Semantics in Biproduct Dagger Categories: a quantum logic for natural language

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Abstract

Biproduct dagger categories serve as models for natural language. They link the extensional models of predicate calculus with the intensional models of quantum logic. The morphisms representing the extensional meanings of a grammatical string are translated to projectors representing the intensional meanings such that truth is preserved. Pregroup grammars serve as the tool that transforms a grammatical string into a morphism. The chosen linguistic examples concern negation, relative noun phrases, comprehension and quantifiers.

Keywords: Categorical logic, quantum logic, compact bilinear logic, compact bicategories, two-sorted functional first order logic, compositional semantics, pregroup grammars, proof graphs

1 Introduction

Biproduct dagger categories have been studied extensively in quantum logic, [Selinger, 2007], [Abramsky and Coecke, 2004], [Heunen and Jacobs, 2010]. They also constitute a natural candidate as a foundation of natural language semantics, because they formalise two logical abstractions present in the great majority of natural languages, count words (biproduct) and relative pronouns (dagger).[1]

These two operations are powerful enough to comprehend the structure of a compact closed category and with it the representation of morphisms by graphs that represent information flow. Information flow is the categorical version of the grammatical notions of dependency and control. The biproduct and dagger combined also capture the intuitive geometrical representation of linguistic and logical notions, say similarity and negation, via the inner product (cosine) and orthogonality.

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[1] A language that has no count words nor relative pronouns is the amazonian Pirahã, [Everett, 2005]

Natural language processing involves both syntactical and logical representation. The syntactical analysis is formulated in the language of compact bicategories, definitionally equivalent to monoidal, not necessarily symmetric categories in which every object has a right and a left adjoint. Throughout this paper, the *syntactical category* is the free compact bicategory $\mathcal{C}(\mathcal{B})$ generated by some category $\mathcal{B}$. The logical analysis takes place in *semantical categories*, i.e. biproduct dagger categories in which each object is finite dimensional.

Biproduct dagger categories include the categories of real respectively complex Hilbert spaces, which accommodate the semantic vector models, popular in Information Retrieval. They also include the category $\mathcal{2SF}$ of two-sorted functions, a model of two-sorted first order predicate logic.

The relevance of two-sorted first order logic for natural language resides in the fact that it is equivalent to second order logic with general models, see [Benthem and Doets, 1983], and in the common belief that second order logic suffices for natural language semantics.

The material of this chapter is organised in three sections. Section 2 presents the basic properties of biproduct dagger categories with an emphasis on the class of projectors called ' intrinsic ', because their matrix representation is the same in any biproduct dagger category. They include the morphisms arising from grammatical strings. Section 3 establishes the equivalence between the quantum logic of intrinsic projectors and the logic of two-sorted predicates. The essential characteristic of a two-sorted predicate is that it assigns truth values both to individuals and sets of individuals. Section 4 starts with a cut free axiomatisation of compact bilinear logic, [Lambek, 1993], and shows how pregroup grammars, [Lambek, 1999], construct syntactical analysis and semantical representation in compact bicategories, based on the proof graphs of [Preller and Lambek, 2007]. The section concludes with a few linguistic examples linking relative noun phrases and comprehension as well as quantifiers and negation.

2 Basic properties

This section recalls definitions and properties frequently intervening in quantum logic, see for example [Abramsky and Coecke, 2004], [Heunen and Jacobs, 2010], [Selinger, 2007]. Only the emphasis on 'intrinsic' morphisms is new.

2.1 Biproduct dagger categories

A *dagger category* is a category $\mathcal{C}$ together with a contravariant involution functor *dagger* $\dagger : \mathcal{C} \to \mathcal{C}$ that is the identity on objects. This means that the following equalities hold for any object V and morphisms $f : V \to W$, $g : W \to C$

$$\begin{aligned} V^\dagger &= V \\ 1_V^\dagger &= 1_V \\ (g \circ f)^\dagger &= f^\dagger \circ g^\dagger : C \to A \\ f^{\dagger\dagger} &= f : V \to W\,. \end{aligned}$$

Call $f^\dagger$ the *adjoint* of f.

In a dagger category colimits and limits coincide. For example, a coproduct of V and W with canonical injections q_1 and q_2 is also a product of V and W with canonical projections $q_1^\dagger$ and $q_2^\dagger$. Hence coproducts are biproducts in a dagger category.

An initial object 0 with a unique morphism $0_V : 0 \to V$ is also a terminal object with unique morphism $0_V^\dagger : V \to 0$. Hence 0 is a zero object where $0_{VW} = 0_W^\dagger \circ 0_V : V \to W$ is the unique morphism that factors through 0. The subscripts may be dropped, context permitting.

Definition 1. *A* biproduct dagger category *is a dagger category $\mathcal{C}$ that has an initial object* 0 *and binary coproducts such that the canonical injections $q_1 : V \to V \oplus W$ and $q_2 : W \to V \oplus W$ satisfy*

$$q_i^\dagger \circ q_i = 1, q_j^\dagger \circ q_i = 0 \text{ for } i,j = 1,2, \ i \neq j\,. \tag{1}$$

Note that $V \oplus 0 \simeq V$. Indeed, $q_1 : V \to V \oplus 0$ and $q_1^\dagger : V \oplus 0 \to V$ are inverse of each other.

Given $g_j : U \to V_j$, denote $\langle g_1, g_2 \rangle : U \to V_1 \oplus V_2$ the unique morphism satisfying

$$q_j^\dagger \circ \langle g_1, g_2 \rangle = g_j \text{ for } j = 1,2\,.$$

Similarly, for $h_i : W_i \to E$ denote $[h_1, h_2] : W_1 \oplus W_2 \to E$ the morphism determined by

$$[h_1, h_2] \circ q_i = h_i \text{ for } i = 1,2\,.$$

Finally, for $f_i : V_i \to W_i$, denote $f_1 \oplus f_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$ the unique morphism such that

$$q_i^\dagger \circ (f_1 \oplus f_2) \circ q_i = f_i \text{ and } q_i^\dagger \circ (f_1 \oplus f_2) \circ q_j = 0_{V_j W_i}, \text{ for } i,j = 1,2,\, i \neq j\,.$$

We have for any $g : U' \to U$ and $h : E \to E'$

$$\begin{array}{ll} \langle g_1, g_2 \rangle \circ g & = \langle g_1 \circ g, g_2 \circ g \rangle, \\ h \circ [h_1, h_2] & = [h \circ h_1, h \circ h_2] \\ (f_1 \oplus f_2) \circ \langle g_1, g_2 \rangle & = \langle f_1 \circ g_1, f_2 \circ g_2 \rangle \\ [h_1, h_2] \circ (f_1 \oplus f_2) & = [h_1 \circ f_1, h_2 \circ f_2] \end{array} \tag{2}$$

Any morphism $f : V_1 \oplus V_2 \to W_1 \oplus W_2$ is uniquely determined by the four morphisms $q_i^\dagger \circ f \circ q_j$, for $i,j = 1,2$. These four morphisms may be displayed in the form of a matrix

$$\begin{pmatrix} q_1^\dagger \circ f \circ q_1 & q_1^\dagger \circ f \circ q_2 \\ q_2^\dagger \circ f \circ q_1 & q_2^\dagger \circ f \circ q_2 \end{pmatrix}.$$

Proposition 1. *The following equalities hold in a biproduct dagger category*

$$\begin{array}{rl} 0_{VW}^\dagger & = 0_{WV} \\ \langle f_1, f_2 \rangle^\dagger & = [f_1^\dagger, f_2^\dagger] \\ (f_1 \oplus f_2)^\dagger & = f_1^\dagger \oplus f_2^\dagger\,. \end{array} \tag{3}$$

Any biproduct category $\mathcal{C}$ is enriched over abelian monoids, i.e. the binary operation defined on each hom-set $\mathcal{C}(V,W)$ by

$$f_1 + f_2 = [1_W, 1_W] \circ (f_1 \oplus f_2) \circ \langle 1_V, 1_V \rangle, \text{ for } f_1, f_2 : V \to W$$

is associative and commutative, with the unit 0_{VW}.

Moreover, addition is bilinear

$$h \circ (f_1 + f_2) \circ g = h \circ f_1 \circ g + h \circ f_2 \circ g, \text{ for } g : V' \to V, h : W \to W' \tag{4}$$

and

$$\begin{aligned} (f_1 + f_2)^\dagger &= f_1^\dagger + f_2^\dagger \\ q_1 \circ q_1^\dagger + q_2 \circ q_2^\dagger &= 1_{V \oplus W} . \end{aligned} \tag{5}$$

It follows that any biproduct category has a *matrix calculus*, i.e. the following equalities hold

$$M_{f+g} = M_f + M_g \text{ and } M_{g \circ f} = M_g M_f . \tag{6}$$

Define the n-ary biproduct

$$\begin{aligned} &V_1 \oplus \ldots \oplus V_0 := 0 \\ &V_1 \oplus \ldots \oplus V_n := (V_1 \oplus \ldots \oplus V_{n-1}) \oplus V_n , \end{aligned}$$

with the appropriate definitions of the injections $q_i : V_i \to V_1 \oplus \ldots \oplus V_n$ and the projections $q_i^\dagger : V_1 \oplus \ldots \oplus V_n \to V_i$, for $i = 1, \ldots, n$. In the case where $V_i = V$ for all $i = 1, \ldots, n$, write $n \cdot V = V_1 \oplus \ldots \oplus V_n$. Adopt a similar convention for $n \cdot f$, where $f : V \to W$.

Equalities (1) - (5) generalise to n-ary biproducts. Together, they constitute the generalised *Dagger Biproduct Calculus*. For example, the generalised version of (1) is

$$q_i^\dagger \circ q_i = 1_{V_i}, \; q_i^\dagger \circ q_j = 0_{V_j V_i}, \text{ for } i, j = 1, \ldots, n, i \neq j .$$

Any morphism $f : V_1 \oplus \ldots \oplus V_m \to W_1 \oplus \ldots \oplus W_n$ is completely determined by the nm morphisms $q_i^\dagger \circ f \circ q_j$, for $j = 1, \ldots, m$, $i = 1, \ldots, n$. Hence, $f = g$ if and only if the following *Matrix Equalities* hold

$$q_i^\dagger \circ g \circ q_j = q_i^\dagger \circ f \circ q_j, \text{ for } j = 1, \ldots, m, \; i = 1, \ldots, n . \tag{7}$$

The Equalities (3) - (6) also generalise to arbitrary biproducts.

Several geometrical notions intervening in vector spaces can be defined in dagger bipropduct categories

Definition 2. *Morphisms $f : U \to W$ and $g : V \to W$ are said to be* orthogonal *in W if $f^\dagger \circ g = 0$. A* projector *is an idempotent and self-adjoint morphism $p : V \to V$, i.e. $p \circ p = p$ and $p^\dagger = p$.*

Orthogonality is a symmetric relation. Every morphism is orthogonal to 0. In general, a morphism can have several distinct orthogonal morphisms.

The rest of this subsection is an argument that iterated biproducts of any object $V \not\simeq 0$ internalise propositions and finite subsets. Projectors will play the role of propositions, the canonical injections $q_i : V \to n \cdot V$ the role of individuals. Note that the canonical injections q_i and q_j are distinct for $i \neq j$, because $1_V \neq 0_{VV}$. Subsets of individuals are internalised as sums of distinct canonical injections.

A morphism $f : V \to W$ is *unitary* if $f^\dagger \circ f = 1_V$. A unitary f is necessarily monic and its adjoint is epic. In the case where f is an isomorphism, f is unitary if and only if $f \circ f^\dagger = 1_W$ if and only if $f^\dagger = f^{-1}$.

Proposition 2. *Let V be any object of $\mathcal{C}$. Assume $K = \{i_1, \ldots, i_k\}$ and $M = \{l_1, \ldots, l_m\}$ are disjoint subsets of $\{1, \ldots, n\}$.*

Then $q_K = [q_{i_1}, \ldots q_{i_k}] : k \cdot V \to n \cdot V$ is unitary and orthogonal to $q_M = [q_{l_1}, \ldots q_{l_m}] : m \cdot V \to n \cdot V$.

The endomorphism $p_K = q_K \circ q_K^\dagger : n \cdot V \to n \cdot V$ is a projector and

$$p_K + p_M = p_{K \cup M} \,. \tag{8}$$

Proof. First, recall that $[q_{i_1}, \ldots q_{i_k}]^\dagger = \langle q_{i_1}^\dagger, \ldots q_{i_k}^\dagger \rangle$. Hence, the Matrix Equalities (7) characterise p_K as the unique morphism satisfying

$$q_i^\dagger \circ p_K \circ q_j = \begin{cases} 1_V & \text{if } i = j \text{ and } j \in K \\ 0_{VV} & \text{else} \end{cases} , \text{ for } i, j = 1, \ldots, n \,. \tag{9}$$

Next, use the Dagger Biproduct Calculus and the Matrix Equalities to show that

$$\begin{aligned} \langle q_{i_1}^\dagger, \ldots q_{i_k}^\dagger \rangle \circ [q_{i_1}, \ldots q_{i_k}] &= 1_{k \cdot V} \\ \langle q_{i_1}^\dagger, \ldots q_{i_k}^\dagger \rangle \circ [q_{l_1}, \ldots q_{l_m}] &= 0 \,. \end{aligned} \tag{10}$$

This proves that q_K is unitary and orthogonal to q_M .

Finally, check that p_K is self-adjoint, via the equality recalled initially, and that it is idempotent, via the first equality of (10). Equality (8) follows from the Matrix Equalities and bilinearity of addition. □

Corollary 1. *If $V \not\simeq 0$, the map $K \mapsto p_K$ is a one-one correspondence between subsets $K \subseteq \{1, \ldots, n\}$ and the projectors p_K.*

Proof. Use the characterising equalities (9) and the fact that $1_V \neq 0_{vv}$. □

Corollary 2. *If $K \cap M = \emptyset$ and $K \cup M = \{1 \ldots, n\}$ then*

$$p_K + p_M = 1_{n \cdot V} = q_1 \circ q_1^\dagger + \cdots + q_n \circ q_n^\dagger \,.$$

Proof. The equality $p_{K \cup M} = 1_{n \cdot V}$ is a special case of (10). Hence, $p_K + p_M = 1_{n \cdot V}$ follows by (8). □

Recall that a morphism $g : U \to V$ is a *kernel* of $f : V \to W$ if it satisfies $f \circ g = 0$ and is universal for this property. Universality means that for any $h : X \to V$ with $f \circ h = 0$ there is a unique $h' : X \to U$ with $h = g \circ h$.

Proposition 3. *Let $K \subseteq \{1, \ldots, n\} = N$ and $M = N \setminus K$. Then q_M is a unitary kernel of p_K and $q_K^\dagger$. Moreover, q_K is the image of p_K.*

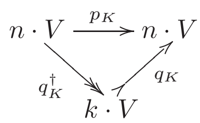
Proof. The equality $p_K \circ q_M = 0$ is a particular case of (10). To prove the universality of q_M, assume that $g : U \to n \cdot V$ satisfies $p_K \circ g = 0$. Let $h := q^\dagger \circ g : U \to m \cdot V$. Then

$$g = (p_K + p_M) \circ g = p_K \circ g + q_M \circ q_M^\dagger \circ g = q_M \circ q_M^\dagger \circ g = q_M \circ h \,.$$

This proves that $ker(p_K) = q_M$. We also have $ker(q_K^\dagger) = q_M$, because $q_K^\dagger \circ g = 0$ implies $p_K \circ g = 0$.

Finally, using the definition of [Heunen and Jacobs, 2010], compute

$$im(p_K) := ker((ker(p_K^\dagger))^\dagger) = ker((ker(p_K))^\dagger) = ker(q_M^\dagger) = q_K \,.$$



□

Note that $v : W \to n \cdot V$ is left invariant by p_K exactly when it factorizing through q_K. Indeed, $v = p_K \circ v$ is implies $v = q_K \circ (q_K^\dagger \circ v)$. Conversely, $v = q_K \circ g$ implies $v = q_K \circ (q_K^\dagger \circ q_K) \circ g = p_K \circ v$.

Define the negation of the projector p_K thus

$$\neg p_K := p_K^\perp = ker(p_K) \circ (ker(p_K))^\dagger = p_{N \setminus K} \,. \tag{11}$$

Proposition 4. *The projectors p_K, p_L satisfy for $K, L \subseteq \{1, \ldots, n\}$*

$$p_K \circ p_L = p_L \circ p_K = p_{K \cap L} \,.$$

The relation given by

$$p_K \le p_L \Leftrightarrow p_K \circ p_L = p_K \tag{12}$$

is a partial order with smallest element 0 and largest element 1 that defines a lattice structure on the projectors p_K.

Under the assumption that $V \not\cong 0$, the map $K \mapsto p_K$ is a negation preserving lattice isomorphism. In particular, $p_L \le \neg p_K$ if and only if $p_K \cap p_L = 0$.

Proof. Partition $K \cup L$ into the three disjoint subsets $M = K \cap L$, $M' = K \setminus (K \cap L)$, and $M'' = L \setminus (K \cap L)$. By Proposition (2), $p_K = p_M + p'_M$ and $p_L = p_M + p''_M$ and the mixed terms $p_M \circ p_{M''}$, $p_{M'} \circ p_M$, $p_{M'} \circ p_{M''}$ are equal to 0. Therefore

$$p_K \circ p_L = (p_M + p_{M'}) \circ (p_M + p_{M''}) = p_M \circ p_M = p_M \,.$$

Similarly, $p_L \circ p_K = p_M$. This proves the first assertion. The rest is now straight forward. □

Proposition 5. *The partial order of the projectors p_K is isomorphic to the partial order of their canonical images $im(p_K) = q_K$.*

More precisely, for arbitrary subsets $K = \{i_1, \ldots, i_k\}$ and $M = \{j_1, \ldots, j_m\}$ of $\{1, \ldots, n\}$ the following equivalence holds

$$p_K \leq p_M \text{ if and only if } q_K \leq q_M \text{ as subobjects.}$$

Proof. Recall that $p_K \leq p_M$ is equivalent to $K \subseteq M$ by (12). Assume that $q_K \leq q_M$ as subobjects and let $g : k \cdot V \to m \cdot M$ be the morphism such that $q_K = q_M \circ g$. Then $q_K = q_M \circ g$ implies $q_{i_l} = q_K \circ q_l = q_M \circ g \circ q_l$ and therefore $i_l \in M$, by Proposition 3, and this for $l = 1, \ldots, k$. Hence, $K \subseteq M$. Conversely, the inclusion $K \subseteq M$ provides an obvious factorisation $q_K = q_M \circ g$. □

2.2 Finite dimensional spaces

Definition 3. *A* finite dimensional biproduct dagger category*,* semantic category *for short, is a biproduct dagger category that has a distinguished object $I \not\simeq 0$ satisfying*

- $\alpha \circ \beta = \beta \circ \alpha$ *for all* $\alpha, \beta : I \to I$
- *for every object V there is an integer $n \geq 0$ and a unitary isomorphism $b_V : n \cdot I \to V$.*

In the category $\mathit{2SF}$ of two-sorted functions, I is a distinguished singleton set. In the category $\mathcal{RI}$ of semi-modules over the real interval $[0, 1]$, the distinguished object is the real interval $I = [0, 1]$. For real Hilbert spaces, $I = \mathbb{R}$, for complex Hilbert spaces, $I = \mathbb{C}$.

A *space* is an object V of $\mathcal{C}$ together with a unitary isomorphism $b_V : n \cdot I \to V$, called the *base* of the space. The integer n is the *dimension* of the space. A *vector* of V is a morphism from I to V. A *scalar* is an endomorphism of I . The scalars form a commutative semiring where multiplication is composition and addition is defined by Proposition 1.

Scalar multiplication is defined for any scalar $\alpha : I \to I$ and vector $v : I \to V$ by

$$\alpha v = v \circ \alpha \, .$$

Scalar multiplication is associative and commutes with addition

$$(\alpha\beta)v = \alpha(\beta v) \text{ and } \alpha(v + w) = \alpha v + \alpha w \, .$$

For any morphism $f : V \to W$ and vector $v : I \to V$, the *value* $f(v)$ of f at v is

$$f(v) = f \circ v \, .$$

The morphisms of $\mathcal{C}$ are *linear*, that is to say for $f : V \to W$, $v, w : I \to V$ and $\alpha, \beta : I \to I$

$$f(\alpha v + \beta w) = \alpha f(v) + \beta f(w) \, . \tag{13}$$

Assume that $b_V : m \cdot I \to V$ is a chosen base of V. The vectors $a_j = b_V \circ q_j : I \to V$, $j = 1, \dots, m$, are *the basis vectors* of V and $A = \{a_1, \dots, a_m\}$ is *the basis* of V. The basis vectors satisfy

$$a_i^\dagger \circ a_j = \delta_{ij}, \text{ for } i, j = 1, \dots, m, \tag{14}$$

where $\delta_{ii} = 1_I$ and $\delta_{ij} = 0_{II}$ for $i \neq j$.

There are exactly m distinct basis vectors, because otherwise we would have $1_I = 0_{II}$, which contradicts $I \not\simeq 0$. The Equalities (14) mean that the basis vectors are unitary and pairwise orthogonal.

Proposition 6. *Every vector of V can be written uniquely as a linear combination of the chosen basis vectors.*

Proof. Let $\{a_1, \dots, a_m\}$ be the basis of V and $v : I \to V$ and

$$\alpha_i = q_i^\dagger \circ b_V^\dagger \circ v, \text{ for } i = 1, \dots, m.$$

Recall that $q_1 \circ q_1^\dagger + \dots + q_m \circ q_m^\dagger = 1_{m \cdot I}$, by (8). Hence

$$\begin{aligned} v &= b_V \circ (q_1 \circ q_1^\dagger + \dots + q_m \circ q_m^\dagger) \circ b_V^\dagger \circ v \\ &= b_V \circ q_1 \circ q_1^\dagger \circ b_V^\dagger \circ v + \dots + b_V \circ q_m \circ q_m^\dagger \circ b_V^\dagger \circ v \\ &= a_1 \circ \alpha_1 + \dots + a_m \circ \alpha_m = \alpha_1 a_1 + \dots + \alpha_m a_m . \end{aligned}$$

This proves the existence.

To see the unicity, assume $v = a_1 \circ \beta_1 + \dots + a_m \circ \beta_m$. Multiplying both sides of the equality on the left by $q_i^\dagger \circ b_V$, we get $q_i^\dagger \circ b_V \circ v = \beta_i$, for $i = 1, \dots, m$. □

Corollary 3. *Let $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, $v : I \to k \cdot I$ and $j \in \{1, \dots, n\}$. Then $q_K \circ v = q_j$ implies $j \in K$.*

Proof. Recall that $q_K = [q_{i_1}, \dots q_{i_k}] : k \cdot I \to n \cdot I$ and therefore $q_K \circ q_l = q_{i_l}$ for $l = 1, \dots, k$. Assume $v : I \to k \cdot I$ and $q_K \circ v = q_j$. Write $v = \sum_{l=1}^k \alpha_l q_l$, where $\alpha_l : I \to I$. Then $q_j = q_K \circ (\sum_{l=1}^k \alpha_l q_l) = \sum_{l=1}^k \alpha_l (q_K \circ q_l) = \sum_{l=1}^k \alpha_l q_{i_l}$. Coordinates are unique, thus $j = i_l$ and $\alpha_l = 1$ for some $l \leq k$ and $\alpha_{l'} = 0$ for $l' \neq l$. Finally, $q_j = q_{i_l}$ implies $j = i_l$, which terminates the proof. □

Refer to the (unique) scalars α_i, $i = 1, \dots, m$, such that $v = \alpha_1 a_1 + \dots + \alpha_m a_m$ as the *coordinates* of v. Use V_A to express that A is the basis of the space V.

2.3 The category $\mathcal{2SF}$ of two-sorted functions

Two-sorted first order logic has two sorts of variables, one for elements x, and one for sets X. Besides an equality symbol for each sort, there is a binary symbol $\in$ requiring elements on the left and sets on the right, $x \in X$. There are two sorts of quantifiers, $\forall_x$, $\forall_X$ etc. Functional symbols accept both sorts as arguments.

Models interpret every function symbol by a *two-sorted function* $f : A \to B$ satisfying

$$\begin{array}{rcl} f(\{x\}) & = & f(x) \text{ for } x \in A \\ f(\emptyset) & = & \emptyset \\ f(X \cup Y) & = & f(X) \cup f(Y) \text{ for } X, Y \subseteq A\,. \end{array} \tag{15}$$

The category $\mathcal{2SF}$ of two-sorted functions and finite sets is a biproduct dagger category. A two-sorted function is determined by its values on elements, because all sets are finite. The adjoint $f^\dagger : B \to A$ of $f : A \to B$ is given by

$$f^\dagger(b) = \{a \in A \,:\, f(a) = b \text{ or } b \in f(a)\}\ .$$

The biproduct is the disjoint union of sets, with $\emptyset$ as the zero object and a singleton set as generating object $I = \{*\}$. There exactly two scalars, namely the identity map and the zero map, which sends the unique element of I to the empty set.

The sum of $f, g : A \to B$ is the set-theoretical union $(f+g)(x) = f(x) \cup g(x)$.

The righthand side of the last equality involves an abuse of notation: if $f(x)$ or $g(x)$ is an element, we should have used the corresponding singleton set. English makes the same abuse. Compare '*apples and pairs*' with '*the teacher and the students.*

2.4 The category $\mathcal{RI}$ of semimodules over a real interval

Recall that the linear order on the real numbers in $[0, 1]$ induces a distributive and implication-complemented lattice structure on $[0, 1]$, namely

$$\begin{array}{c} \alpha \vee \beta = \max\{\alpha, \beta\} \text{ and } \alpha \wedge \beta = \min\{\alpha, \beta\} \\ \alpha \to \beta = \max\{\gamma \in I :\ \alpha \wedge \gamma \leq \beta\} \\ \neg\alpha = \alpha \to 0\,. \end{array}$$

This lattice is not Boolean, because $\neg\neg\alpha = 1 \neq \alpha$ for $0 < \alpha < 1$.

The lattice operations define a semiring structure on $I = [0, 1]$ with neutral element 0 and unit 1 by

$$\alpha + \beta = \alpha \vee \beta \qquad \alpha \cdot \beta = \alpha \wedge \beta\,.$$

The category $\mathcal{RI}$ of free semi-modules over the real interval $[0, 1]$, generated by a finite set is biproduct dagger category. The biproduct of two spaces is the space generated by the disjoint union of the two generating spaces. Every scalar is its own adjoint, $\alpha = \alpha^\dagger$. All scalars are positive. The matrix of the adjoint of a linear map is the transpose of the matrix of the linear map.

The categories $\mathcal{H}_\mathbb{R}$ and $\mathcal{H}_\mathbb{C}$ of real respectively complex Hilbert spaces are biproduct dagger categories. The adjoint of a scalar is the conjugate of the scalar. Thus all real scalars are self adjoint. The matrix of the adjoint of a linear map is the transpose of the conjugate matrix of the linear map.

2.5 Computing with scalars

All results of the previous section have a scalar version. The rest of this subsection recalls the most frequent ones.

Proposition 7. *Every morphism is uniquely determined by its values on the basis vectors.*

Proof. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_n\}$ and suppose that $f, g : V_A \to W_B$ coincide on the basis vectors $a_j = b_V \circ q_j$ for $j = 1, \dots, m$. Then

$$q_i^\dagger \circ b_W^\dagger \circ f \circ b_V \circ q_j = q_i^\dagger \circ b_W^\dagger \circ g \circ b_V \circ q_j \text{ for } i = 1, \dots, n, j = 1, \dots, m\,.$$

Hence, $b_W^\dagger \circ f \circ b_V = b_W^\dagger \circ g \circ b_V$, which implies $f = g$. □

Proposition 7 has a converse

Proposition 8 (Explicit Definitions). *Given vectors $w_1, \dots, w_m$ in W_B, there is a unique morphism $f : V_A \to W_B$ satisfying*

$$f \circ a_j = w_j, \text{ for } j = 1, \dots, m\,. \tag{16}$$

Proof. The coordinates of $w_j = \phi_{1j} b_1 + \cdots + \phi_{nj} b_n$, for $j = 1, \dots, m$, define a unique morphism $g : m \cdot I \to n \cdot I$ such that $q_i^\dagger \circ g \circ q_j = \phi_{ij}$, for $i = 1, \dots, n$, $j = 1, \dots, m$. Then $f = b_W \circ g \circ b_V^\dagger$ satisfies (16). □

Proposition 8 can be rephrased by saying that semantic categories admit *Explicit Definitions*. The morphism f is explicitly defined by equalities in (16).

By Proposition 8, every morphism $f : V_A \to W_B$ determines and is determined by the nm scalars

$$\phi_{ij} = q_i^\dagger \circ b_W^\dagger \circ f \circ b_V \circ q_j, \text{ for } i = 1, \dots, n, j = 1, \dots, m\,.$$

The scalars $\phi_{ij}^\dagger$, $j = 1, \dots, m$, $i = 1, \dots, n$ then determine $f^\dagger$, namely

$$M_f = \begin{pmatrix} \phi_{11} & \dots & \phi_{1m} \\ \vdots & & \vdots \\ \phi_{n1} & \dots & \phi_{nm} \end{pmatrix} \quad M_{f^\dagger} = \begin{pmatrix} \phi_{11}^\dagger & \dots & \phi_{n1}^\dagger \\ \vdots & & \vdots \\ \phi_{1m}^\dagger & \dots & \phi_{nm}^\dagger \end{pmatrix}\,.$$

The Dirac notation can be introduced with its usual properties: Assign to any vector $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$ of $V = V_B$ a row matrix and a column matrix

$$\langle v| = M_{v^\dagger} = \begin{pmatrix} \alpha_1^\dagger & \dots & \alpha_n^\dagger \end{pmatrix}, \quad |v\rangle = M_v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}\,.$$

The *inner product* of v and $w = \beta_1 b_1 + \cdots + \beta_n b_n : I \to V$ is

$$\langle v|w\rangle := M_{v^\dagger} M_w = \alpha_1^\dagger \beta_1 + \cdots + \alpha_n^\dagger \beta_n\,,$$

and the *outer product* of any vector $u = \gamma_1 a_1 + \cdots + \gamma_m a_m$ of $U = U_A$ and w

$$|w\rangle\langle u| := M_w M_{u^\dagger} = \begin{pmatrix} \beta_1\gamma_1^\dagger & \dots & \beta_1\gamma_m^\dagger \\ \vdots & & \vdots \\ \beta_n\gamma_1^\dagger & \dots & \beta_n\gamma_m^\dagger \end{pmatrix} .$$

Otherwise said, $\langle v|w\rangle$ is the matrix of $v^\dagger \circ w$ and $|w\rangle\langle u|$ is the matrix of $b_V \circ w \circ u^\dagger \circ b_U^\dagger : U \to V$.

The outer product of a basis vector $b_i = \sum_{k=1}^n \delta_{ki} b_k$ of V_B and a basis vector $a_j = \sum_{l=1}^m \delta_{jl} a_l$ of U_A is

$$|b_i\rangle\langle a_j| = (\delta_{kl}^{ij}) ,$$

where $\delta_{ij}^{ij} = 1$ and $\delta_{kl}^{ij} = 0$ for $k \neq i$ or $l \neq j$, $k = 1, \dots, n$, $l = 1, \dots, m$. Indeed, $\delta_{kl}^{ij} = \delta_{ki}\delta_{jl}^\dagger = \delta_{ki}\delta_{jl}$. In particular, the outer product $|b_i\rangle\langle b_i|$ is the matrix of the projector $p_{\{i\}}$, for $i = 1, \dots, n$.

Definition 2 can now be reformulated for vectors in terms of the inner product. Vectors are orthogonal if and only if their inner product equals 0. A vector is unitary if the inner product of the vector with itself equals 1.

2.6 Compact closed categories

Recall that a *symmetric monoidal category* consists of a category $\mathcal{C}$, a bifunctor $\otimes$, a distinguished object I and natural isomorphisms $\sigma_{VW} : V \otimes W \to W \otimes V$, $\alpha_{VWU} : (V \otimes W) \otimes U \to V \otimes (W \otimes U)$, $\lambda_V : V \to I \otimes V$ and $\rho_V : V \to V \otimes I$ subject to the coherence conditions of [Mac Lane, 1971].

For notational convenience, the associativity isomorphisms α_{VWU} and the unit isomorphisms λ_V and ρ_V are replaced by identities, e.g. $(V \otimes W) \otimes V = V \otimes (W \otimes U)$, $V = I \otimes V$ and $V \otimes I$.

The tensor product is definable in semantic categories. It plays the role of a book keeping device and 'internalises' matrices as vectors of a tensor product space.

Let $b_V : m \cdot I \to V$ and $b_W : n \cdot I \to W$ be spaces with chosen basis vectors $a_j = b_V \circ q_j$, $b_i = b_W \circ q_i$, where $q_j : I \to m \cdot I$, $q_i : I \to n \cdot I$, $j = 1, \dots, m$, $i = 1, \dots, n$, are the canonical injections.

The *tensor product of* V and W and the dagger isomorphism $b_{V\otimes W} : n \cdot (m \cdot I) \to V \otimes W$ are

$$V \otimes W := n \cdot V, \quad b_{V\otimes W} := n \cdot b_V .$$

Let $q_i' : m \cdot I \to n \cdot (m \cdot I)$, $i = 1, \dots, n$, be the canonical injections. The *tensor product of* a_j *and* b_i is the vector

$$a_j \otimes b_i := b_{V\otimes W} \circ q_i' \circ q_j : I \to V \otimes W .$$

The canonical associativity isomorphism of the tensor product is the appropriate associativity isomorphism of the biproduct. The unit isomorphism is defined in a straight forward manner. The symmetry isomorphism σ_{VW} : $V \otimes W \to W \otimes V$ has the explicit definition

$$\sigma_{VW} \circ (a_j \otimes b_i) = b_i \otimes a_j, i = 1, \ldots, n, j = 1, \ldots, m\,.$$

Under these definitions, the tensor product distributes over the dagger and the biproduct

$$\begin{array}{ll}(f \otimes g)^\dagger & = f^\dagger \otimes g^\dagger : W \otimes D \to V \otimes U \\ V \otimes (W \oplus U) & \simeq (V \otimes W) \oplus (V \otimes U)\,.\end{array}$$

A *compact closed category* is a symmetric monoidal category $\mathcal{C}$ together with a contra-variant functor * and maps $\eta_V : I \to V^* \otimes V$ and $\epsilon_V : V \otimes V^* \to I$, called *unit* and *counit* respectively, such that

$$\begin{array}{ll}(\epsilon_V \otimes 1_V) \circ (1_V \otimes \eta_V) & = 1_V \\ (\epsilon_{V^*} \otimes 1_{V^*}) \circ (1_{V^*} \otimes \eta_{V^*}) & = 1_{V^*}\,.\end{array}$$

Proposition 9. *Finite dimensional biproduct dagger categories are compact closed.*

Indeed, follow the construction of the dual space in [Abramsky and Coecke, 2004] for the category of complex Hilbert spaces. First, introduce the *dual scalar multiplication* for $v : I \to V_A$ and $\alpha : I \to I$

$$\alpha * v := v \circ \alpha^\dagger = \alpha^\dagger v\,.$$

This definition creates a dual version of Proposition 6: Every vector can be written uniquely as the sum of dual scalar multiples of basis vectors. Indeed, let $\beta_i = {\alpha_i}^\dagger$ for $j = 1, \ldots, m$. Then

$$\sum_{i=1}^m \alpha_i a_i = \sum_{i=1}^m {\alpha_i}^{\dagger\dagger} a_i = \sum_{i=1}^m \beta_i * a_i\,.$$

The *dual space* ${V_A}^*$ is the space V_A where vectors are given as sums of dual scalar multiples of basis vectors. In the case of $\mathcal{2SF}$, $\mathcal{RI}$ and real Hilbert spaces, we have ${V_A}^* = V_A$, because $\alpha^\dagger = \alpha$ for all $\alpha : I \to I$.

Given $f : V_A \to W_B$, use the principle of Explicit Definitions to introduce the morphisms $f_* : {V_A}^* \to {W_B}^*$ and the *dual* $f^* : {W_B}^* \to {V_A}^*$ such that

$$\begin{array}{lll}f_*(a_j) & = \sum_{i=1}^n \phi_{ij} * b_i & = \sum_{i=1}^n \phi_{ij}^\dagger b_i, \text{ for } j = 1, \ldots, m \\ f^*(b_i) & = \sum_{j=1}^m {\phi_{ij}}^\dagger * a_j & = \sum_{j=1}^m \phi_{ij} a_j, \text{ for } i = 1, \ldots, n\,.\end{array}$$

Then

$$f^* = {f_*}^\dagger = {f^\dagger}_* : {W_B}^* \to {V_A}^*\,.$$

The unit $\eta_V : I \to V^* \otimes V$ and counit $\epsilon_V : V \otimes V^* \to I$ are the morphisms defined explicitly thus

$$\begin{array}{rl}\eta_V(1_I) & = \sum_{i=1}^n a_i \otimes a_i \\ \epsilon_V(a_i \otimes a_j) & = \delta_{ij}, \text{ for } i, j = 1, \ldots, n\,.\end{array}$$

The definitions above satisfy the axioms of compact closure. Moreover,

$$\eta_V = \sigma_{AA^*} \circ \epsilon_V^\dagger, \text{ for all } V\,.$$

3 The internal logic of semantic categories

An internal logic of a category consists of a class of morphisms, the propositions, and a set of equalities expressing the truth of propositions.

The internal logic of semantic categories follows quantum logic in choosing the projectors as propositions. Logical connectives are defined in such a way that they form an ortho-complemented lattice with the identity as the largest element. But there is another approach via two-sorted predicates that is to say morphisms with values in a space of 'truth-values'.

In both cases, the basis vectors of the domain play the role of individuals. Basis vectors are generalised to 'Boolean vectors' to capture the plurals of natural language. A vector $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$ is said to be *Boolean* if $\alpha_i = 0$ or $\alpha_i = 1$, for all $i = 1, \ldots, n$.

Every Boolean vector $v : I \to V_B$ determines a unique subset $K = \{j_1, \ldots, j_k\}$ of $\{1, \ldots, n\}$ such that

$$v = \sum_{i \in K} b_i = v_K \,.$$

The propositional connectives are lifted from subsets $K \subseteq \{1, \ldots, n\}$ to Boolean vectors in such a way that

$$v_K \wedge v_L = v_{K \cap L},\ v_K \vee v_L = v_{K \cup L}\ , \text{ etc.}$$

hold. Hence, the map $K \mapsto v_K$ is a Boolean isomorphism. The Boolean vectors form a Boolean algebra with largest element $\overrightarrow{1} = \sum_{i=1}^{n} b_i$ and smallest element $\overrightarrow{0} = v_\emptyset = 0_{IV_B}$.

If convenient use subsets of B to describe Boolean vectors. Given a subset of basis vectors $A = \{b_{i_1}, \ldots, b_{i_k}\}$ let $K = \{i_1, \ldots, i_k\}$ and define

$$v_A := b_{i_1} + \cdots + b_{i_k} = v_K \,.$$

3.1 The logic of intrinsic projectors

Intrinsic projectors will stand for propositions in this subsection. The truth of a proposition p is expressed by the equality $p = 1_V$.

Given a space V with base $b_V : n \cdot I \to V$, the morphisms p_K, q_K of $n \cdot I$ lift to the morphisms $b_V \circ p_K \circ b_V^\dagger,\ b_V \circ q_K \circ b_V^\dagger$ with the same properties. If the context permits, we use p_K instead of $b_V \circ p_K \circ b_V^\dagger$ etc.

Definition 4 (**Intrinsic morphism**). *A morphism of $\mathcal{C}$ is* intrinsic *if it sends every basis vector to a basis vector or to the null vector.*

The identity 1_V and the diagonal $d_V : V \to V \otimes V$, which maps any basis vector b of V to $b \otimes b$, are intrinsic. Intrinsic morphisms are ubiquitous in natural language. Determiners, relative pronouns and verbs, to mention but a few, are interpreted by intrinsic morphisms.

Observe the following properties, which are are straight forward except possibly (17), which is proved in [Preller, 2012].

Proposition 10. *In any semantic category $\mathcal{C}$, the following holds*

- *if f is intrinsic then $f^\dagger = f$*
- *intrinsic morphisms are closed under composition and tensor products*
- *a projector $p : V_B \to V_B$ is intrinsic if and only if*

$$p(b_i) = b_i \text{ or } p(b_i) = \overrightarrow{0}, \text{ for } i = 1, \ldots, n \tag{17}$$

- *the entries of the matrix $(\pi_{ij})_{ij}$ of an intrinsic projector p satisfy*

$$\begin{aligned} \pi_{ij} &= 1, \text{ if } i = j \text{ and } p(b_i) = b_i \\ \pi_{ij} &= 0, \text{ else} \end{aligned}$$

- *intrinsic projectors map Boolean vectors to Boolean vectors*
- *every intrinsic projector p has the form $b_V \circ p_K \circ b_V^\dagger$, where*

$$K = \{i : p(b_i) = b_i \text{ and } 1 \leq i \leq n\}$$

- *the morphism $b_V \circ p_K \circ b_V^\dagger$ is an intrinsic projector of V for every $K \subseteq \{1, \ldots, n\}$.*

Hence, an arbitrary set of basis vectors $A = \{b_{i_1}, \ldots, b_{i_k}\}$ gives rise to the projector

$$p_A = b_V \circ p_K \circ b_V^\dagger, \text{where } K = \{i_1, \ldots, i_k\}\ .$$

Context permitting, we use p_K instead of p_A.

Intrinsic projectors are in a one-one correspondence with Boolean vectors. Indeed, let $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$ be any vector of V_B. Define a morphism $p_v : V_B \to V_B$ by its values on the basis vectors thus

$$p_v(b_i) = \alpha_i b_i, \text{ for } i = 1, \ldots, n\,. \tag{18}$$

Proposition 11. *If $v = b_{i_1} + \cdots + b_{i_k}$ then $p_v = p_K$ and the following properties hold*

- *$p_v(w) = v \wedge w$ for every Boolean vector w; in particular $p_v(\overrightarrow{1}) = v$*
- *$p_{\overrightarrow{1}} = 1_{V_B}$*
- *the map $v \mapsto p_v$ is a negation preserving lattice isomorphism from the Boolean vectors onto the intrinsic projectors of V_B*
- *intrinsic projectors are monotone increasing on Boolean vectors.*

Equalities (18) define a projector for every vector in the category $\mathcal{RI}$.

3.2 Predicates and two-sorted logic

Predicates and two-sorted truth are definable in an arbitrary semantic category. Let $S \simeq I \oplus I$ be a fixed two-dimensional space with basis vectors $\top = b_S \circ q_1$ and $\bot = b_S \circ q_2$. The vectors of S are called *truth values*.

Use set-theoretical notation to highlight the analogy between categorical and set-theoretical logic, e.g. $p(v) = \top$ instead of $p \circ v = \top$, etc.

The two-sorted connectives introduced below are morphisms and as such they are determined by their values on the basis vectors.

The *two-sorted negation* $\mathtt{not}_S : S \to S$ is defined explicitly by

$$\mathtt{not}_S(\top) = \bot, \ \mathtt{not}_S(\bot) = \top\,.$$

Recall that the full vector $\overrightarrow{1}$ of S satisfies $\overrightarrow{1} = \top + \bot$. Then

$$\mathtt{not}_S(\overrightarrow{1}) = \overrightarrow{1} \text{ and } \mathtt{not}_S(\overrightarrow{0}) = \overrightarrow{0}\,.$$

More generally, let $k \cdot v$ denote the k-fold sum of vector v, for any non-negative integer k. Then

$$\mathtt{not}_S(k_1 \cdot \top + k_2 \cdot \bot) = k_2 \cdot \top + k_1 \cdot \bot\,.$$

The *two-sorted conjunction* $\mathtt{and}_S : S \otimes S \to S$ and *two-sorted disjunction* $\mathtt{or}_S : S \otimes S \to S$ are defined explicitly on the four basis vectors of $S \otimes S$ thus

$$\mathtt{and}_S(\top \otimes \top) = \top, \ \mathtt{and}_S(\bot \otimes \top) = \mathtt{and}_S(\top \otimes \bot) = \mathtt{and}_S(\bot \otimes \bot) = \bot$$
$$\mathtt{or}_S(\bot \otimes \bot) = \bot, \ \mathtt{or}_S(\bot \otimes \top) = \mathtt{or}_S(\top \otimes \bot) = \mathtt{or}_S(\top \otimes \top) = \top\,.$$

Note that the two-sorted connectives are different form the set-theoretical connectives introduced for Boolean vectors at the beginning of this section.

Proposition 12. *The two-sorted connectives define a Boolean structure on the vectors of S. In particular, for arbitrary vectors $v : I \to S$ and $w : I \to S$ the following holds*

$$\mathtt{not}_S \circ \mathtt{not}_S \circ v = v, \ \mathtt{not}_S \circ \mathtt{and}_S \circ (v \otimes w) = \mathtt{or}_S \circ (\mathtt{not}_S \circ v \otimes \mathtt{not}_S \circ w)\,.$$

Proof. Use the fact that morphisms commute with addition and scalar multiplication. □

A two-sorted predicate is any intrinsic morphism with codomain S. As we are not discussing disambiguation, only predicates that never take the value $\overrightarrow{0}$ for a basis vector are considered.

A morphism $p : V \to S$ is a *two-sorted predicate on* V if it maps basis vectors of V to basis vectors of S, i.e.

$$p(x) = \top \text{ or } p(x) = \bot, \text{ for every basis vector } x \text{ of } V\,. \tag{19}$$

By an *n-ary two-sorted predicate on E* we mean a two-sorted predicate on $V = E \otimes \ldots \otimes E$.

Identify the basis vectors with the 'individuals of the universe of discourse'. A two-sorted predicate takes individuals (sort one) and sets of individuals (sort two) as arguments. For a basis vector there are only two possible truth values, namely $\top$ and $\bot$. The values that a predicate may assign to sums of basis vectors depends on the properties of scalar addition.

Proposition 13. *Let $p : V_B \to S$ be a two-sorted predicate on V_B and $A = \{b_{i_1}, \dots, b_{i_k}\}$ a subset of basis vectors. Then there are non-negative integers k_1 and k_2 such that*

$$k_1 + k_2 = k \text{ and } p(\sum_{x \in A} x) = k_1 \cdot \top + k_2 \cdot \bot \,. \tag{20}$$

If scalar addition is idempotent, and in particular in 2$\mathcal{SF}$, identify A with $\sum_{x\in A} x$. Then the following holds
[Fundamental Property]

$$\begin{array}{ll} p(A) = \overrightarrow{0} & \Leftrightarrow p(x) = \overrightarrow{0} \text{ for all } x \in A \\ p(A) = \top & \Leftrightarrow p(x) = \top \text{ for all } x \in A \text{ and } A \neq \emptyset \\ p(A) = \bot & \Leftrightarrow p(x) = \bot \text{ for all } x \in A \text{ and } A \neq \emptyset \\ p(A) = \overrightarrow{1} & \Leftrightarrow p(x) = \top \text{ and } p(y) = \bot \text{ for some } x, y \in A \,. \end{array} \tag{21}$$

Proof. The proof of (20) rests on the fact that p separates A into two disjoint subsets A_1 and A_2 such that $p(x) = \top$ for all $x \in A_1$ and $p(x) = \bot$ for all $x \in A_2$.

To show the Fundamental Property, use linearity (13), and $\overrightarrow{1} = \top + \bot$. □

In Hilbert spaces, the value a two-sorted predicate assigns to a set A consists of two 'counts', one of the number of elements of A for which the predicate is true, another one for which it is false. This suggests a similarity with probability distributions, a subject beyond the scope of this paper.

Proposition 14. *The two-sorted predicates are closed under composition with the two-sorted connectives.*

More precisely, assume that $p : V_B \to S$ and $r : V_B \to S$ are two-sorted predicates on V . Then the morphisms

$$\mathtt{not}_S \circ p,\ \mathtt{and}_S \circ (p \otimes r),\ \mathtt{or}_S \circ (p \otimes r)$$

are again two-sorted predicates on V_B respectively on $V_B \otimes V_B$ and satisfy

$$\begin{array}{c} \mathtt{not}_S \circ \mathtt{not}_S \circ p = p \\ \mathtt{not}_S \circ \mathtt{and}_S \circ (p \otimes r) = \mathtt{or}_S \circ ((\mathtt{not}_S \circ p) \otimes (\mathtt{not}_S \circ r)) \,. \end{array} \tag{22}$$

For any $x \in B$, $A \subseteq B$

$$\begin{array}{lll} p(x) & = \bot & \Leftrightarrow \mathtt{not}_S(p(x)) = \top \\ p(\sum_{x\in A} x) & = k_1 \cdot \top + k_2 \cdot \bot & \Leftrightarrow \mathtt{not}_S(p(\sum_{x\in A} x)) = k_2 \top + k_1 \cdot \bot \end{array}$$

Whereas $\mathtt{not}_S(p(x)) = \top$ is equivalent to $p(x) \neq \top$, this no longer holds for arbitrary sets A . From $p(\sum_{x\in A} x) \neq \top$ does not follow that $\mathtt{not}_S \circ p(\sum_{x\in A} x) = \top$. For the counter example, let a and b be two distinct basis vectors such that $p(a) = \top$ and $p(b) = \bot$ we have $p(a + b) = \top + \bot \neq \bot$.

The predicates $\mathtt{and}_S \circ (p \otimes r)$ and $\mathtt{or}_S \circ (p \otimes r)$ are predicates on $V \otimes V$. Composing them with the diagonal $d_V : V \to V \otimes V$, we obtain predicates $\mathtt{and}_S \circ (p \otimes r) \circ d_V$ and $\mathtt{or}_S \circ (p \otimes r) \circ d_V$ on V such that the equalities (22) still hold. Hence, the two-sorted predicates on a given space form a Boolean algebra.

Definition 5. *Let $p : V \to S$ be a two-sorted predicate on V. A Boolean vector v_A is said to satisfy p if $p(v_A) = k \cdot \top$ for some positive integer k.*

Assume that $\mathcal{C}$ is $\mathit{2SF}$, $\mathcal{RI}$ and or the category of real/complex Hilbert spaces. Then v_A satisfies p if and only if the set of basis vectors A is not empty and $p(x) = \top$ for all $x \in A$. In the case of Hilbert spaces, k is the number of elements in A.

3.3 Intrinsic projectors and two-sorted predicates

Let $\mathcal{C}$ be an arbitrary semantic category and $V = V_B$ be a space of $\mathcal{C}$ with basis B. For every intrinsic projector $p : V \to V$, define a two sorted predicate $\widehat{p} : V \to S$ by the condition

$$\widehat{p}(x) = \begin{cases} \top & \text{if } p(x) = x \\ \bot & \text{else} \end{cases}, \text{ for all } x \in B. \tag{23}$$

Conversely, given a two-sorted predicate $p : V \to S$ on V, define an intrinsic projector $\widetilde{p} : V \to V$ by

$$\widetilde{p}(x) = \begin{cases} x & \text{if } p(x) = \top \\ \overrightarrow{0} & \text{else} \end{cases}, \text{ for all } x \in B. \tag{24}$$

Proposition 15. *The map $p \mapsto \widehat{p}$ is a Boolean isomorphism from the intrinsic projectors of V onto the two-sorted predicates on V satisfying*

$$\begin{aligned} \mathtt{not}_S \circ \widehat{p} &= \widehat{\neg p} \\ \mathtt{and}_S \circ (\widehat{p} \otimes \widehat{r}) \circ d_V &= \widehat{p \wedge r} \\ \mathtt{or}_S \circ (\widehat{p} \otimes \widehat{r}) \circ d_V &= \widehat{p \vee r}. \end{aligned} \tag{25}$$

Moreover, if $\mathcal{C} = \mathit{2SF}$, $\mathcal{RI}$, $\mathcal{H}_{\mathbb{R}}$ or $\mathcal{H}_{\mathbb{C}}$ then for any Boolean vector $w : I \to V$ and any intrinsic projector $p : V \to V$

$$\begin{aligned} p(w) = w \quad &\Leftrightarrow \ \widehat{p}(w) = k \cdot \top \text{ for some integer } k \geq 0 \\ p(w) = \overrightarrow{0} \quad &\Leftrightarrow \ \widehat{p}(w) = k \cdot \bot \text{ for some integer } k \geq 0. \end{aligned} \tag{26}$$

Proof. It is sufficient to verify (25) for basis vectors, an easy exercise.

The equalities (26) follow from (20). For example, when proving the second equality of (26), let $w = v_A$ and assume that $\widehat{p}(v_A) = k \cdot \bot$. Then $\widehat{p}(x) = \bot$ for all $x \in A$. This means that $p(x) \neq x$ for all $x \in A$, by (23). Thus, $p(x) = \overrightarrow{0}$ for all $x \in A$, because p is intrinsic. Hence, $p(v_A) = \sum_{x \in A} p(x) = \sum_{x \in A} \overrightarrow{0} = \overrightarrow{0}$. This shows the implication from left to right. For the converse implication, note that $p(v_A) = \sum_{x \in A} p(x) = \overrightarrow{0}$ implies $p(x) = \overrightarrow{0}$ for each $x \in A$, because p is intrinsic. Thus, $\widehat{p}(x) = \bot$ for all $x \in A$. Finally, let k be the number of elements of A. The equality $\widehat{p}(v_A) = k \cdot \bot$ follows by (20). ☐

The switch between two-sorted predicates and intrinsic projectors is common in natural language. Typically, an adjective in attributive position is interpreted as an intrinsic projector $\mathtt{big}_a : V_A \to V_A$. The same adjective, when in predicative position, defines a binary predicate $\mathtt{big}_p : V_B \to S$ such that

$$\mathtt{big}_a(x) = x \Leftrightarrow \mathtt{big}_p(x) = \top, \text{ for all } x \in B\,.$$

The transformation (24) of a predicate into a projector is implemented by the relative pronoun, Section 4.3.

4 Compositional semantics

4.1 The syntactical category

The description of the syntactical category given below is a notational variant of the description in [Preller and Lambek, 2007].

Call *syntactical category* any free compact bicategory $\mathcal{C}(\mathcal{B})$ with a single 0-cell, generated by some category $\mathcal{B}$. Think of the objects of $\mathcal{B}$ as basic types and of the morphisms of $\mathcal{B}$ as axioms. For simplicity, the canonical associativity and unit isomorphisms of the tensor product (1-cell composition) are replaced by identities, for example $A{\otimes}(B{\otimes}C) = A{\otimes}B{\otimes}C = (A{\otimes}B){\otimes}C$, $A{\otimes}I = A = I{\otimes}A$. The iterated tensor products are assimilated to strings of objects.

Saying that $\mathcal{C}(\mathcal{B})$ is compact means that every 1-cell (object) Γ has a left adjoint Γ^ℓ and a right adjoint Γ^r. Then Γ is a right adjoint to its left adjoint Γ^ℓ, thus $\Gamma^{\ell r} \simeq \Gamma$. Hence the objects (1-cells) of $\mathcal{C}(\mathcal{B})$ are the unit I, the objects of $\mathcal{B}$, their iterated right of left adjoints and the strings built from these. An iterated adjoint $A^{(z)}$ is *even* if $z = (2n)\ell$ or $z = (2n)r$. It is *odd* if $z = (2n+1)\ell$ or $z = (2n+1)r$. By convention, $A^{(0)} = A$. A similar convention applies to the morphisms of $\mathcal{B}$. Capital latin letters designate objects of $\mathcal{B}$, capital greek letters objects of $\mathcal{C}(\mathcal{B})$.

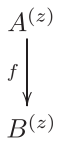
The *morphisms*, i.e. the 2-cells, of $\mathcal{C}(\mathcal{B})$, are represented by graphs where the vertices are labelled by iterated adjoints of objects of $\mathcal{B}$ and the oriented links are labelled by morphisms of $\mathcal{B}$.

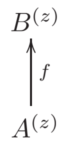
The first four rules constitute a cut-free axiomatisation of Compact Bilinear Logic. They imply the fifth, the Cut rule. In the presentation below, each rule comes with the corresponding morphism and its proof-graph. The first four rules generate all morphisms of $\mathcal{C}(\mathcal{B})$.

Axioms

$$\frac{}{\vdash} \qquad \begin{array}{c} I \\ {\scriptstyle 1_I}\big\downarrow \\ I \end{array}$$

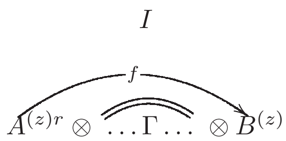
if z is even $\dfrac{f:A\to B\in\mathcal{B}}{A^{(z)}\vdash B^{(z)}}$ if z is odd $\dfrac{f:A\to B\in\mathcal{B}}{B^{(z)}\vdash A^{(z)}}$

$f^{(z)} =$ 

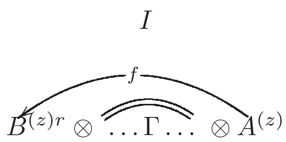
$f^{(z)} =$ 

Units for $g : I \to \Gamma,\ f : A \to B$

if z is even $\dfrac{\vdash\Gamma \quad A\vdash B}{\vdash A^{(z)r}\otimes\Gamma\otimes B^{(z)}}$

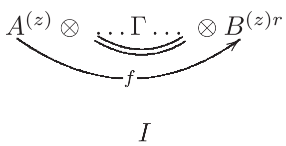
$\eta_{f^{(z)}} \circ (1_{A^{(z)r}} \otimes g \otimes 1_{B^{(z)}}) =$ 

if z is odd $\dfrac{\vdash\Gamma \quad A\vdash B}{\vdash B^{(z)r}\otimes\Gamma\otimes A^{(z)}}$

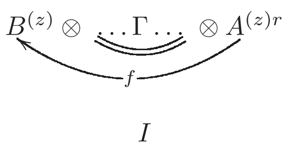
$\eta_{f^{(z)}} \circ (1_{B^{(z)r}} \otimes g \otimes 1_{A^{(z)}}) =$ 

Counits for $g : \Gamma \to I,\ f : A \to B$

if z is even $\dfrac{\Gamma\vdash \quad f:A\to B}{A^{(z)r}\otimes\Gamma\otimes B^{(z)}\vdash I}$

$\epsilon_{f^{(z)}} \circ (1_{A^{(z)}} \otimes g \otimes 1_{B^{(z)r}}) =$ 

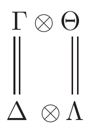
if z is odd $\dfrac{\Gamma\vdash \quad f:A\to B}{B^{(z)}\otimes\Gamma\otimes A^{(z)r}\vdash I}$

$\epsilon_{f^{(z)}} \circ (1_{B^{(z)}} \otimes g \otimes 1_{A^{(z)r}}) =$ 

1-Cell Composition

for $g : \Gamma \to \Delta,\ h : \Theta \to \Lambda$

$$\frac{\Gamma \vdash \Delta \quad \Theta \vdash \Lambda}{\Gamma \otimes \Theta \ \vdash \Delta \otimes \Lambda}$$

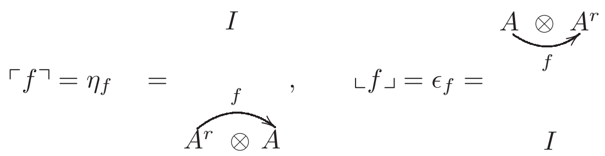
$g \otimes h =$ 

Cut

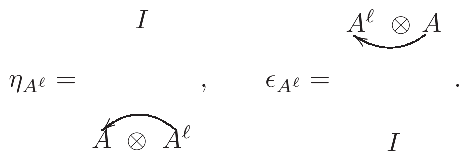
$$\frac{\Gamma \vdash \Delta \quad \Delta \vdash \Theta}{\Gamma \ \vdash \ \Theta} \qquad \begin{array}{c} \text{for } f : \Gamma \to \Delta,\ g : \Delta \to \Theta \\ g \circ f : \Gamma \to \Theta \end{array}$$

NOTE: *A double line stands for the collection of links in some previously constructed graph. Graphs display the domain of the morphism (2-cell) above, the codomain below. The labels of links are morphisms of* $\mathcal{B}$. *The tail of the link is the domain and the head is the codomain of the label. In the case where the label is an identity, it is in general omitted.*

Making $g = 1_I$ in the Unit and Counit rules we obtain the name and the coname of f

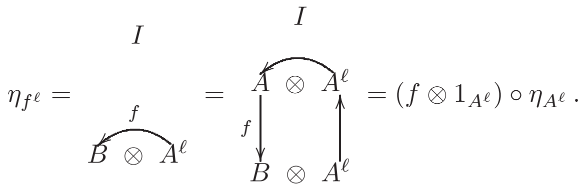


In the particular case where $f = 1_A$, the result is the unit $\eta_A : I \to A^r \otimes A$ and the counit $\epsilon_A : A \otimes A^r \to I$ for the right adjunction. Recalling that $A = A^{\ell r}$, we obtain the unit and counit of the left adjunction

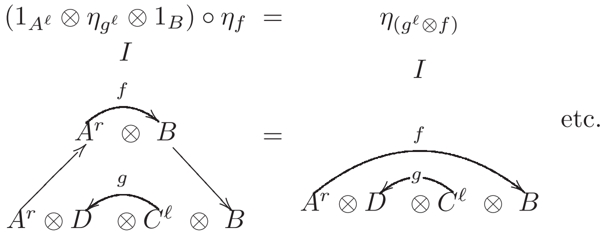


Composition of morphisms is computed by connecting the graphs at the joint interface and walking paths, picking up and composing the labels in the order in which they appear.

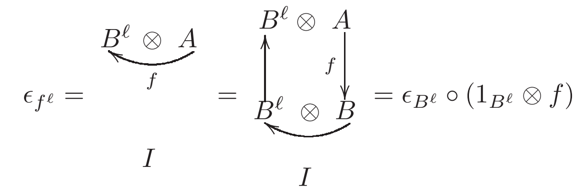
For example, let $f : A \to B$ and make $g = 1_I$ in the unit rule. Then

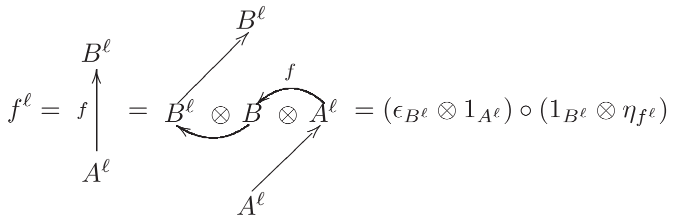


Units of adjunction give rise to 'nested' graphs. The same holds for counits. For example, let $f : A \to B$, $g : C \to D$



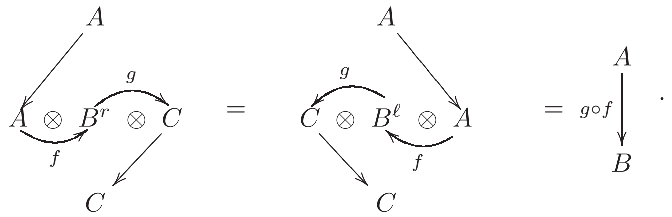
Other examples concern the composition of units and counits of adjunction. For $f : A \to B$





Assume $g : B \to C$. Then

$$(\epsilon_f \otimes 1_C) \circ (1_A \otimes \eta_g) = (1_C \otimes \epsilon_{f^\ell}) \circ (\eta_{g^\ell} \otimes 1_A) \quad = g \circ f$$



The benefit of orienting and labelling links becomes evident when computing the meaning of strings of words where the graphs are given by the grammar in Section 4.2.

4.2 Meanings via pregroup grammars

Like other categorial grammar, a pregroup grammar has a lexicon and a calculus, namely compact bilinear logic, also known as pregroup calculus. The initial category $\mathcal{B}$ is a partially ordered set. Its elements stand for grammatical notions.

The free compact bicategory $\mathcal{C}(\mathcal{B})$ has an equivalent definition as the free, not necessarily symmetric monoidal compact closed category generated by $\mathcal{B}$. In particular, every functor from $\mathcal{B}$ into a symmetric compact closed monoidal category $\mathcal{C}$ extends to a functor $\mathcal{F} : \mathcal{C}(\mathcal{B}) \to \mathcal{C}$ that maps right and left adjoints to duals

$$\mathcal{F}(T^\ell) = \mathcal{F}(T)^* = \mathcal{F}(T^r)$$

and every derivation of compact bilinear logic to a morphism of $\mathcal{C}$.

A *lexicon* is a finite list of entries. An *entry* is a triple $w : T :: m$, where w is a word, T a type and m a *meaning* expression in the language of compact closed categories. It depends functionally on the word and the type in the entry.

This description differs from the original one in [Lambek, 1999]. There, only pregroup dictionaries are considered where the entries are pairs $w : T$ of words and types. The meaning must be added explicitly, because the functional semantics of higher order types has been lost by the pregroup types.

Consider the following entries

$$\begin{array}{llll} \textit{all} & : \boldsymbol{n}_2\, \boldsymbol{c}_2{}^{\ell} :: I \xrightarrow{\overline{\texttt{all}}} E \otimes E^* & \textit{birds}: \boldsymbol{c}_2 & :: I \xrightarrow{\overline{\texttt{bird}}} E \\ \textit{some} & : \boldsymbol{n}_2\, \boldsymbol{c}_2{}^{\ell} :: I \xrightarrow{\overline{\texttt{some}}} E \otimes E^* & \textit{fly} \quad : \boldsymbol{n}_2{}^r\, \boldsymbol{s} & :: I \xrightarrow{\overline{\texttt{fly}}} E^* \otimes S \end{array}$$

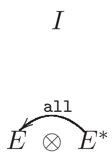
$$\begin{array}{lll} \textit{who} & : \boldsymbol{c}_2{}^r \boldsymbol{c}_2\, \boldsymbol{s}^{\ell} \boldsymbol{n}_2 & :: I \xrightarrow{\overline{\texttt{who}}} E^* \otimes E \otimes S^* \otimes E \\ \textit{do} & : \boldsymbol{n}^r \boldsymbol{s}\, \boldsymbol{i}^{\ell} \boldsymbol{d} & :: I \xrightarrow{\overline{\texttt{do}}} E^* \otimes S \otimes S^* \otimes E \\ \textit{not} & : \boldsymbol{d}^r \boldsymbol{i}\boldsymbol{i}^{\ell} \boldsymbol{d} & :: I \xrightarrow{\overline{\texttt{not}}} E^* \otimes S \otimes S^* \otimes E \end{array}$$

The basic types $\boldsymbol{c}_2, \boldsymbol{n}_2, \boldsymbol{d}, \boldsymbol{i}$ and $\boldsymbol{s}$ stand for 'plural count noun', 'plural noun phrase', 'dummy noun phrase', 'infinitive' and 'sentence', in that order. Moreover, $\boldsymbol{c}_2 < \boldsymbol{n}_2$. The basic types $\boldsymbol{c}_2, \boldsymbol{n}_2, \boldsymbol{d}$ are interpreted by a distinguished space $E = V_B$, where B plays the role of the set of individuals. The basic types $\boldsymbol{i}$ and $\boldsymbol{s}$ are interpreted by a fixed two-dimensional space S, 'the space of truth values'. Thus, the lexicon defines an obvious functor from $\mathcal{B}$ to the semantic category $\mathcal{C}$, which maps the inequality $\boldsymbol{c}_2 < \boldsymbol{n}_2$to 1_E .

We postulate that every vector $\texttt{word} : I \to V$ occurring in the lexicon is Boolean. If the domain of $\texttt{word} : V \to W$ is not isomorphic to I then $\texttt{word}$ is intrinsic. If $W = S$ it is a predicate. If $W = V$, it is a projector.

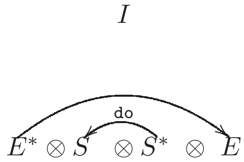
The meaning vector of a lexical entry is determined by its type and a postulate reflecting its logical content. For example,

$\overline{\texttt{all}} = \eta_{\texttt{all}^*}$ =



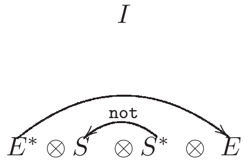
$\texttt{all} = 1_E,$

$\overline{\texttt{do}} = \eta_{(1_E \otimes \texttt{do}^*)}$ =



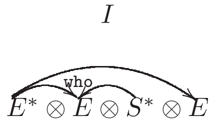
$\texttt{do} = 1_S$

$\overline{\texttt{not}} = \eta_{(1_E \otimes \texttt{not}^*)}$ =



$\texttt{not} = \texttt{not}_S$

$\overline{\texttt{who}}$ =



$$\begin{array}{l} \texttt{who}(b \otimes \top) = b \\ \texttt{who}(b \otimes \bot) = \overrightarrow{0} \end{array} \ , \text{ for } b \in B\,.$$

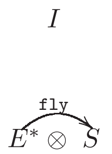
The algebraic expression defining $\overline{\texttt{who}} : I \to E^* \otimes E \otimes S^* \otimes E$ is

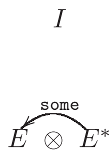
$$\overline{\texttt{who}} = c \circ \ulcorner(1_E \otimes \texttt{who}) \circ (d_E \otimes 1_S)\urcorner,$$

where $c : E^* \otimes S^* \otimes E \otimes E \ \to \ E^* \otimes E \otimes S^* \otimes E$ is the permutation that first switches the last two factors of $E^* \otimes S^* \otimes E \otimes E$ and then the third and the second factor.

If there is no particular logical content the postulate depends only on the type, namely the vector $\texttt{bird} : I \to V$ is Boolean, the morphism $\texttt{fly}$ is a predicate on E, the morphism $\texttt{some} : V \to V$ is an intrinsic projector.

$\overline{\texttt{bird}} =$ 

, $\overline{\texttt{fly}} = \eta_{\texttt{fly}} =$ 

, $\overline{\texttt{some}} = \eta_{\texttt{some}^*} =$ 

.

The meaning of grammatical strings involves besides the meanings of the words a syntactical analysis of the string.

A string of words $w_1 \dots w_n$ is *grammatical* if there are entries $w_1 : T_1 :: m_1, \dots, \ w_n : T_n :: m_n$, and a basic type $\boldsymbol{b}$ such that

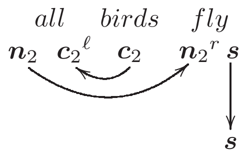
$$T_1 \dots T_n \vdash \boldsymbol{b}$$

is provable in compact bilinear logic. Due to a theorem in [Lambek, 1999] the graph of the proof involves only underlinks. Such a graph is called a *reduction*.

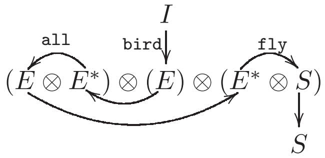
The *meaning* of the grammatical string $w_1 \dots w_n$ is computed by composing the tensor product $m_1 \otimes \dots \otimes m_n$ of the word meanings with the functorial image r' of the reduction $r : T_1 \dots T_n \to \boldsymbol{b}$ defined by the proof, i.e.

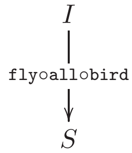
$$m(w_1 \dots w_n) = r' \circ (m_1 \otimes \dots \otimes m_n)\,.$$

The string *all birds fly* has a reduction to the sentence type

$r_0 =$ 

Hence, taking into account the postulate $\texttt{all} = 1_E$, the meaning vector of the sentence is

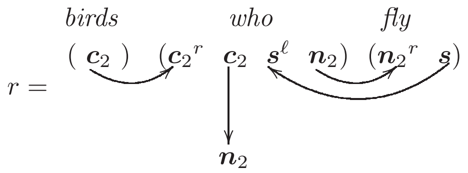
$r_0' \circ (\overline{\texttt{all}} \otimes \overline{\texttt{bird}} \otimes \overline{\texttt{fly}}) =$ 

$=$ 

(27)

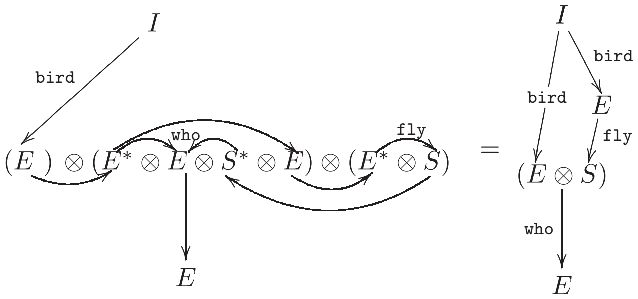
$= \texttt{fly} \circ \texttt{bird}\,.$

The string *birds who fly* has a reduction to the plural noun phrase type, namely



Compose the tensor product of the word vectors with the reduction to obtain

$$m(\textit{birds who fly}) = r' \circ (\overline{\texttt{bird}} \otimes \overline{\texttt{who}} \otimes \overline{\texttt{fly}}) \ =$$

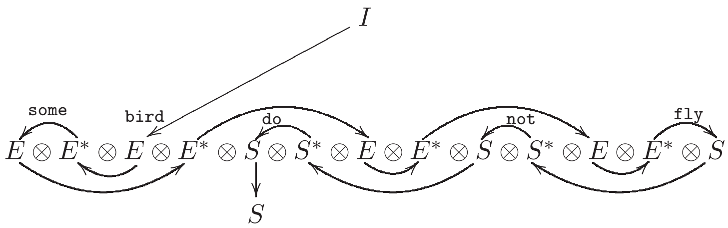


(28)

$$= \texttt{who} \circ (\texttt{bird} \otimes (\texttt{fly} \circ \texttt{bird})) = \texttt{who}(\texttt{bird}, \texttt{fly}(\texttt{bird}))\,.$$

Finally,

$$m(\textit{some birds do not fly}) = r \circ (\overline{\texttt{some}} \otimes \overline{\texttt{bird}} \otimes \overline{\texttt{do}} \otimes \overline{\texttt{not}} \otimes \overline{\texttt{fly}}) =$$



(29)

$$= \texttt{do} \circ \texttt{not} \circ \texttt{fly} \circ \texttt{some} \circ \texttt{bird} = \texttt{not}_S(\texttt{fly}(\texttt{some}(\texttt{bird})))\,.$$

To sum up: All grammatical strings are interpreted by variable free expressions formed by morphisms and vectors.

The computation of the expression involves a syntactical analysis of the string via a pregroup grammar. There are cubic polynomial algorithms that decide whether the string is grammatical and, if it is grammatical, construct a reduction. The reduction includes a choice of type for each word. Forming the tensor product of the corresponding meanings is proportional to the length of the string. Walking the graph is linear in the number of links, which is proportional to the number of words.

4.3 Internal logic in action

The pregroup algorithm interprets nouns and noun phrases by (Boolean) vectors. According to Proposition 11, they may be replaced by intrinsic projectors and application by composition. Indeed, $p(v_K) = p \circ p_K(\overrightarrow{1})$ for any projector p and Boolean vector v_K .

The lexicon list projectors as meanings of determiners and adjectives in attributive positions. Next we show that the relative clause formed by a relative pronoun and a verb phrase also corresponds to a projector.

[Preller and Sadrzadeh, 2011] have shown that the map `who` insures comprehension. Indeed, for every predicate $p : V_B \to S$ on V_B and every subset A of B, the following equality holds in $\mathit{2SF}$

$$\{x \in A \,:\, p(x) = \top\} = \mathtt{who}(A, p(A))\,.$$

Reformulate comprehension in terms of projectors of $\mathcal{C}$.

Proposition 16. *For every predicate p on $V = V_B$, the intrinsic projector $\widetilde{p}$ of V satisfies*

$$\widetilde{p} = \mathtt{who} \circ (1_V \otimes p) \circ d_V\,. \tag{30}$$

Proof. Recall the explicit definition of the morphism $\mathtt{who} : V \otimes S \to V$ in Subsection 4.2, namely

$$\begin{array}{ll} \mathtt{who} \circ (x \otimes \top) & = x \\ \mathtt{who} \circ (x \otimes \bot) & = \overrightarrow{0} \text{ else} \end{array}\;, \text{ for all } x \in B\,.$$

Hence

$$\begin{array}{ll} \mathtt{who} \circ (1_V \otimes p) \circ d_V \circ x = \mathtt{who} \circ \langle x, p(x)\rangle & = x \text{ if } p(x) = \top \\ \mathtt{who} \circ (1_V \otimes p) \circ d_V \circ x = \mathtt{who} \circ \langle x, p(x)\rangle & = \overrightarrow{0} \text{ else} \end{array}\;, \text{ for all } x \in B\,.$$

Hence, Equality (30) now follows from the explicit definition of $\widetilde{p}$ in Subsection 3.3 . ☐

The projector $\mathtt{who} \circ (1_V \otimes p) \circ d_V$ is the interpretation of the relative clause formed with the verb phrase p, e. g. $x \mapsto \mathtt{who} \circ \langle x, \mathtt{fly}(x)\rangle$ interprets the relative clause *who fly*.

For example, the meaning vector $\mathtt{who} \circ \langle \mathtt{bird}, \mathtt{fly} \circ \mathtt{bird}\rangle$ of the noun phrase *birds who fly* satisfies

$$\begin{array}{ll} \mathtt{who} \circ \langle \mathtt{bird}, \mathtt{fly} \circ \mathtt{bird}\rangle & = \{x \in \mathtt{bird} \,:\, \mathtt{fly}(x) = \top\} \text{ in } \mathit{2SF} \\ \mathtt{who} \circ \langle \mathtt{bird}, \mathtt{fly} \circ \mathtt{bird}\rangle & = \widetilde{\mathtt{fly}} \circ \mathtt{bird} \text{ in any semantic category.} \end{array}$$

By definition, $\widetilde{\mathtt{fly}} \circ \mathtt{bird}$ is the sum of the basis vectors $x \leq \mathtt{bird}$ left invariant by the projector $\widetilde{\mathtt{fly}}$.

The pregroup grammar interprets all noun phrases as projectors. Adjectives in predicative position and verbs, however, are interpreted by predicates. Hence, the meaning expression of a sentence is a Boolean combination of predicates applied to Boolean vectors. Replacing predicates and vectors by the corresponding projectors the result involves only projectors, say

$$\begin{aligned}
\texttt{fly} \circ \texttt{all} \circ \texttt{bird} &\mapsto \widetilde{\texttt{fly}} \circ \texttt{all} \circ \texttt{bird} \\
\texttt{fly} \circ \texttt{some} \circ \texttt{bird} &\mapsto \widetilde{\texttt{fly}} \circ \texttt{some} \circ \texttt{bird} \\
\texttt{not}_S \circ \texttt{fly} \circ \texttt{bird} &\mapsto \widetilde{\texttt{fly}}^{\perp} \circ \texttt{bird}.
\end{aligned}$$

We just saw that the first righthand expression above is the meaning of a noun phrase. The meaning of the sentence *all birds fly* is rendered by the assumption that it is true. The meanings of the sentences above are the equivalent equalities

$$\begin{aligned}
\texttt{fly} \circ \texttt{all} \circ \texttt{bird} = \top &\Leftrightarrow \widetilde{\texttt{fly}} \circ \texttt{all} \circ \texttt{bird} = \texttt{all} \circ \texttt{bird} \\
\texttt{fly} \circ \texttt{some} \circ \texttt{bird} = \top &\Leftrightarrow \widetilde{\texttt{fly}} \circ \texttt{some} \circ \texttt{bird} = \texttt{some} \circ \texttt{bird} \\
\texttt{not}_S \circ \texttt{fly} \circ \texttt{bird} = \top &\Leftrightarrow \widetilde{\texttt{fly}}^{\perp} \circ \texttt{bird} = \texttt{bird}\,.
\end{aligned}$$

The equivalence of the equalities is a particular case of Proposition 15, where the lefthand side is interpreted in $\mathcal{2SF}$ to avoid counting of elements.

We discuss the meaning of the second sentence, because the projector `some` acts differently from the existential quantifier $\exists_X$. Note that

$$p(\texttt{some}(A)) = \top \Rightarrow \exists_X (X \neq \emptyset \,\&\, X \subseteq A \,\&\, \texttt{fly}(X))\,.$$

The determiner *some* acts in natural language like a witness and only as a consequence like an existential quantifier, e.g. *some birds do not fly, they have no wings.* On the other hand, the interpretation of `some` may change from one occurrence to the next, for instance *some birds fly and some birds do not fly.*

The solution to the latter problem is to index the occurrences of *some.* This results in the following meaning of the latter sentence

$$\texttt{and}_S(\texttt{fly}(\texttt{some}_1(\texttt{bird})), \texttt{not}_S(\texttt{fly}(\texttt{some}_2(\texttt{bird})))) = \top\,.$$

By the Fundamental Property, $\texttt{some}_1$ selects a non-empty set of birds that fly and $\texttt{some}_2$ selects a non-empty set of birds that do not fly.

The fact that `some` acts like a witness is built into our categorical semantics. The discourse *Some birds fly. They have wings* is represented by the three expressions $\texttt{fly}(\texttt{some}(\texttt{bird}))$, $\texttt{have}(\texttt{they}, \texttt{wing})$, $\texttt{they} = \texttt{some}(\texttt{bird})$.

The interpretation of *some bird* as a generalised quantifier, see for example [Barwise and Cooper, 2002], takes into account the change of meaning with occurrences, but it does not construct the set to which the noun phrase refers.

5 Conclusion

The logic of natural language can be captured with two simple categorical operators, the biproduct and the dagger. Combined, they define the logic of subobjects in dagger kernel categories of [Heunen and Jacobs, 2010] and of compound systems in compact closed categories of [Abramsky and Coecke, 2004], two recent paradigms of quantum logic. The biproduct, with the ensuing matrix calculus, is present in the main intended models of quantum computing. It is also behind the explicit definitions of functions, ubiquitous in logical models and in programming languages.

A promising line of future investigations is the step from counting predicates to measuring predicates in Hilbert spaces to capture the notion of truth in probability of quantum logic and its interaction with classical logic.

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