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# Pattern Avoidance and HDOL Words

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WORDS 2011

# Patterns

Pattern: finite word over  $\{A, B, C, \dots\}$ .

Word: over  $\Sigma_k = \{0, 1, \dots, k-1\}$ .

Occurrence: obtained from the pattern by replacing each letter by a non-empty word.

## Example

- 10111011 is an occurrence of ABCACB with  $A = 10$ ,  $B = 1$ ,  $C = 1$ .
- 00101001 contains 3 occurrences of ABBA: 001010, 010100 and 1001.

$w$  avoids  $P$  if  $w$  contains no occurrence of  $P$  as a factor.

Avoidability index:  $\lambda(P)$  is the smallest  $k$  such that there exists an infinite word over  $\Sigma_k$  avoiding  $P$ .

## Avoidability index (1)

- $\lambda(AA) = 3$ : infinite ternary square-free words.
- $\lambda(AAA) = 2$ : infinite binary cube-free words.

Remark:  $\lambda(P) \geq 2$

Divisibility:

*ABCABCAAC* contains an occurrence *AABB*

$\implies \lambda(ABCABCAAC) \leq \lambda(AABB)$

$\implies$  partial order among patterns

## Avoidability index (2)

- $\dots \leq \lambda(ABCABC) \leq \lambda(ABAB) \leq \lambda(AA) = 3.$
- $\dots \leq \lambda(AABBCC) \leq \lambda(AABB) \leq \lambda(AA) = 3.$

We have:

- $\lambda(ABCABC) = 2, \lambda(ABAB) = 3.$
- $\lambda(AABBCC) = 2, \lambda(AABB) = 3.$

## Bounds on the avoidability index

Lower bounds: To prove that  $\lambda(P) \geq k$ , check that all words over  $\Sigma_{k-1}$  are finite by lexicographic enumeration. This is the easy part.

Upper bounds: To prove that  $\lambda(P) \leq k$ , construct an infinite word over  $\Sigma_k$  and prove that it avoids  $P$ .

# Square-free words and square-free morphisms

$\lambda(AA) = 3$ : infinite ternary square-free word.

square-free morphisms:

- $0 \mapsto 01201, 1 \mapsto 020121, 2 \mapsto 0212021.$
- $0 \mapsto 0120212012102120210,$   
 $1 \mapsto 1201020120210201021,$   
 $2 \mapsto 2012101201021012102.$

More than  $1.3^n$  ternary square-free words of size  $n$ .

## Morphic words

Pure morphic word (DOL):

fixed point  $m^\infty(a)$  of a morphism  $m$  such that  $m(a) = aw$ .

Morphic word (HDOL):

image  $h(m^\infty(a))$  of a pure morphic word by a morphism  $h$ .

- $\lambda(AA) = 3$ , avoided by pure morphic words.
- $s: 0 \mapsto 0101, 1 \mapsto 0011, 2 \mapsto 1000$ .  
The image by  $s$  of a square-free word avoids  $ABCABC$ .
- $\lambda(ABCABC) = 2$ , avoided by a morphic word, but no pure morphic word.
- Exponentially many binary words avoid  $ABCABC$ .

Julien Cassaigne's conjecture:

$\lambda(P) = k$  if and only if there exists a morphic word over  $\Sigma_k$  avoiding  $P$ .



## A square-free morphic word

$m_1: 0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1.$

$m_1^\infty(0)$  is the only infinite ternary word avoiding

- squares, 010, and 212.
- squares and  $0u1u0$  for  $u \in \Sigma_3^*$ .

$m_1$  is not square-free:

$m_1(0u1u0) = 012m_1(u)02m_1(u)012$  contains  $(2m_1(u)0)^2$ .

# Characterization

- A set  $S$  of forbidden patterns and factors.
- A morphic word  $w$  over  $\Sigma_k$ .

$S$  characterizes  $w$  if  $w$  avoids  $S$  and for every finite factor  $f$  of  $w$ ,  $S \cup f$  is unavoidable over  $\Sigma_k$ .

## Example

- $\{AA, 010, 212\}$  characterizes  $m_1^\infty(0)$ .
- $\{ABABA, 000, 111\}$  characterizes the Thue-Morse word.

This notion is defined to handle extendability issues.  
Makes no difference between a pattern and the associated formula.

## Other characterizations by Thue (1)

$m_2: 0 \mapsto 130402, 1 \mapsto 132, 2 \mapsto 1304, 3 \mapsto 1402, 4 \mapsto 1404.$

$h_1: 0 \mapsto 012102120210120212,$

$1 \mapsto 01210212,$

$2 \mapsto 01210212021,$

$3 \mapsto 01210120212,$

$4 \mapsto 0121012021.$

$h_2: 0 \mapsto 0210120102012,$

$1 \mapsto 021012,$

$2 \mapsto 02101201,$

$3 \mapsto 02102012,$

$4 \mapsto 0210201.$

- $\{AA, 010, 020\}$  characterizes  $h_1(m^\infty(1))$
- $\{AA, 121, 212\}$  characterizes  $h_2(m^\infty(1))$

## Other characterizations by Thue (2)

$m_2$ :  $0 \mapsto 130402$ ,  $1 \mapsto 132$ ,  $2 \mapsto 1304$ ,  $3 \mapsto 1402$ ,  $4 \mapsto 1404$

- $\{AA, 010, 020\}$  characterizes  $h_1(m_2^\infty(1))$
- $\{AA, 121, 212\}$  characterizes  $h_2(m_2^\infty(1))$

$m_2^\infty(1)$  appears in both. Interesting ?

- Take a ternary word avoiding  $\{AA, 010, 020\}$ , delete each letter after a 0, you get a word avoiding  $\{AA, 121, 212\}$ .
- $\{AA, 01, 03, 10, 12, 20, 23, 24, 31, 34, 42, 43, 141, 302, 414, 2132, 3213\}$  characterizes  $m_2^\infty(1)$ .

## Characterizations using $m_1^\infty(0)$

$h_3$ : 0  $\mapsto$  0010110111011101001,  
1  $\mapsto$  00101101101001,  
2  $\mapsto$  00010.

$h_4$ : 0  $\mapsto$  1001001101011001101001011001001101  
100101101001101100100110100101100110101,  
1  $\mapsto$  100100110100101,  
2  $\mapsto$  1001001101100101101001101.

- $\{AABBCABBA, 0011, 1100\}$  characterizes  $h_3(m_1^\infty(0))$ .
- Containing only 10101 as a  $2^+$ -power and 11 distinct squares characterizes  $h_4(m_1^\infty(0))$ .

## More examples by Golnaz Badkobeh

$m_3: 0 \mapsto 032, 1 \mapsto 04, 2 \mapsto 12, 3 \mapsto 14, 4 \mapsto 1432.$

$h_5: 0 \mapsto 1100, 1 \mapsto 110, 2 \mapsto 100, 3 \mapsto 10, 4 \mapsto 101100.$

$h_6: 0 \mapsto 100110, 1 \mapsto 10010110, 2 \mapsto 0110, 3 \mapsto 01,$   
 $4 \mapsto 0100110.$

- Containing only 01010 and 10101 as  $2^+$ -powers and 8 distinct squares characterizes  $h_5(m_3^\infty(0))$ .
- Containing only 1001001 as a  $2^+$ -power and 14 distinct squares characterizes  $h_6(m_3^\infty(0))$ .

## A third example with $m_3$

$h_7$ : 0  $\mapsto$  001110010110001101,  
1  $\mapsto$  00111001011000110100010110,  
2  $\mapsto$  00111001011101,  
3  $\mapsto$  0011100101110100111000110100010110,  
4  $\mapsto$  0011100101110100111000110100010110001101.

$\{AABBCC, ABCABC, 0000, 1111, 0001110010110,$   
 $1110001101001, 0110100111000, 1001011000111\}$   
characterizes  $h_7(m_3^\infty(0))$ .

## Remarks

- $m_1^\infty(0)$  appears in many characterizations and we understand why.
- $m_2^\infty(1)$  appears in two similar characterizations and we see the link.
- $m_3^\infty(0)$  appears in three characterizations, seemingly by chance.

We hope to:

- Discover new useful (pure) morphic words and find their properties.
- Find links between characterizations using a same pure morphic word.
- Use them to quickly find and prove new characterizations.



## Dejean words VS Pure morphic words

To obtain the avoidability index of patterns:

- I used morphic images of Dejean words.
  - Proves exponential factor complexity.
  - Automated proofs.
- Julien Cassaigne used (pure) morphic words.
  - More theoretical insights.
  - Mandatory for patterns with polynomial factor complexity.

The only word avoiding  $AB.AC.BA.BC.CA$  is the fixed point  $\Omega$  of  $0 \mapsto 01, 1 \mapsto 21, 2 \mapsto 03, 3 \mapsto 23$ , up to permutation of letters.

$\implies \{AB.AC.BA.BC.CA, 02, \dots\}$  characterizes  $\Omega$ .

## One last case by Golnaz Badkobeh

$m_4$ : 0  $\mapsto$  0102,  
1  $\mapsto$  1013,  
2  $\mapsto$  40135,  
3  $\mapsto$  51024,  
4  $\mapsto$  1024,  
5  $\mapsto$  0135.

$h_8$ : 0  $\mapsto$  10011,  
1  $\mapsto$  01100,  
2  $\mapsto$  01001,  
3  $\mapsto$  10110,  
4  $\mapsto$  0110,  
5  $\mapsto$  1001.

Containing only 0110110 and 1001001 as  $2^+$ -powers and 12 distinct squares characterizes  $h_8(m_4^\infty(0))$ .

## Proof technique

To prove that  $S = \mathcal{P} \cup \mathcal{F}$  characterizes a morphic word  $h(m^\infty(0))$ , where  $m$  is  $\Sigma_{k'} \rightarrow \Sigma_{k'}$ ,  $h$  is  $\Sigma_{k'} \rightarrow \Sigma_k$ , and  $k' > k$ :

- Prove that extendable words in  $S$  are  $h$ -images of words over  $\Sigma_{k'}$ .
- Find a set  $S' = \mathcal{P} \cup \mathcal{F}'$  that seem to characterize  $m^\infty(0)$  over  $\Sigma_{k'}$ .
- Prove that  $S'$  characterizes  $m^\infty(0)$  over  $\Sigma_{k'}$ :
  - Prove that extendable words in  $S'$  are  $m$ -images of words over  $\Sigma_{k'}$ .
  - Prove that the pre-images by  $m$  of words in  $S'$  are also in  $S'$ .

## Remarks

- Conjecture: If  $\lambda(P) = k$ , then there exists a finite set  $S$  of forbidden patterns and factors containing  $P$  that characterizes a morphic word over  $\Sigma_k$ .
- The converse (every morphic word has a characterization) does not seem to hold: consider the Fibonacci word.
- For all these characterizations, the set can be put in the form  $\{AB \cdots XAB \cdots X\} \cup \mathcal{F}$ . It would be nice to have characterizations with other patterns.
- Strategy “from above” to prove avoidability.

Thank you for your attention !