| = |N(P)| - 1$ and $|N'(P^S) \cap N'(P^T)| = |N(P^S) \cap N(P^T)|$ so the SIM-HC is violated for P in G'.

Rule 2c: Symmetric to Rule 2b.

Rule 2d: If (u,v) violates Rule 2d for a SIM-HALLSET P then $|N'(P)| = |N(P)| - 1$ so the SIM-HC is violated for P in G'.

Rule 3a: If (u,v) violates Rule 3a for an A-SIM-HALLSET P then $|N'(P)| = |N(P)| - 1$ and $|N'(P^S) \cap N'(P^T)| = |N(P^S) \cap N(P^T)| - 1$, so $|N'(P)| + |N'(P^S) \cap N'(P^T)| = |P| - 1$ and the SIM-HC is violated for P in G'.

Completeness. Second, we show that Rules 2a- 3a are *complete*. We will show that we can use any edge (u,v) in a maching by showing that the graph $G'_{(u,v)}$ satisfies the SIM-HC, thus has a SIM-BM.

Suppose there is a set P that violates the SIM-HC in G' but not in G so that

$$|N'(P)| + |N'(P^S) \cap N'(P^T)| < |P| \tag{1}$$

and

$$|N(P)| + |N(P^S) \cap N(P^T)| \geq |P| \tag{2}$$

Note that $N'(P) = N(P)\setminus\{v\}$ and $N'(P^S)\cap N'(P^T) = N(P^S)\cap N(P^T)\setminus\{v\}$. Hence, $|N'(P)| \geq |N(P)| - 1$ and $|N'(P^S)\cap N'(P^T)| \geq |N(P^S)\cap N(P^T)| - 1$.

There are three cases to consider for P in G, when P is a loose set, a SIM-HALLSET and an A-SIM-HALLSET in G. These cases are similar, so we consider only the most difficult case. Let P be an A-SIM-HALLSET in G. If $u \in A^{ST}$ then $v \notin N(P^S)\cap N(P^T)$ by Rule 3a. Hence $N'(P^S)\cap N'(P^T) = N(P^S)\cap N(P^T)$, so $|N'(P)| + |N'(P^S)\cap N'(P^T)| \geq |N(P)| + |N(P^S)\cap N(P^T)| - 1 \geq |P|$ and therefore (1) and (2) cannot both be true.

If $u \in A^S$ ($u \in A^T$ is symmetric) then $v \in N(P^S)\cap N(P^T)$ or its complement. In the first case $N'(P) = N(P)$, while in the second $N'(P^S)\cap N'(P^T) = N(P'^S)\cap N(P'^T)$. In both cases, $|N'(P)| + |N'(P^S)\cap N'(P^T)| \geq |N(P)| + |N(P^S)\cap N(P^T)| - 1 \geq |P|$ so (1) and (2) cannot both be true. □

Running example. *Consider again our running example. We show that Rules 2a-3a remove every edge that can not be extended to a matching. Consider the set* $P = \{X_2, X_3, X_4, X_6\}$. *This is a* SIM-HALLSET *as* $N(P) = \{1, 2, 3\}$, $N(P^S)\cap N(P^T) = \{2\}$ *and* $4 = |P| = |N(P)| + |N(P^S)\cap N(P^T)| = 4$. *Hence, by Rule 2d we prune* $1, 2, 3$ *from* X_5 *and by Rule 2a we prune* 2 *from* X_1 *and* X_7. *Now consider the set* $P = \{X_2, X_3, X_6\}$. *This is an* A-SIM-HALLSET. *By Rule 3a we prune* 2 *from* X_4. *The set* $P = \{X_2, X_4, X_6\}$ *is also an* A-SIM-HALLSET *and, by Rule 3a, we prune* 2 *from* X_3. *Next consider the set* $P = \{X_3, X_4\}$ *which is a* SIM-HALLSET. *By Rules 2b and 2c we prune* $1, 3$ *from* X_1, X_2, X_6 *and* X_7. *Now,* $\{X_1\}$ *is a* SIM-HALLSET *and* 4 *is pruned from* X_5 *by Rule 2d. Finally, from the simultaneous Hall set* $\{X_5\}$, *we prune* 5 *from* X_7 *using Rule 2c and we are now at the fixpoint.*

		1	2	3	4	5
$A^S = S \setminus T$	X_1				*	
	X_2		*			
$A^{ST} = S \cap T$	X_3	*		*		
	X_4	*		*		
	X_5					*
$A^T = T \setminus S$	X_6		*			
	X_7				*	

The overlapping ALLDIFFERENT constraint

We now uses these results to build a propagator.

Definition 6. OVERLAPPINGALLDIFF$([X], S, T)$ *where* $S \subseteq X$, $T \subseteq X$, $S \cup T = X$ *holds iff* ALLDIFFERENT(S) *and* ALLDIFFERENT(T) *hold simultaneously .*

Enforcing DC on the OVERLAPPINGALLDIFF constraint is NP-hard (Bessiere *et al.* 2007). We consider instead enforcing just BC. This relaxation is equivalent to the simultaneous matching problem on a bipartite convex variable-value graph. Our main result is an algorithm that enforces BC on the OVERLAPPINGALLDIFF constraint in $O(nd^3)$ time. The algorithm is based on the decomposition of the OVERLAPPINGALLDIFF constraint into a set of arithmetic constraints derived from Rules 2b–3a. It is inspired by a decomposition of ALLDIFFERENT (Bessiere *et al.* 2009). As there, we introduce Boolean variables a_{ilu}, b_{il} to represent whether X_i takes a value in the interval $[l, u]$ and the variables C^S, C^{ST} and C^T to represent bounds on the number of variables from $S \setminus T$, $T \setminus S$ and $S \cap T$ that may take values in the interval $[l, u]$. We introduce the following set of constraints for $1 \leq i \leq n$, $1 \leq l \leq u \leq d$ and $u - l < n$:

$$b_{il} = 1 \iff X_i \leq l \tag{3}$$

$$a_{ilu} = 1 \iff (b_{i(l-1)} = 0 \wedge b_{iu} = 1) \tag{4}$$

$$C^{ST}_{lu} = \sum_{i \in S\cap T} a_{ilu} \tag{5}$$

$$C^{S}_{lu} = \sum_{i \in S\setminus T} a_{ilu} \tag{6}$$

$$C^{T}_{lu} = \sum_{i \in T\setminus S} a_{ilu} \tag{7}$$

$$C^{ST}_{1u} = C^{ST}_{1l} + C^{ST}_{(l+1)u} \tag{8}$$

$$C^{ST}_{lu} + C^{S}_{lu} \leq u - l + 1 \tag{9}$$

$$C^{ST}_{lu} + C^{T}_{lu} \leq u - l + 1 \tag{10}$$

We also introduce a dummy variable $C^{ST}_{10} = 0$ to simplify the following lemma and theorems.

Lemma 1. *Consider a sequence of values* $v_1, v_2, \ldots, v_k$. *Enforcing* BC *on* (8) *ensures* $ub(C^{ST}_{1v_k-1}) \leq \sum_{i:v_i<v_{i+1}} ub(C^{ST}_{v_iv_{i+1}-1}) - \sum_{i:v_i>v_{i+1}} lb(C^{ST}_{v_{i+1}v_i-1})$.

Proof. For every i such that $v_i < v_{i+1}$, constraint (8) ensures that $ub(C^{ST}_{1v_{i+1}-1}) \leq ub(C^{ST}_{1v_i-1}) + ub(C^{ST}_{v_iv_{i+1}-1})$. For every i such that $v_i > v_{i+1}$ constraint (8) ensures that $ub(C^{ST}_{1v_{i+1}-1}) \leq ub(C^{ST}_{1v_i-1}) - lb(C^{ST}_{v_{i+1},v_i-1})$. The left side of each inequality can be substituted into the right side of another inequality until one obtains $ub(C^{ST}_{1v_k-1}) \leq \sum_{i:v_i<v_{i+1}} ub(C^{ST}_{v_iv_{i+1}-1}) - \sum_{i:v_i>v_{i+1}} lb(C^{ST}_{v_{i+1}v_i-1})$. □

Theorem 4. *Enforcing* BC *on* (3)-(10) *detects bound disentailment of* OVERLAPPINGALLDIFF *in* $O(nd^2)$ *time but does not enforce* BC *on* OVERLAPPINGALLDIFF.

Proof. First we derive useful upper bounds for the variables C^{ST}_{lu}. Consider a set P and an interval $[a, b]$ such that $N(P) = [a, b]$. Let $[c_1, d_1] \cup \ldots \cup [c_k, d_k]$ be a set of intervals that tightly contain variables from P^S so that $\forall i, [c_i, d_i] \in N(P^S)$, $[e_1, f_1] \cup \ldots \cup [e_m, f_m]$ be a set of intervals that tightly contain variables from P^T so that $\forall i, [e_i, f_i] \in N(P^T)$, and $I_1 \cup \ldots \cup I_p$ are intersection intervals between intervals $[c_i, d_i]$ and $[e_i, f_i]$, i.e., $\forall i, I_i \in N(P^S)\cap N(P^T)$.

We first remove all intervals $[c_i, d_i]$ ($[e_i, f_i]$) that are completely inside an interval $[e_j, f_j]$ ($[c_j, d_j]$). For any of these intervals $[c_i, d_i]$ ($[e_i, f_i]$ is similar) there exists an intersection interval I_j such that $I_j = [c_i, d_i]$. We denote the set of removed intervals RI. The remaining intervals are $\{[c_1, d_1], \ldots, [c_{k'}, d_{k'}]\}$, $[e_1, f_1] \ldots, [e_{m'}, f_{m'}]\}$ and $I_1, \ldots, I_{p'}$. For any interval $[c_i, d_i]$, (9) ensures that $ub(C^{ST}_{c_id_i}) \leq d_i - c_i + 1 - lb(C^{S}_{c_id_i})$. Similarly for an interval $[e_i, f_i]$, we have $ub(C^{ST}_{e_if_i}) \leq f_i - e_i + 1 - lb(C^{T}_{e_if_i})$.

We sort the union of remaining intervals $[c_i, d_i]$ and $[e_i, f_i]$ by their lower bounds and list them as semi-open intervals $[g_1, g_2), [g_3, g_4), \ldots, [g_{k'+m'-1}, g_{k'+m'})$. Using the sequence $a, (g_1, g_2, \ldots, g_{k'+m'}, b+1, a)^x$ where $(.)^x$ indicates a repetition of x times the same sequence, Lemma 1 provides the inequality $ub(C^{ST}_{1a-1}) \leq ub(C^{ST}_{1a-1}) + x(ub(C^{ST}_{ag_1-1}) + ub(C^{ST}_{g_{k'+m'}b}) + \sum_{i=1}^{k'} ub(C^{ST}_{c_i d_i}) + \sum_{i=1}^{m'} ub(C^{ST}_{e_i f_i}) - lb(C^{ST}_{a,b}))$. Substituting the inequalities that we already defined, we obtain $ub(C^{ST}_{1a-1}) \leq a - 1 + x(b - a + 1 + \sum_{i=1}^{k'} lb(C^{S}_{c_i d_i}) - \sum_{i=1}^{m'} lb(C^{T}_{e_i f_i}) + \sum_{i=1}^{p'} |I_i| - lb(C^{ST}_{a,b}))$.

For any removed interval $I_j \in RI$ we have $(|I_j| - lb(C^{S}_{min(I_j)max(I_j)})) \geq 0$ or $(|I_j| - lb(C^{T}_{min(I_j)max(I_j)})) \geq 0$. We reintegrate all removed intervals into the inequation to get $ub(C^{ST}_{1a-1}) \leq a - 1 + x(b - a + 1 - \sum_{i=1}^{k} lb(C^{S}_{c_i d_i}) - \sum_{i=1}^{m} lb(C^{T}_{e_i f_i}) + \sum_{i=1}^{p} |I_i| - lb(C^{ST}_{ab}))$. Note that $\sum_{i=1}^{p} |I_i| = |N(P^S) \cap N(P^T)|$, $lb(C^S_{ab}) = \sum_{i=1}^{k} lb(C^{S}_{c_i d_i})$ and $lb(C^T_{ab}) = \sum_{i=1}^{m} lb(C^{T}_{e_i f_i})$. Hence

$$ub(C^{ST}_{1a-1}) \leq a - 1 + x(b - a + 1 - lb(C^S_{ab}) - lb(C^T_{ab}) + |N(P^S) \cap N(P^T)| - lb(C^{ST}_{ab})) \qquad (*)$$

Bound disentailment. Suppose, for the purpose of contradiction, that OVERLAPPINGALLDIFF is bound disentailed and that constraints (3)-(10) are bound consistent. Then, there exists a set P, such that $N(P)$ is an interval and $|N(P)| + |N(P^S) \cap N(P^T)| < |P|$. As P fails SIM-HC, it holds that $lb(C^{ST}_{ab}) + lb(C^S_{ab}) + lb(C^T_{ab}) \geq |P| > |N(P)| + |N(P^S) \cap N(P^T)| = b - a + 1 + |N(P^S) \cap N(P^T)|$ or $lb(C^{ST}_{ab}) \geq b - a + 2 - lb(C^S_{ab}) - lb(C^T_{ab}) + |N(P^S) \cap N(P^T)|$. Substituting the last inequality in (*) gives $ub(C^{ST}_{1a-1}) \leq a - 1 - x$. Choosing a large enough value for x (say a) gives the contradiction $ub(C^{ST}_{1a-1}) < 0$.

Bound consistency. To show that this decomposition does not enforce BC, consider the conjunction of ALLDIFFERENT $([X_1, X_2, X_3])$ and ALLDIFFERENT $([X_2, X_3, X_4])$ with $D(X_1) = [2, 3]$, $D(X_2) = [2, 4]$, $D(X_3) = [1, 3]$, $D(X_4) = [1, 2]$. Enforcing BC on (3)-(10) does not remove the bound inconsistent value $X_2 = 2$.

Complexity. There are $O(nd)$ constraints (3) that can be invoked $O(d)$ times at most. There are $O(nd^2)$ constraints (4) that can be invoked $O(1)$ times. There are $O(d^2)$ constraints (8) that can be invoked $O(n)$ times. There are $O(d^2)$ constraints (5)– (7) that can be propagated in $O(n)$. The remaining constraints take $O(nd^2)$ to propagate. The total time complexity is $O(nd^2)$. □

It follows immediately that the simultaneous matching problem is polynomial on bipartite convex graphs.

Theorem 5. *A simultaneous matching can be found in polynomial time on an overlapping convex bipartite graph.*

Next, we present an algorithm to enforce BC. We show that constraints (3)–(10) together with the following two constraints enforce all but one of the rules from Theorem 3.

$$C^T_{1u} = C^T_{1l} + C^T_{(l+1)u} \qquad 1 \leq l \leq u \leq d \qquad (11)$$

$$C^S_{1u} = C^S_{1l} + C^S_{(l+1)u} \qquad 1 \leq l \leq u \leq d \qquad (12)$$

Theorem 6. *Constraints* (3)-(12) *enforce Rules 2a– 2d.*

Proof Sketch. Based on Lemma 1, similar to the proof of Theorem 4, we show that all intervals that contain variables from a SIM-HALLSET P become saturated intervals, so that the lower bounds of the corresponding variables C^{ST}, C^S and C^T equal to their upper bounds. Hence, these values are pruned from domains of variables outside the set P. □

Theorem 7. *Suppose constraints* (3)-(12) *together with* $C^{ST}_{lu} = C^{ST}_{lk} + C^{ST}_{(k+1)u}$, $2 \leq l \leq k \leq u \leq d$ *have reached their fixpoint. Rule 3a can now be enforced in* $O(nd^3)$ *time.*

Proof Sketch. Let P is an A-SIM-HALLSET, $N(P) = [a, b]$. Similar to the proof of Theorem 4, we can obtain that $lb(C^{ST}_{ab}) + 1 \geq ub(C^{ST}_{ab})$. Hence, we can identify intervals, that might contain an A-SIM-HALLSET P. Next, we observe that if we add a dummy variable Z, $D(Z) = [a, b]$ to the set P so that $P' = P \cup \{Z\}$, $Z \in P^{ST'}$ then P' is a SIM-HALLSET. This allows us to identify the set $N(P^S) \cap N(P^T)$ by simulating constraints (3)-(12) inside the interval $[a, b]$ taking into account the variable Z. There are $O(d^2)$ intervals. Finding $N(P^S) \cap N(P^T)$ takes $O(n + d)$ time inside an interval. Enforcing the rule takes $O(nd)$ time. Hence, the total time complexity is $O(nd^3)$. □

From Theorems 6 and 7 it follows that

Theorem 8. *BC on* OVERLAPPINGALLDIFF *can be enforced in* $O(nd^3)$ *time.*

Running example. *We demonstrate the action of constraints* (3)-(12). *The interval* $[1, 4]$ *contains a* SIM-HALLSET $P = \{X_2, X_3, X_4, X_6\}$. ***Rule 2d.*** $lb(C^S_{12}) \geq 1$ *and* $lb(C^T_{23}) \geq 1 \Rightarrow$ (9), (10) $\Rightarrow ub(C^{ST}_{12}) \leq 1$ *and* $ub(C^{ST}_{23}) \leq 1 \Rightarrow$ (8) $\Rightarrow ub(C^{ST}_{13}) \leq 2$. *The interval* $[1, 3]$ *is saturated, as* $lb(C^{ST}_{13}) = ub(C^{ST}_{13})$. *Hence, by* (3)-(5), $[1, 3]$ *is removed from* $D(X_5)$. ***Rules 2b,2c.*** *As* $lb(C^{ST}_{13}) = 2 \Rightarrow$ (9) $\Rightarrow ub(C^S_{13}) \leq 1 \Rightarrow$ (12) $\Rightarrow ub(C^S_{12}) \leq 1$. *The interval* $[1, 2]$ *is saturated, as* $lb(C^S_{12}) = ub(C^{ST}_{12})$. *Hence, by* (3)-(4),(6), $[1, 2]$ *is removed from* $D(X_1)$. *Similarly,* $[2, 3]$ *is removed from* $D(X_7)$. ***Rule 2a.*** *This is satisfied as* 2 *is removed from all variables outside* P.

Exponential separation

We now give a pathological problem on which our new propagator does exponentially less work than existing methods.

Theorem 9. *There exists a class of problems such that enforcing BC on* OVERLAPPINGALLDIFF *immediately detects unsatisfiability while a search method that enforces DC on the decomposition into* ALLDIFFERENT *constraints explores an exponential search tree regardless of branching.*

Proof. The instance $\mathcal{I}_n$ is defined as follows $\mathcal{I}_n =$ ALLDIFFERENT$([X \cup Y])$ $\wedge$ ALLDIFFERENT$([Y \cup Z])$, $D(X_i) = [1, 2n-1]$, $i = 1, \ldots, n$, $D(Y_i) = [1, 4n-1]$, $i = 1, \ldots, 2n$ and $D(Z_i) = [2n, 4n-1]$, $i = 1, \ldots, n$.

OVERLAPPINGALLDIFF. Consider the interval $[1, 4n-1]$. $|P| = 4n$, $|N(P)| = 4n-1$ and $|N(P^S) \cap N(P^T)| = 0$. By Theorem 2, we detect unsatisfiability.

Decomposition. Consider any ALLDIFFERENT constraint. A subset of n or fewer variables has at least $2n-1$ values in their domains and a subset of $n+1$ to $3n$ variables has $4n-1$ values in their domains. Thus, to obtain a Hall set and prune, we must instantiate at least $n-1$ variables. □

Experimental results

To evaluate the performance of our decomposition we carried out an experiment on random problems. We used Ilog 6.2 on an Intel Xeon 4 CPU, 2.0 GHz, 4GB RAM. We compare the performance of the DC, BC (Lopez-Ortiz *et al.* 2003) propagators and our decomposition into constraints (3)-(12) for the OVERLAPPINGALLDIFF constraint (oBC). We use randomly generated problems with three global constraints: ALLDIFFERENT$(X \cup W)$, ALLDIFFERENT$(Y \cup W)$ and ALLDIFFERENT$(Z \cup W)$, and a linear number of binary ordering relations between variables in X, Y and Z. We use a random variable ordering and run each instance with 50 different seeds. As Table 1 shows, our decomposition reduces the search space significantly, is much faster and solves more instances overall.

Table 1: Random problems. n is the size of X, Y and Z, o is the size of W, d is the size of variable domains. Number of instances solved in 300 sec out of 50 runs / average backtracks/average time to solve.

n,d,o	BC	DC	oBC
	#s / #bt / t	#s / #bt / t	#s / #bt / t
4, 15, 10	14 /2429411 / 61.8	41 /1491341 / 52.1	**42 / 17240 / 32.5**
4, 16, 11	6 /5531047 / 153.7	22 /1745160 / 67.9	**31 / 8421 / 19.5**
4, 17, 12	1 / 17 / 0	6 /2590427 / 100.9	**24 / 8185 / 21.5**
5, 16, 10	11 /3052298 / 82.0	37 /1434903 / 58.2	**42 / 20482 / 48.5**
5, 17, 11	2 /3309113 / 94.5	19 /2593819 / 114.6	**26 / 4374 / 15.8**
5, 18, 12	1 / 17 / 0	4 /2666556 / 133.1	**22 / 3132 / 12.2**
6, 17, 10	11 /2845367 / 79.1	31 /1431671 / 66.3	**40 / 6796 / 21.9**
6, 18, 11	4 / 199357 / 6.6	16 /1498128 / 80.2	**31 / 4494 / 17.5**
6, 19, 12	4 /3183496 / 110.0	5 /1035126 / 66.2	**27 / 3302 / 15.5**
TOTALS			
sol/total	54 /450	181 /450	**285** /450
avg time for sol	78.072	70.551	**24.689**
avg bt for sol	2818926	1666568	**9561**

Conclusions

We have generalized Hall's theorem to simultaneous matchings in a bipartite graph. This generalization suggests a polynomial time algorithm to find a simultaneous matching in a convex bipartite graph. We applied this to a problem in constraint programming of propagating conjunctions of ALLDIFFERENT constraints. Initial experimental results suggest that reasoning about such conjunctions can significantly reduce the size of the explored search space. There are several avenues for future research. For example, the algorithmic techniques proposed in (Puget 1998) and (Lopez-Ortiz *et al.* 2003) may be generalizable to simultaneous bipartite matchings, giving more efficient propagators. Further, matchings are used to propagate other constraints such as NVALUE (Bessiere *et al.* 2006). It may be possible to apply similar insights to develop propagators for conjunctions of other global constraints, or to improve existing propagators for global constraints that decompose into overlapping constraints like SEQUENCE (Brand *et al.* 2005). Finally, we may be able to develop polynomial time propagators for otherwise intractable cases if certain parameters are fixed (Bessiere *et al.* 2008).

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