On Essentially Conditional Information Inequalities

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Abstract—In 1997, Z. Zhang and R.W. Yeung found the first example of a conditional information inequality in four variables that is not "Shannon-type". This linear inequality for entropies is called conditional (or constraint) since it holds only under condition that some linear equations are satisfied for the involved entropies. Later, the same authors and other researchers discovered several unconditional information inequalities that do not follow from Shannon's inequalities for entropy.

In this paper we show that some non Shannon-type conditional inequalities are "essentially" conditional, i.e., they cannot be extended to any unconditional inequality. We prove one new essentially conditional information inequality for Shannon's entropy and discuss conditional information inequalities for Kolmogorov complexity.

I. INTRODUCTION

Let $(X_1, \ldots, X_n)$ be jointly distributed random variables on a finite domain. For this collection of random variables there are $2^n - 1$ non-empty subsets and for each subset we have a value of Shannon's entropy. We call this family of entropies the *entropy profile* of the distribution $(X_1, \ldots, X_n)$. Thus, to every n-tuple of jointly distributed random variables there corresponds its entropy profile which is a vector of values in $\mathbb{R}^{2^n-1}$. We say that a point in $\mathbb{R}^{2^n-1}$ is *constructible* if it is a vector of entropies for some distribution.

All constructible points satisfy different *information inequalities* that characterize the range of all entropies for X_i. The most known and understood are so-called *Shannon-type* inequalities, i.e., linear combinations of basic inequalities of type $I(U : V|W) \geq 0$, where U, V, W are any (possibly empty) subsets of the given family of random variables.

In 1998 Z. Zhang and R.W. Yeung proved the first example of an unconditional *non Shannon-type* information inequality, which was a linear inequality for entropies of (X_1, X_2, X_3, X_4) that cannot be represented as a combination of basic inequalities [5]. Since this seminal paper of Zhang and Yeung was published, many (in fact, *infinitely many*) non Shannon-type linear information inequalities were proven, see, e.g., [7], [8], [9], [12], [13]. Another very curious *piecewise linear* information inequality was proven in [15]. These new inequalities were applied in problems of network coding [14], secret sharing [16], etc. However, these inequalities and their 'physical meaning' are still not very well understood.

In this paper we discuss *conditional* (constraint) information inequalities. That is, we are interested in linear information inequalities that are true only given some linear constraint for entropies. Trivial examples of conditional inequalities can be easily derived from (unconditional) basic inequalities, e.g.,

if $H(X_1) = 0$ then $H(X_1, X_2) \leq H(X_2)$. However, some conditional inequalities cannot be obtained as a corollary of Shannon-type inequalities. The first example of a nontrivial conditional inequality was proven in [4] (even before the first example of an unconditional non Shannon-type inequality):

$$\text{if } I(A : B) = I(A : B|C) = 0, \text{ then}$$
$$I(C : D) \leq I(C : D|A) + I(C : D|B) \tag{1}$$

Another conditional inequality

$$\text{if } I(A : B|C) = I(B : D|C) = 0, \text{ then}$$
$$I(C : D) \leq I(C : D|A) + I(C : D|B) + I(A : B) \tag{2}$$

was proven by F. Matúš in [6].

In [7] it was conjectured that (1) can be extended to some unconditional inequality

$$I(C : D) \leq I(C : D|A) + I(C : D|B) +$$
$$+ \kappa(I(A : B) + I(A : B|C)) \tag{3}$$

(for some constant $\kappa > 0$). In this paper we prove that this conjecture is *wrong*: for any coefficient κ, inequality (3) is not true for some distributions. So, inequality (1) is "essentially conditional"; it cannot be extended to an unconditional information inequality. A similar statement can be proven for (2).

In this paper we also prove one new conditional linear inequality that cannot be extended to any unconditional inequality. So, now we have three examples of essentially conditional information inequality.

It is known that the class of unconditional linear information inequalities are the same for Shannon's entropy and for Kolmogorov complexity. The situation with conditional inequalities is more complicated: the known technique used to prove constraint information inequalities for Shannon's entropy cannot be directly adapted for Kolmogorov complexity. In fact, it is not even clear how to formulate Kolmogorov's version of constraint inequalities. However, we prove for Kolmogorov complexities some counterpart of inequality (1); this inequality holds only for some special tuples of words.

The paper is organized as follows. In Section II we use the technique from [4] and prove one new conditional information inequality. In Section III we prove that this new inequality as well as (1) and (2) cannot be extended to any unconditional inequalities. In Section IV we prove some version of conditional inequality for Kolmogorov complexities.

II. NONTRIVIAL CONDITIONAL INFORMATION INEQUALITIES

The very first example of an inequality that does not follow from basic (Shannon type) inequalities was the following result of Z. Zhang and R. W. Yeung:

Theorem 1 (Zhang–Yeung, [4])**.** *For all random variables A, B, C, D, if $I(A : B|C) = I(A : B) = 0$ then*

$$I(C : D) \leq I(C : D|A) + I(C : D|B).$$

With the same technique F. Matúš proved another conditional inequality (2), see [6]. Using a similar method, we prove one new conditional inequality:

Theorem 2. *For all random variables A, B, C, D if*

$$H(C|A, B) = I(A : B|C) = 0,$$

then $I(C : D) \leq I(C : D|A) + I(C : D|B) + I(A : B)$.

Proof: The argument consists of two steps: *enforcing conditional independence* and *elimination of conditional entropy*. Let us have a joint distribution of random variables A, B, C, D. The first trick of the argument is a special transformation of this distribution: we keep the same distribution of the triples (A, C, D) and (B, C, D) but make A and B independent conditional on (C, D). Intuitively it means that we first choose at random (using the old distribution) values of C and D; then given fixed values of C, D we independently choose at random A and B (the conditional distributions of A given (C, D) and B given (C, D) are the same as in the original distribution).

More formally, we construct a new distribution $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. If $\text{Prob}[A = a, B = b, C = c, D = d]$ is the original distribution, then the new distribution is defined as follows:

$$\text{Prob}[\tilde{A} = a, \tilde{B} = b, \tilde{C} = c, \tilde{D} = d] =$$
$$\frac{\text{Prob}[A = a, C = c, D = d] \cdot \text{Prob}[B = b, C = c, D = d]}{\text{Prob}[C = c, D = d]}$$

(with the convention $\frac{0}{0} = 0$ for all values a, b, c, d of the four random variables). From the construction ($\tilde{A}$ and $\tilde{B}$ are independent given $\tilde{C}, \tilde{D}$) it follows that

$$H(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = H(\tilde{C}, \tilde{D}) + H(\tilde{A}|\tilde{C}, \tilde{D}) + H(\tilde{B}|\tilde{C}, \tilde{D})$$

Since $(\tilde{A}, \tilde{C}, \tilde{D})$ and $(\tilde{B}, \tilde{C}, \tilde{D})$ have exactly the same distributions as the original (A, C, D) and (B, C, D) respectively, we have

$$H(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = H(C, D) + H(A|C, D) + H(B|C, D)$$

The same entropy can be bounded in another way:

$$H(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \leq H(\tilde{D}) + H(\tilde{A}|\tilde{D}) + H(\tilde{B}|\tilde{D}) + H(\tilde{C}|\tilde{A}, \tilde{B})$$

Notice that the entropies $H(\tilde{D})$, $H(\tilde{A}|\tilde{D})$ and $H(\tilde{B}|\tilde{D})$ are equal to $H(D)$, $H(A|D)$ and $H(B|D)$ respectively (we again use the fact that $\tilde{A}, \tilde{D}$ and $\tilde{B}, \tilde{D}$ have the same distributions as

A, D and B, D respectively in the original distribution). Thus, we get

$$H(C, D) + H(A|C, D) + H(B|C, D) \leq$$
$$H(D) + H(A|D) + H(B|D) + H(\tilde{C}|\tilde{A}, \tilde{B})$$

It remains to estimate the value $H(\tilde{C}|\tilde{A}, \tilde{B})$. We will show that it is zero (and this is the second trick used in the argument).

Here we will use the two conditions of the theorem. We say that some values a, c (respectively, b, c or a, b) are *compatible* if in the original distribution these values can appear together, i.e., $\text{Prob}[A = a, C = c] > 0$ (respectively, $\text{Prob}[B = b, C = c] > 0$ or $\text{Prob}[A = a, B = b] > 0$). Since A and B are independent given C, if some values a and b (of A and B) are compatible with the same value c of C, then these a and b are compatible with each other.

In the new distribution $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ values of $\tilde{A}$ and $\tilde{B}$ are compatible with each other *only if* they are compatible with one and the same value of $\tilde{C}$; hence, these values must be also compatible with each other in the original distribution (A, B). Further, since $H(C|A, B) = 0$, for each pair of compatible values of A, B there exists only one value of C. Thus, for each pair of values $(\tilde{A}, \tilde{B})$ with probability 1 there exists only one value of $\tilde{C}$. In a word, in the new distribution $H(\tilde{C}|\tilde{A}, \tilde{B}) = 0$.

Summarizing our arguments, we get

$$H(C, D) + H(A|C, D) + H(B|C, D) \leq$$
$$H(D) + H(A|D) + H(B|D),$$

which is equivalent to

$$I(C : D) \leq I(C : D|A) + I(C : D|B) + I(A : B).$$

∎

III. CONDITIONAL INEQUALITIES THAT CANNOT BE EXTENDED TO ANY UNCONDITIONAL INEQUALITIES

In [7] it was conjectured that the conditional inequality from Theorem 1 is a corollary of some *unconditional* information inequality (which was not discovered yet):

Conjecture 1 ([7])**.** *For some constant $\kappa > 0$ inequality (3) is true for all random variables A, B, C, D.*

Obviously, if such an inequality could be proven, it would imply the statement of Theorem 1. Similar conjectures could be formulated for (2) and the conditional inequality from Theorem 2. We prove that these conjectures are false, i.e., these three conditional inequalities cannot be converted into unconditional inequalities:

Theorem 3. *(a) For any κ the inequality (3) is not true for some distributions (A, B, C, D).*
(b) For any κ the inequality

$$I(C : D) \leq I(C : D|A) + I(C : D|B) + I(A : B) + \quad (4)$$
$$+ \kappa(I(A : B|C) + H(C|A, B))$$

is not true for some distributions (A, B, C, D).

(c) For any κ the inequality

$$I(C:D) \leq I(C:D|A) + I(C:D|B) + I(A:B) + \\ + \kappa(I(A:B|C) + H(B:D|C)) \quad (5)$$

is not true for some distributions (A,B,C,D). Thus, (2) cannot be extended to an unconditional inequality.

Proof: (a) For all $\varepsilon \in [0,1]$ we us consider the following joint distribution of binary variables (A,B,C,D):

$$
\begin{array}{lll}
\text{Prob}[A=0,\ B=0,\ C=0,\ D=1] & = & (1-\varepsilon)/4, \\
\text{Prob}[A=0,\ B=1,\ C=0,\ D=0] & = & (1-\varepsilon)/4, \\
\text{Prob}[A=1,\ B=0,\ C=0,\ D=1] & = & (1-\varepsilon)/4, \\
\text{Prob}[A=1,\ B=1,\ C=0,\ D=1] & = & (1-\varepsilon)/4, \\
\text{Prob}[A=1,\ B=0,\ C=1,\ D=1] & = & \varepsilon.
\end{array}
$$

For each value of A and for each values of B, the value of at least one of variables C,D is uniquely determined: if $A=0$ then $C=0$; if $A=1$ then $D=1$; if $B=0$ then $D=1$; and if $B=1$ then $C=0$. Hence, $I(C:D|A) = I(C:D|B) = 0$. Also it is easy to see that $I(A:B|C) = 0$. Thus, if (3) is true, then $I(C:D) \leq \kappa I(A:B)$.

Denote the right-hand and left-hand sides of this inequality by $L(\varepsilon) = I(C:D)$ and $R(\varepsilon) = \kappa I(A:B)$. Both functions $L(\varepsilon)$ and $R(\varepsilon)$ are continuous, and $L(0) = R(0) = 0$ (for $\varepsilon = 0$ both sides of the inequality are equal to 0). However the asymptotics of $L(\varepsilon)$ and $R(\varepsilon)$ as $\varepsilon \to 0$ are different: it is not hard to check that $L(\varepsilon) = \Theta(\varepsilon)$, but $R(\varepsilon) = O(\varepsilon^2)$. From (3) we have $\Theta(\varepsilon) \leq O(\varepsilon^2)$, which is a contradiction.

(b) For every value of $\varepsilon \in [0,1]$ we consider the following joint distribution of binary variables (A,B,C,D):

$$
\begin{array}{lll}
\text{Prob}[A=1,\ B=1,\ C=0,\ D=0] & = & 1/2 - \varepsilon, \\
\text{Prob}[A=0,\ B=1,\ C=1,\ D=0] & = & \varepsilon, \\
\text{Prob}[A=1,\ B=0,\ C=1,\ D=0] & = & \varepsilon, \\
\text{Prob}[A=0,\ B=0,\ C=1,\ D=1] & = & 1/2 - \varepsilon.
\end{array}
$$

The argument is similar to the proof if (a). First, it is not hard to check that $I(C:D|A) = I(C:D|B) = H(C|AB) = 0$ for every ε. Second,

$$
\begin{array}{lll}
I(A:B) & = & 1 + (2 - 2/\ln 2)\varepsilon + 2\varepsilon \log \varepsilon + O(\varepsilon^2), \\
I(C:D) & = & 1 + (4 - 2/\ln 2)\varepsilon + 2\varepsilon \log \varepsilon + O(\varepsilon^2),
\end{array}
$$

so $I(A:B)$ and $I(C:D)$ both tend to 1 as $\varepsilon \to 0$, but their asymptotics are different. Similarly,

$$I(A:B|C) = O(\varepsilon^2).$$

It follows from (4) that

$$2\varepsilon + O(\varepsilon^2) \leq O(\varepsilon^2) + O(\kappa\varepsilon^2),$$

and with any κ we get a contradiction for small enough ε.

(c) For the sake of contradiction we consider the following joint distribution of binary variables (A,B,C,D) for every value of $\varepsilon \in [0,1]$:

$$
\begin{array}{lll}
\text{Prob}[A=0,\ B=0,\ C=0,\ D=0] & = & 3\varepsilon, \\
\text{Prob}[A=1,\ B=1,\ C=0,\ D=0] & = & 1/3 - \varepsilon, \\
\text{Prob}[A=1,\ B=0,\ C=1,\ D=0] & = & 1/3 - \varepsilon, \\
\text{Prob}[A=0,\ B=1,\ C=0,\ D=1] & = & 1/3 - \varepsilon.
\end{array}
$$

We substitute this distribution in (5) and obtain

$$I_0 + O(\varepsilon) \leq I_0 + 3\varepsilon \log \varepsilon + O(\varepsilon) + O(\kappa\varepsilon),$$

where I_0 is the mutual information between C and D for $\varepsilon = 0$ (which is equal to the mutual information between A and B for $\varepsilon = 0$). We get a contradiction as $\varepsilon \to 0$. ∎

The theorem above implies that the set of all linear information inequalities for 4-tuples must have rather complicated structure. Let us remind that a point $\vec{a} \in \mathbb{R}^{15}$ is called constructible if there exists a joint distribution (A,B,C,D) such that

$$\vec{a} = (H(A), H(B), \ldots, H(A,B,C,D))$$

($\vec{a}$ consists of entropies of all non-empty tuples of random variables A,B,C,D). Further, a point $\vec{a}$ is called *asymptotically constructible* if for every $\varepsilon > 0$ in ε-neighborhood of $\vec{a}$ there exists an constructible point $\vec{a}'$. In a similar way the set of (asymptotically) constructible points is defined for any number of random variables (in $\mathbb{R}^{2^n - 1}$ for n-tuples of random variables). It is known (see, e.g., [5], [18]) that for every n the set of asymptotically constructible points representable by n-tuples of random variables make a closed convex cone in $\mathbb{R}^{2^n - 1}$. The dual representation of this cone is the set of all linear information inequalities. We will show that for $n \geq 4$ the structure of this cone is not trivial.

From Theorem 3 we get a new proof of the result by F. Matúš [9]: the set of linear information inequalities for 4 random variables is not finitely generated.

Theorem 4 ([9])**.** *The cone of asymptotically constructible points for 4 random variables is not polyhedral (equivalently, the set of linear information inequalities for 4-tuples of random variables is not finitely generated).*

Proof: For the sake of contradiction we assume that the cone in $\mathbb{R}^{15}$ that consist of all asymptotically constructible points for 4 random variables (A,B,C,D) is polyhedral. The constraints $I(A:B) = I(A:B|C) = 0$ specify some face (of co-dimension 2) on the boundary of this polyhedron. The corresponding conditional inequality (from Theorem 1) specifies a non-degenerate linear functionals, which is non-negative on the corresponding faces. Technically, this functional is defined by the linear form $g = I(C:D|A) + I(C:D|B) - I(C:D)$, which is non-negative on this face of the cone. With the standard linear programming technique it can be proven that this functional g can be extended to a linear functional g' such that (a) g' is non-negative on the entire cone of asymptotically constructible points, and (b) g' coincides with g on the subspace of co-dimension 2 defined by the condition $I(A:B) = I(A:B|C) = 0$ (see Proposition 17 in [1]). In coordinates such a functional g' must have form

$$g' = I(C:D|A) + I(C:D|B) - I(C:D) \\ + d_1 I(A:B) + d_2 I(A:B|C).$$

(with some reals d_1 and d_2). It follows that $g' \geq 0$ for all constructible points, so we get (3) (where κ is equal to

maximum of d_1 and d_2). This contradicts Theorem 3, and we are done. ∎

IV. CONSTRAINT INEQUALITY FOR KOLMOGOROV COMPLEXITY

Kolmogorov complexity of a finite binary string X is defined as the length of the shortest program that generates X; similarly, Kolmogorov complexity of a string X given another string Y is defined as the length of the shortest program that generates X given Y as an input. More formally, for any programming language L, Kolmogorov complexity $K_L(X|Y)$ is defined as

$$K_L(X|Y) = \min\{|p| \ : \ \text{program } p \text{ prints } X \text{ on input } Y\},$$

and unconditional complexity $K_L(X)$ is defined as complexity of X given the empty Y. The basic fact of Kolmogorov complexity theory is the invariance theorem: there exists a *universal* programming language U such that for any other language L we have $K_U(X|Y) \leq K_L(X|Y) + O(1)$ (the $O(1)$ depends on L but not on X and Y). We fix such a universal language U; in what follows we omit the subscript U and denote Kolmogorov complexity by $K(X)$, $K(X|Y)$. We refer the reader to an excellent book [10] for a survey of properties of Kolmogorov complexity.

Kolmogorov complexity was introduced in [2] as an algorithmic version of measure of information in an individual object. In some sense, properties of Kolmogorov complexity are quite similar to properties Shannon's entropy. For example, for the property of Shannon's entropy $H(A, B) = H(A) + H(B|A)$ there is a Kolmogorov's counterpart

$$K(A, B) = K(A) + K(B|A) + O(\log K(A, B)) \qquad (6)$$

(the Kolmogorov–Levin theorem, [3]). This result justifies the definition of the mutual information, which is an algorithmic version of the standard Shannon's definition: the mutual information is defined as $I(A : B) := K(A) + K(B) - K(A, B)$, and the conditional mutual information is defined as

$$I(A : B|C) := K(A, C) + K(B, C) - K(A, B, C) - K(C).$$

From the Kolmogorov–Levin theorem it follows that $I(A : B)$ is equal to $K(A) - K(A|B)$, and the conditional mutual information $I(A : B|C)$ is equal to $K(A|C) - K(A|B, C)$ (all these equations hold only up to logarithmic terms).

In fact, we have a much more deep and general parallel between Shannon's and Kolmogorov's information theories; for every linear inequality for Shannon's entropy there exists a Kolmogorov's counterpart:

Theorem 5 ([11]). *For each family of coefficients $\{\lambda_W\}$ the inequality*

$$\sum_i \lambda_i H(\alpha_i) + \sum_{i<j} \lambda_{ij} H(\alpha_i, \alpha_j) + \ldots \geq 0$$

is true for every distribution $\{\alpha_i\}$ if and only if for some constant C the inequality

$$\sum_i \lambda_i K(a_i) + \sum_{i<j} \lambda_{ij} K(a_i, a_j) + \ldots C \log N \geq 0$$

is true for all tuples of strings $\{a_i\}$, $N = K(a_1, a_2, \ldots)$ (C does not depend on a_i).

Thus, the class of unconditional inequalities valid for Shannon's entropy coincides with the class of (unconditional) inequalities valid for Kolmogorov complexity. What about conditional inequalities?

In the framework of Kolmogorov complexity we cannot say that some information quantity *exactly* equals zero. Indeed, even the definition of Kolmogorov complexity makes sense only up to an additive term that depends on the choice of the universal programming language. Moreover, such a natural basic statement as the Kolmogorov–Levin theorem (6) holds only up to a logarithmic term. So, if we want to prove a sensible conditional inequality for Kolmogorov complexity, the linear constraints must be formulated with some reasonable precision. A natural version of Theorem 1 is the following conjecture:

Conjecture 2. *There exist functions $f(n)$ and $g(n)$ such that $f(n) = o(n)$ and $g(n) = o(n)$, and for all strings A, B, C, D satisfying $I(A : B|C) \leq f(N)$, $I(A : B) \leq f(N)$ it holds $I(C : D) \leq I(C : D|A) + I(C : D|B) + g(N)$ (where $N = K(A, B, C, D)$).*

There is no hope to prove Conjecture 2 with $f(n)$ and $g(n)$ of order $\Theta(\log n)$. Indeed, using a counterexample from the proof of Theorem 3(a), we can construct binary strings A, B, C, D such that the quantities $I(A : B|C)$, $I(A : B)$, $I(C : D|A)$, and $I(C : D|B)$ are bounded by $O(\log N)$, but $I(C : D) = \Omega(\sqrt{N \log N})$. However, even if Conjecture 2 is false in general, similar conditional inequalities (even with logarithmic precision) can be true for some special tuples A, B, C, D. In what follows we show how to prove such an inequality for one natural example of strings A, B, C (and any D).

Let $\mathbb{F}_n$ be the finite field of 2^n elements. We consider the affine plane over $\mathbb{F}_n$. Let C be random line in this plane, and A and B be two points incident to this line. To specify the triple $\langle A, B, C \rangle$ we need at most $4n + O(1)$ bits of information: a line in a plane can be specified by two parameters in $\mathbb{F}_n$; to specify each point in a given line we need additional n bits of information.

We take a triple of strings $\langle A, B, C \rangle$ as specified above with maximal possible Kolmogorov complexity, i.e., such that $K(A, B, C) = 4n + O(1)$ (it follows from a simple counting argument that such a triple exists; moreover, there are about $2^{4n+O(1)}$ such triples). For these A, B and C we can easily estimate all their Kolmogorov complexities:

$K(A), \ K(B),$ and $K(C)$ are equal to $2n + O(1)$,
$K(A, C) = 3n + O(1), \ K(B, C) = 3n + O(1),$
$H(A, B) = 4n + O(1).$

For this triple of strings the quantities $I(A : B)$ and $I(A : B|C)$ are negligible (logarithmic). This condition is very similar to the condition on random variables A, B, C in Theorem 1. So, it is not very surprising that Kolmogorov's counterpart of Theorem 1 holds for these strings:

Proposition 1. *For the strings* A, B, C *defined above and for* all *strings* D *we have*

$$I(C : D) \leq I(C : D|A) + I(C : D|B) + O(\log N),$$

where $N = K(A, B, C, D)$.

This statement can be proven by an argument similar to the proof of Theorem 2. Let us explain this argument in full detail.

Proof: We may identify C with a linear function $c_1 x + c_2$ over $\mathbb{F}_n$, where c_1 and c_2 are elements of the field (since Kolmogorov complexity of C is large, it cannot be a *vertical* line on the plane). Further, the points A and B in this line can be represented as pairs $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ such that

$$c_1 \cdot a_1 + c_2 = a_2 \text{ and } c_1 \cdot b_1 + c_2 = b_2$$

(here a_i and b_i are also elements of $\mathbb{F}_n$). By assumption, complexity of the pair (A, B) is close to $4n$. It means that $A \neq B$; hence, $a_1 \neq b_1$. Let i be one of indexes such that the ith bits of a_1 and b_1 are different. W.l.o.g. we assume that the ith bit in a_1 is equal to 0 and the ith bit in b_1 is equal to 1.

Now we split the affine plane over $\mathbb{F}_n$ into two halves: P_0 will consist of all points (x, y) such that the ith bit of x is 0, and P_1 will consist of the points (x, y) such that the ith bit of x is 1. So, point $A = (a_1, a_2)$ belongs to P_0, and $B = (b_1, b_2)$ belongs to P_1.

Now we are going to variate the points A and B: we will substitute A and B by their 'clones' A' and B' so that the triples $\langle A', B', C \rangle$ remain "similar" to the initial one $\langle A, B, C \rangle$. More precisely, we say that A' is a *clone* of A if

- $A' = (a_1', a_2')$ is a point in line C, and $A' \in P_0$ (i.e., $c_1 \cdot a_1' + c_2 = a_2'$, and the ith bit of a_1' is equal to 0);
- complexities $K(A')$, $K(A', C)$, $K(A', D)$, and $K(A', C, D)$ are equal (up to an additive term $O(\log N)$) to the corresponding complexities $K(A)$, $K(A, C)$, $K(A, D)$, and $K(A, C, D)$.

Similarly, we say that B' is a *clone* of B if

- $B' = (b_1', b_2')$ is a point in line C, and $B' \in P_1$, and
- complexities $K(B')$, $K(B', C)$, $K(B', D)$, and $K(B', C, D)$ are equal (up to an additive term $O(\log N)$) to the corresponding complexities $K(B)$, $K(B, C)$, $K(B, D)$, and $K(B, C, D)$.

From a simple counting argument it follows that there exist $2^{K(A|C,D)-O(\log N)}$ different clones of A and $2^{K(B|C,D)-O(\log N)}$ clones of B (see, e.g., [11, Lemma 2] or [17, Lemmas 1–2]).

Let us take a pair of clones A' and B' with maximal complexity given (C, D). Then

$$K(A', B', C, D) =$$
$$K(C, D) + K(A'|CD) + K(B'|CD) + O(\log N) =$$
$$K(C, D) + K(A|C, D) + K(B|C, D) + O(\log N)$$

On the other hand,

$$K(A', B', C, D) \leq K(D) +$$
$$K(A'|D) + K(B'|D) + K(C|A', B') + O(\log N)$$

By definition of clones, complexities $K(A'|D)$ and $K(B'|D)$ are equal (up to $O(\log N)$ term) to $K(A|D)$ and $K(B|D)$ respectively. Since A' and B' belong to P_0 and P_1 respectively, they cannot be equal to each other. Hence, A' and B' uniquely determine line C. So, we get

$$K(C, D) + K(A|CD) + K(B|CD) \leq$$
$$K(D) + K(A|D) + K(B|D) + O(\log N),$$

which is equivalent (by the Kolmogorov–Levin theorem) to

$$I(C : D) \leq I(C : D|A) + I(C : D|B) + O(\log N).$$

■

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