



**HAL**  
open science

## Every strong digraph has a spanning strong subgraph with at most $n+2\alpha-2$ arcs

Stéphane Bessy, Stéphan Thomassé

### ► To cite this version:

Stéphane Bessy, Stéphan Thomassé. Every strong digraph has a spanning strong subgraph with at most  $n+2\alpha-2$  arcs. *Journal of Combinatorial Theory, Series B*, 2003, 87 (2), pp.289-299. 10.1016/S0095-8956(02)00020-5 . lirmm-00738215

**HAL Id: lirmm-00738215**

**<https://hal-lirmm.ccsd.cnrs.fr/lirmm-00738215>**

Submitted on 3 Oct 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Every Strong Digraph has a Spanning Strong Subgraph with at most $n + 2\alpha - 2$ arcs.

Stéphane Bessy  
and  
Stéphan Thomassé

Laboratoire LaPCS, U.F.R. de Mathématiques,  
Université Claude Bernard - Lyon 1, 50, avenue Tony Garnier,  
Bâtiment RECHERCHE [B], Domaine de Gerland  
69367 Lyon Cedex 07, France  
email : bessy@univ-lyon1.fr  
thomasse@univ-lyon1.fr

## Abstract

Answering a question of Adrian Bondy [4], we prove that every strong digraph has a spanning strong subgraph with at most  $n + 2\alpha - 2$  arcs, where  $\alpha$  is the size of a maximum stable set of  $D$ . Such a spanning subgraph can be found in polynomial time. An infinite family of oriented graphs for which this bound is sharp was given by Odile Favaron [3]. A direct corollary of our result is that there exists  $2\alpha - 1$  directed cycles which span  $D$ . Tibor Gallai [6] conjectured that  $\alpha$  directed cycles would be enough.

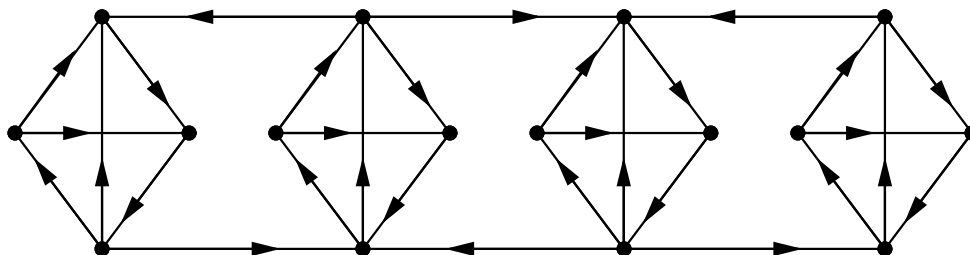
## 1 Introduction and known results.

In this paper, cycles of length two are allowed. Since loops and multiple arcs play no role in this topic, we will simply assume that our digraphs are loopless and simple - when performing a contraction, we will implicitly delete the cycles of length one and reduce the multiple arcs to simple one. Let  $D = (V, E)$  be a strong digraph. We are mainly concerned in this paper by the following problem: What is the minimum number of arcs of a strong spanning subgraph of  $D$ ? This classical problem is known as the MSSS-problem, see for instance [1] for a survey on this topic, see [7] and [11] for its relationship with connectivity and [8], [12] for some approximation algorithm. Let us say that a strong digraph  $D = (V, E)$  is a  $k$ -handle if  $k = |E| - |V| + 1$  (a 0-handle is simply a single vertex). We want to find the minimum  $k$ , for which there exists a  $k$ -handle which is a spanning subgraph of  $D$ . We introduce now the key-definitions in this topic: a *handle* is a directed path  $H := x_1, \dots, x_l$  in which we allow  $x_1 = x_l$ . We denote the restriction of  $H$  to  $\{x_i, x_{i+1}, \dots, x_j\}$  by  $H[x_i, x_j]$ , and  $\overset{\circ}{H} := H \setminus \{x_1, x_l\}$ . The vertex  $x_1$  is the *head* of  $H$  and  $x_l$  is the *tail* of  $H$ . If  $A$  and  $B$  are subgraphs of  $D$ , an  $(A, B)$ -handle of  $D$  is a handle with its head in  $V(A)$ , its tail in  $V(B)$  and its internal vertices and arcs disjoint from  $A \cup B$ . We simply write  $(A)$ -handle instead of  $(A, A)$ -handle. A *handle basis* of  $D$  (or *ear decomposition*, see [1]) is a sequence  $H_0, H_1, \dots, H_k$  of handles of  $D$  such that  $H_0$  is a single vertex,  $H_i$  is a  $(\cup\{H_j : j < i\})$ -handle for all  $i = 1, \dots, k$  and  $D = \cup\{H_i : i = 0, \dots, k\}$ . Clearly, a digraph has a handle basis  $H_0, \dots, H_k$  if and only

if  $D$  is a  $k$ -handle. Moreover, if  $D'$  is a minimum strong spanning subgraph of  $D$ , every  $H_i$  in any handle basis of  $D'$  has at least 2 arcs. It follows directly that  $D$  is spanned by a  $k$ -handle with  $k \leq n - 1$ . Our goal in this paper is to prove the following theorem, where  $\alpha(D)$  is the number of vertices of a maximum stable set of  $D$ , called the *stability* of  $D$ :

**Theorem 1** *Every strong digraph  $D$  is spanned by a  $k$ -handle, with  $k \leq 2\alpha(D) - 1$ .*

To motivate this result, we invite the reader to check that the bound is sharp when  $D$  is chosen in the following family of examples due to O. Favaron and drawn for the illustrative case  $\alpha = 4$ .



Theorem 1 is one of the corollary of the following conjecture of Chen and Manalastas, which is explicitly stated in [1] and [3].

**Conjecture 1** *Every strong digraph with stability  $\alpha$  is spanned by the disjoint union of some  $k_i$ -handles, where  $k_i > 0$  for all  $i$  and the sum of the  $k_i$  being at most  $\alpha$ .*

To see that Conjecture 1 implies Theorem 1, observe that such a disjoint union has exactly  $n + k - c$  arcs where  $c$  is the number of components and  $k$  is the sum of the  $k_i$ . Consequently, making this disjoint union strong requires at most  $2c - 2$  new arcs, and thus  $D$  is spanned by a strong digraph with at most  $n + k + c - 2 \leq n + 2\alpha - 2$  arcs, since  $\alpha \geq c$ . Conjecture 1 also implies the following result [9], once conjectured by Las-Vergnas:

**Theorem 2** *Every strong digraph with stability  $\alpha > 1$  is spanned by the disjoint union of  $\alpha - 1$  paths.*

But the real motivation of Conjecture 1 is to prove the following long-standing conjecture of Gallai [6]: every strong digraph is spanned by the union of  $\alpha$  cycles. For all these reasons, Conjecture 1 seems to be the very challenge of this topic. It is verified for  $\alpha = 1$ , this is the well-known result of Camion: *every strong tournament has a hamilton cycle*. The case  $\alpha = 2$  is the following theorem of Chen and Manalastas [5]: *Every strong digraph with stability 2 is spanned by two cycles, intersecting one another on a (possibly empty) path*. The case  $\alpha = 3$  can be found in [10]. The link between the MSSS-problem and the stability number is the classical Gallai-Milgram's theorem: *every digraph  $D$  is spanned by the disjoint union of  $\alpha(D)$  directed paths*. It suggests that the involved number of handles in a handle basis should be related to  $\alpha$ . In [3], Bondy proposed the following refinement of Gallai-Milgram's theorem. The proof is by induction on  $k$ , and can also be found in [1] and in [2].

**Theorem 3** *Let  $D$  be a digraph and  $\{P_i : 1 \leq i \leq k\}$  be a spanning set of disjoint directed paths of  $D$ . If  $k > \alpha(D)$ , there exists a spanning set of disjoint paths  $\{P'_i : 1 \leq i \leq k - 1\}$  of  $D$  such that every head (resp. tail) of a  $P'_i$  is the head (resp. the tail) of a  $P_j$ .*

This theorem provides the key-operation of this paper - the main snag being that strong connectivity is certainly not preserved under such a path exchange. The difficult part of the proof is to find some structures (the so-called tree-handle systems) on which we can perform Theorem 3.

## 2 Completion.

An *out-arborescence* is an oriented tree in which every vertex has indegree at most 1. The one vertex with indegree 0 is called the *root*. The vertices with outdegree 0 are the *leaves*. The dual definitions hold for *in-arborescence*. A *bi-arborescence*  $A$  is a tree obtained by identifying the root of an in-arborescence  $A_-$  and the root of an out-arborescence  $A_+$ . The vertices of  $A_-$  (resp.  $A_+$ ) are the *in-vertices* (resp. the *out-vertices*) of  $A$ . The vertices of  $A$  with indegree 0 (resp. outdegree 0) are the *in-leaves* (resp. the *out-leaves*) of  $A$ . The common root is both an in-vertex and an out-vertex of  $A$ , we call it the *center* of  $A$ . Observe that the center of  $A$  can also be a leaf of  $A$ , when  $A_-$  or  $A_+$  is a single vertex. The vertices of  $A$  which are not leaves are the *internal* vertices of  $A$ . A bi-arborescence is *plain* if it has at least two in-leaves and two out-leaves. Let  $D = (V, E)$  be a strong digraph and  $S$  a subset of  $V$ . We denote by  $D[S]$  the induced restriction of  $D$  on  $S$ . If  $A$  and  $B$  are two nonempty subsets of  $V$ , an  $(A, B)$ -*completion* is a set  $\{A_1, \dots, A_r\}$  of bi-arborescences such that:

- i) Every  $A_i$  is a subgraph of  $D$ .
- ii) The internal vertices of  $A_i$  do not belong to  $A \cup B$ .
- iii) For all  $i \neq j$ ,  $V(A_i) \cap V(A_j) \subseteq A \cup B$ .
- iv) In the graph  $D[A \cup B] \cup A_1 \cup \dots \cup A_r$  (called *completed graph*), every vertex  $a \in A$  is the head of an  $(a, B)$ -path and every vertex  $b \in B$  is the tail of an  $(A, b)$ -path.

An  $(A, B)$ -completion  $C$  is *minimum* if  $\sum_1^r |V(A_i)|$  is minimum. Observe that if  $C$  is minimum, every leaf  $f$  of  $A_i$  belongs to  $A \cup B$ . Indeed, if  $f \notin A \cup B$ , the vertex  $f$  only belongs to one bi-arborescence  $A_i$  and thus  $(C \setminus \{A_i\}) \cup \{A_i \setminus f\}$  is still a completion, a contradiction to the minimality of  $C$ . It follows from this observation that every vertex  $v \notin A \cup B$  in the completed graph of a minimum  $(A, B)$ -completion is the head of a  $(v, B)$ -path and the tail of an  $(A, v)$ -path: indeed the vertex  $v$  is certainly in a unique bi-arborescence  $A_i$ , therefore there exists a path from  $v$  to an out-leaf  $l$  of  $T$ . If  $l \in B$  we are done, and if  $l \in A$ , by iv), there is a path from  $l$  to  $B$  in the completed graph. Similarly, an  $(A, v)$ -path exists.

**Lemma 1** *Let  $C$  be a minimum  $(A, B)$ -completion of a strong digraph  $D = (V, E)$ . If a bi-arborescence  $T$  of  $C$  has more than one out-leaf, all the out-leaves of  $T$  belong to  $B \setminus A$ . Similarly, if  $T$  has more than one in-leaf, all the in-leaves of  $T$  belong to  $A \setminus B$ .*

**Proof.** Suppose that  $T$  has one out-leaf  $f \in A$  and another out-leaf  $g \in B$ . We consider  $C' = (C \setminus \{T\}) \cup \{T \setminus f\}$ . Since  $T \setminus f$  has an out-leaf in  $B$ , the completed graph of  $C'$  still satisfies the first part of iv). Also, since  $f$  belongs to  $A$ , deleting  $f$  from  $T$  does not affect the second part of iv). Now we suppose that all the out-leaves of  $T$  belong to  $A \setminus B$ . In the completed graph, consider a path  $P$  from the center  $c$  of  $T$  to a vertex  $b$  of  $B$ . We call  $f$  the last out-leaf of  $T$  on the path  $P$ . We claim that  $C' = (C \setminus \{T\}) \cup \{T'\}$ , where  $T' := T_- \cup T[c, f]$ , is an  $(A, B)$ -completion. Indeed, in the completed graph of  $C'$ , every in-leaf of  $T'$  is still the head of a path with tail in  $B$ . The second part of iv) is still satisfied since we deleted out-leaves of  $T$  which belong to  $A$ . We again contradicts the minimality of  $C$ , therefore the leaves of  $T$  form a subset of  $B \setminus A$ . The proof for the in-leaves follows by directional duality.  $\square$

**Lemma 2** *If  $D = (V, E)$  is a strong digraph and  $A, B$  are two nonempty subsets of  $V$ ,  $D$  admits an  $(A, B)$ -completion.*

**Proof.** We proceed by induction on  $|A|$ . If  $A = \{a\}$ , there exists a spanning out-arborescence  $T$  rooted at  $a$ . Now we consider the set of sub-arborescences  $A_i$  of  $T$  which have no internal vertices in  $A \cup B$  and are maximal with respect to inclusion for this property. This set is clearly an  $(A, B)$ -completion since it satisfies i), ii) and iii) by construction, and its completed graph contains  $T$  as a subgraph, therefore it satisfies iv). Now, we suppose that  $|A| > 1$ , and, for some  $a \in A$ , we apply the induction hypothesis in order to find an  $(A \setminus \{a\}, B)$ -completion  $\{A_1, \dots, A_n\}$ . We assume without loss of generality that this

completion is minimum and denote its completed graph by  $D_c$ . If the vertex  $a$  is a vertex of  $D_c$ , we are done since every vertex  $x$  of  $D_c$  is the head of an  $(x, B)$ -path. Otherwise, we consider a shortest directed  $(a, D_c)$ -path  $P$  in  $D$ . We denote by  $t$  the tail of  $P$ . If  $t \in A \cup B$ , the set  $\{A_1, \dots, A_n, P\}$  is an  $(A, B)$ -completion. If  $t$  is an in-vertex of  $A_i$ , the set  $\{A_1, \dots, A_{i-1}, A_i \cup P, A_{i+1}, \dots, A_n\}$  is an  $(A, B)$ -completion. If  $t$  is an out-vertex of  $A_i$  and  $A_i \cup P$  is a bi-arborescence (with new center  $t$ , this can only happen when  $A_i$  is the union of an in-arborescence and a path which contains  $t$  as an internal vertex),  $\{A_1, \dots, A_{i-1}, A_i \cup P, A_{i+1}, \dots, A_n\}$  is an  $(A, B)$ -completion. If  $t$  is an out-vertex of  $A_i$  and  $A_i \cup P$  is not a bi-arborescence,  $A_i$  has more than one out-leaf, and thus, by Lemma 1 all its out-leaves belong to  $B$ . We denote by  $(A_i)_t$  the sub-out-arborescence of  $A_i$  with root  $t$ , and by  $A'_i$  the bi-arborescence  $A_i \setminus (A_i)_t$ . Finally,  $\{A_1, \dots, A_{i-1}, A'_i, (A_i)_t \cup P, A_{i+1}, \dots, A_n\}$  is an  $(A, B)$ -completion.  $\square$

### 3 Spanning a neighbourhood.

In this part, we show that, given a vertex  $w$  of a strong digraph  $D$ , there exists a  $k$ -handle which spans  $w$  and the neighbours of  $w$ , where  $k$  is at most the stability of the neighbourhood of  $w$ . This result is the core of our proof, we introduce for this the notion of tree-handle system. Given a vertex  $v$  in a digraph  $D$ , we denote by  $N_D^+(v)$  the set of out-neighbours of  $v$  in  $D$ , and by  $N_D^-(v)$  the set of in-neighbours of  $v$  in  $D$ . We write also the  $d_D^+(v) := |N_D^+(v)|$  and  $d_D^-(v) := |N_D^-(v)|$ .

**Theorem 4** *If  $D = (V, E)$  is a strong digraph and  $w$  is a vertex of  $D$ , the set  $\{w\} \cup N_D^+(w) \cup N_D^-(w)$  is contained in a  $p$ -handle  $D'$ , where  $D'$  is a subgraph of  $D$  and  $p \leq \alpha(D[\{w\} \cup N_D^+(w) \cup N_D^-(w)])$ .*

**Proof.** We will simply denote  $N_D^+(w)$  by  $w^+$ ,  $N_D^-(w)$  by  $w^-$  and  $\alpha(D[\{w\} \cup w^+ \cup w^-])$  by  $\alpha$ . We proceed by induction on  $E$ . To simplify a bit, we first treat the case  $w^+ \cap w^- \neq \emptyset$ . Assume for this that a vertex  $v$  belongs to  $w^+ \cap w^-$ , and that strong connectivity is lost when the arc  $vw$  or the arc  $wv$  is deleted (otherwise we simply remove the arc - the stability is unchanged). In this case,  $D$  consists of the union of two strong digraphs  $D_1$  and  $D_2$ , such that  $w \in D_1$  and  $v \in D_2$  and the unique arcs between  $D_1$  and  $D_2$  are  $vw$  and  $wv$ . By the induction hypothesis,  $\{w\} \cup N_{D_1}^+(w) \cup N_{D_1}^-(w)$  is spanned by at most an  $(\alpha - 1)$ -handle, to which we add the handle  $wvw$ .

From now on, we suppose that  $w^+$  and  $w^-$  are disjoint sets. Let  $C := \{B_1, \dots, B_r\}$  be a minimum  $(w^+ \cup \{w\}, w^- \cup \{w\})$ -completion. Observe that the completed graph is strong, therefore, we may suppose that the completed graph is exactly  $D$ , otherwise we apply the induction hypothesis. If one of the  $B_i$  is an out-arborescence, say with root  $r$  and set of leaves  $L$ . We construct a digraph  $D^*$  by removing from  $D$  the internal vertices of  $B_i$  and adding the set  $S$  consisting of all  $(r, L)$ -arcs. Observe that  $B_i$  has at least one internal vertex, otherwise  $\{B_1, \dots, B_r\} \setminus \{B_i\}$  would be a  $(w^+ \cup \{w\}, w^- \cup \{w\})$ -completion since the leaves of  $B_i$  belong to  $\{w\} \cup w^+ \cup w^-$ . Thus, we can apply the induction hypothesis to  $D^*$  and span  $\{w\} \cup w^+ \cup w^-$  by a  $k$ -handle  $H$ , where  $k \leq \alpha(D^*[\{w\} \cup w^+ \cup w^-]) \leq \alpha$ . Since  $H$  is strong and  $C$  is minimum, the set of arcs  $S$  is included in the arc set of  $H$ , thus  $H' := (H \setminus S) \cup B_i$  is a  $k$ -handle and satisfies the conclusion of Theorem 4. We can now assume that every  $B_i$  is a plain bi-arborescence. If for some  $i$ ,  $B_i$  has at least two internal vertices, we can consider instead of  $D$  the digraph  $D^*$  in which all the internal vertices of  $B_i$  are contracted to a single vertex. Again, we apply the induction hypothesis to  $D^*$  to conclude. From now on, we assume that every  $B_i$  is plain and has a single internal vertex  $b_i$ . From Lemma 1, it follows that the out-leaves of  $B_i$  belong to  $w^-$  and the in-leaves of  $B_i$  belong to  $w^+$ .

We introduce the notion of *tree-handle system* of  $D$ . We define it as a set

$$TH = \{W, A_1, A_2, \dots, A_k | P_1, P_2, \dots, P_l\}$$

where  $W$  and  $A_i$ ,  $1 \leq i \leq k$ , are some bi-arborescences whose centers are respectively  $w$  and  $a_i$ , and  $P_j$ ,  $1 \leq j \leq l$ , are some handles (possibly arcs) with the additional conditions:

- i) The sets  $V(W), V(A_1), \dots, V(A_k), V(\overset{\circ}{P}_1), \dots, V(\overset{\circ}{P}_l)$  are pairwise disjoint.
- ii) The digraph  $\bigcup\{W, A_1, A_2, \dots, A_k, P_1, P_2, \dots, P_l\}$  is a spanning subgraph of  $D$ . We call it the *realization* of  $TH$ , and we denote it by  $R$ .
- iii) The head (resp. the tail) of  $P_j$ ,  $1 \leq j \leq l$ , is an out-vertex (resp. an in-vertex) of an  $A_i$  or  $W$ .
- iv) Every vertex  $x$  of  $D$ , except possibly  $w$ , verifies  $d_R^+(x) \geq 1$  and  $d_R^-(x) \geq 1$ .
- v) For all  $i$ ,  $1 \leq i \leq k$ , the out-neighbours (resp. the in-neighbours) of  $a_i$  in  $R$  are in-neighbours (resp. out-neighbours) of  $w$  in  $D$ .

We call  $l$  and  $k$  the *handle index* and the *tree index* of  $TH$ , respectively. Observe that in  $R$ , every vertex  $x$  different from  $w$  is the tail of an  $(w^+, x)$ -path and the head of an  $(x, w^-)$ -path. Thus, by the minimality of the completion  $C$ , every arc of  $B_i$ ,  $1 \leq i \leq r$ , must be an arc of  $R$ . In particular, every center of  $B_i$  is also the center of an  $A_j$ . We will call *special* such a bi-arborescence  $A_j$  (to say it differently,  $A_j$  is special if its center does not belong to  $\{w\} \cup w^+ \cup w^-$ , and, conversely, if a vertex is not in  $\{w\} \cup w^+ \cup w^-$ , it is the center of a special bi-arborescence). Keep in mind that a special bi-arborescence is necessarily plain. Let us prove now that  $D$  admits a tree-handle system:

An *out-fork* is an out-arborescence with height exactly 1 (i.e. consists of one root and a non empty set of leaves), an *in-fork* is defined analogously. We denote by  $X$  the subset of vertices of  $w^+$  which have an out-neighbour in  $w^-$  and by  $Y$  the subset of vertices of  $w^-$  which have an in-neighbour in  $w^+$ . In particular, every vertex of  $X$  has an out-neighbour in  $Y$ , and every vertex of  $Y$  has an in-neighbour in  $X$ . It is routine to check that  $X \cup Y$  is spanned by a disjoint union of out-forks with root in  $X$  and leaves in  $Y$  and in-forks with root in  $Y$  and leaves in  $X$ . We denote by  $F$  this union of forks. Since  $D$  is strong, for every vertex  $y \in w^-$ , there exists an  $(u, y)$ -path in  $D[w^-]$  with  $u \in Y$  or  $u$  is an out-leaf of some  $B_i$ . Equivalently, there exists a disjoint union  $O$  of out-arborescences with set of roots  $Y \cup \{b_i : i = 1, \dots, r\}$  and set of vertices  $w^- \cup \{b_i : i = 1, \dots, r\}$ . By a similar argument, there exists a disjoint union  $I$  of in-arborescences with set of roots  $X \cup \{b_i : i = 1, \dots, r\}$  and set of vertices  $w^+ \cup \{b_i : i = 1, \dots, r\}$ . Now  $F \cup O \cup I$  is a disjoint union of bi-arborescences whose centers are the roots of the forks of  $F$  and the  $\{b_i : i = 1, \dots, r\}$ . We denote by  $A_1, \dots, A_k$  the subset of these bi-arborescences whose center is not a leaf. We denote by  $A'_1, \dots, A'_p$  the bi-arborescences whose center is a leaf. Every in-leaf of  $A_i$  or  $A'_j$  is in  $w^+$ , and every out-leaf of  $A_i$  or  $A'_j$  is in  $w^-$ . We denote by  $IL$  the set of  $wl$  arcs where  $l$  is an in-leaf of  $A_i$  and by  $OL$  the set of  $lw$  arcs where  $l$  is an out-leaf of  $A_i$ , for all  $i = 1, \dots, k$ . We also denote by  $IL'$  the set of  $wl$  arcs where  $l$  is an in-leaf but not the center of  $A'_j$  and by  $OL'$  the set of  $lw$  arcs where  $l$  is an out-leaf but not the center of  $A'_j$ , for all  $j = 1, \dots, p$ . Finally, we denote by  $CI$  the set of  $wl$  arcs where  $l$  is an in-leaf and the center of  $A'_j$  and  $CO$  the set of  $lw$  arcs where  $l$  is an out-leaf and the center of  $A'_j$ . Observe that  $W := \{w\} \cup A'_1 \cup \dots \cup A'_p \cup CI \cup CO$  is a bi-arborescence with center  $w$ . We have the tree-handle system  $TH = \{W, A_1, \dots, A_k | IL \cup OL \cup IL' \cup OL'\}$ , all the handles of which are arcs.

Our next goal is to prove that there exists a tree-handle system with handle index at most  $\alpha$ . We consider for this a tree-handle system  $TH = \{W, A_1, A_2, \dots, A_k | P_1, P_2, \dots, P_l\}$  which satisfies the following conditions:

- a)  $l$  is minimum.
- b) Subject to a),  $k$  is minimum.
- c) Subject to a) and b),  $|V(P_1)| + |V(P_2)| + \dots + |V(P_l)|$  is maximum.

Let us prove that  $TH$  is *complete*, that is, it verifies the property:

vi) Except in the case  $x = w$ , every out-leaf  $x$  of  $A_i$  or  $W$  satisfies  $d_R^+(x) \geq 2$  and every in-leaf  $x$  of  $A_i$  or  $W$  satisfies  $d_R^-(x) \geq 2$ . In other words,  $x$  is the head or the tail of at least two handles of  $TH$ .

Consider for this an out-leaf  $x$  of a bi-arborescence of  $TH$  and assume that  $x$  is the head of a unique handle  $P$  of  $TH$ . Without loss of generality, we can suppose that  $P = P_1$ . We consider several cases:

1. Assume that  $x$  belongs to the bi-arborescence  $W$ . If  $x = w$ , the condition vi) holds vacuously. If  $x \neq w$ ,  $x$  has an in-neighbour  $x'$  in  $W$  which is an out-vertex of  $W$ . In this case, we extend  $P$  with  $x'$ , i.e. we consider  $TH' = \{W \setminus \{x\}, A_1, A_2, \dots, A_k | x'x \cup P_1, P_2, \dots, P_l\}$ . Note that the realization of  $TH$  is exactly  $\bigcup TH'$ , thus  $TH'$  is still a tree-handle system. However the total length of the handles of  $TH'$  has increased, a contradiction to the condition c).
2. Now, assume that  $x$  belongs to a bi-arborescence  $A_i$  for some  $i$ , say  $A_1$ . If  $x \neq a_1$ , we conclude as previously in the case  $x \neq w$ . If  $x = a_1$ , we denote by  $t$  the tail of  $P_1$ :

- The simplest case arises when  $t$  does not belong to  $A_1$ , say  $t \in A_2$ . Note that the bi-arborescence  $A_1$  has no out-vertex except  $a_1$  and that  $a_1$  is the head of the unique handle  $P_1$ . Consider  $TH' = \{W, A_1 \cup P_1 \cup A_2, A_3, \dots, A_k | P_2, \dots, P_l\}$  and observe that  $TH$  and  $TH'$  have the same realization which implies, as previously, that  $TH'$  is a tree-handle system. However, the handle index has strictly decreased and this contradicts the condition a). The same argument holds if  $t \in A_i$ ,  $3 \leq i \leq k$  or if  $t \in W$ .

- Suppose now that  $t$  belongs to  $A_1$ , in particular  $t$  is an in-vertex of  $A_1$ . Again, we modify  $TH$  in order to find a contradiction. There exists a path  $Q$  from  $t$  to  $a_1$  in  $A_1$ . The union  $Q \cup P_1$  forms a cycle  $C$  in  $R$  which contains  $a_1$ . Note that, by the property v) in the definition of tree-handle systems, the in-neighbour and the out-neighbour of  $a_1$  in  $C$  respectively belong to  $w^+$  and to  $w^-$ . Since  $w^+ \cap w^- = \emptyset$ , it follows that  $C$  has at least three vertices, all of these apart possibly  $a_1$  being neighbours of  $w$ . Indeed, the vertices of  $D \setminus (\{w\} \cup w^+ \cup w^-)$  are the centers of special bi-arborescences, so  $C \setminus \{a_1\} \subseteq w^- \cup w^+$ . Thus, we can exhibit two vertices  $y$  and  $z$  in  $C \setminus \{a_1\}$  such that  $yz$  is an arc of  $C$ ,  $y \in w^-$  and  $z \in w^+$ . To conclude, observe that  $A = (A_1 \cup P_1 \cup yw) \setminus yz$  is an in-arborescence rooted at  $w$  and that  $wz$  forms a handle from  $W$  to  $A$ . Now, we consider  $TH' = \{W \cup A, A_2, \dots, A_k | wz, P_2, \dots, P_k\}$  and check that  $TH'$  is a tree-handle system. Conditions i) and ii) clearly hold. The unique added handle is the arc  $wz$ , since  $w$  is an out-vertex of  $W \cup A$  and  $z$  is an in-vertex of  $W \cup A$ , the property iii) is satisfied. To check property iv), observe that  $z$  is the unique leaf possibly created by our modifications, and in this case  $z$  is an in-leaf of  $W \cup A$  and the tail of the handle  $wz$ . Finally, we have not created new arborescence, which implies that property v) still holds. Consequently,  $TH'$  is a tree-handle system with the same handle index than  $TH$  but with lower tree index, a contradiction to the condition b).

We proceed similarly if  $x$  is an in-leaf of a bi-arborescence. Since all these cases give a contradiction,  $TH = \{W, A_1, A_2, \dots, A_k | P_1, P_2, \dots, P_l\}$  is a complete tree-handle system. Now we want to achieve our bound, that is we want to prove that  $l$ , the handle index of  $TH$ , is at most  $\alpha$ . First observe that if one of the handles  $P_j$  is an arc, we can simply remove it from  $TH$  and still have a tree-handle system: the reason for this is simply that removing  $P_j$  cannot harm the condition iv) in the definition of tree-handle system since  $TH$  is complete. By minimality of  $l$ , all the handles have length at least 2. Suppose for contradiction that  $l > \alpha$ . Since  $\mathcal{P}$  is a subset of  $w^+ \cup w^-$ , its stability is at most  $\alpha$ , therefore we can apply Theorem 3 to the set of disjoint paths  $\mathcal{P} := \{\overset{\circ}{P}_1, \overset{\circ}{P}_2, \dots, \overset{\circ}{P}_l\}$ , in order to get a set of disjoint paths  $\mathcal{P}' := \{P'_1, P'_2, \dots, P'_{l-1}\}$ . Since the head of  $P'_i$  is the head of some  $\overset{\circ}{P}_a$  and the tail of  $P'_i$  is the tail of some

$\overset{\circ}{P}_b$ , the path  $P'_i$  extends naturally to a handle  $H_i := hP'_i t$  where  $h$  is the head of  $P_a$  and  $t$  is the tail of  $P_b$ . Let us show that  $TH' := \{W, A_1, A_2, \dots, A_k | H_1, H_2, \dots, H_{l-1}\}$  is a tree-handle system. Conditions i) and ii) are still satisfied. Since the sets of heads and tails of  $\mathcal{P}'$  are subsets of the sets of heads and tails of  $\mathcal{P}$ , the condition iii) holds for  $TH'$ . Since  $TH$  is complete and exactly one head and one tail of  $\mathcal{P}$  are lost, the condition iv) holds. Finally, no new out or in-neighbours of any  $a_i$  is created in  $TH'$ , so the condition v) still holds. Thus  $TH'$  has handle index  $l - 1$ , a contradiction to the condition a). Consequently, the handle index  $l$  of  $TH$  is at most  $\alpha$ . Our last step is to span  $D$  with an  $l$ -handle. By minimality, we recall that every handle of TH is non trivial (i.e. has length at least 2).

Consider for this a subgraph  $D'$  of  $D$  with vertex set  $V'$  and arc set  $E'$ , which is maximal with respect to  $|V'|$  and verifies the following conditions:

- I) For some  $p \in \{0, \dots, l\}$ ,  $D'$  is a  $p$ -handle and contains the vertices of at least  $p$  handles of  $TH$ .
- II) For all  $j = 1, \dots, l$ , either  $V(\overset{\circ}{P}_j) \cap V' = \emptyset$  or  $V(\overset{\circ}{P}_j) \subseteq V'$ .
- III) For all  $i = 1, \dots, k$ , either  $V(A_i) \cap V' = \emptyset$  or  $D'[V(A_i) \cap V']$  is a sub-bi-arborescence of  $A_i$  which contains  $a_i$ . Moreover  $D'[V(W) \cap V']$  is a sub-bi-arborescence of  $W$  which contains  $w$ .

Since the singleton digraph  $\{w\}$  satisfies I),II) and III), such a  $D'$  exists. We prove that  $D'$  necessarily spans  $D$ , and thus achieve our goal:

- Let us assume that there exists a  $(V')$ -handle  $H$  in  $R$  which is not an arc. We denote its head by  $h$  and its tail by  $t$ . By the condition III),  $H$  is not contained in a bi-arborescence  $A_i$  or  $W$ . Therefore  $H$  contains an internal vertex of some handle  $P$  of  $TH$ . By the condition II), it follows that  $P$  is contained in  $H$ . Thus,  $D' \cup H$  contains at least one handle of  $TH$  which is not in  $D'$ , in particular  $D' \cup H$  satisfies the condition I). If  $\overset{\circ}{H}$  contains an internal vertex  $v$  of some handle  $P_v$ , the whole handle  $P_v$  is contained in  $D' \cup H$ , so the condition II) is also satisfied. To check that condition III) is still satisfied, suppose that  $\overset{\circ}{H}$  contains a vertex of some bi-arborescence  $A_i$  which is disjoint from  $D'$ . Denote by  $a$  the first vertex of  $\overset{\circ}{H} \cap A_i$  along  $H$  and by  $b$  the last one. Since  $H$  is included in  $R$ ,  $a$  is an in-vertex of  $A_i$ ,  $b$  is an out-vertex of  $A_i$ , and  $H[a, b]$  is included in  $A_i$ , and thus forms a sub-bi-arborescence which contains  $a_i$ . Now if  $\overset{\circ}{H}$  contains a vertex  $v$  of some bi-arborescence  $A_i$  (resp.  $W$ ) which meets  $D'$ , assume without loss of generality that  $v$  is an in-vertex of  $A_i$  (resp.  $W$ ). Since  $H$  is included in  $R$  and  $a_i \in D'$  (resp.  $w \in D'$ ), the path  $H[v, t]$  is included in the set of in-vertices of  $A_i$  (resp.  $W$ ). In both of these two cases,  $D' \cup H$  satisfies the condition III).

- If there is no  $(V')$ -handle in  $R$  and  $V \neq V'$ , since both the in and the out-degree in  $R$  of any vertex different from  $w$  are greater than one,  $R$  contains a cycle  $C$  which is disjoint from  $D'$ . The cycle  $C$  contains the center  $a_i$  of a bi-arborescence  $A_i$  of  $TH$ . Let us denote by  $a_j$  the next center of an  $A_j$  on  $C$ , that is, no internal vertex of  $C[a_i, a_j]$  is the center of some bi-arborescence of  $TH$ . If  $a_i$  is the unique center which is contained in  $C$ , we simply choose  $a_j := a_i$ . Since  $C[a_i, a_j]$  contains a handle of  $TH$ , it has length at least two. By the property v) of a tree-handle system, the out-neighbour  $a_i^+$  of  $a_i$  in  $C$  belongs to  $w^-$ , and the in-neighbour  $a_j^-$  of  $a_j$  in  $C$  belongs to  $w^+$ . Since  $w^+$  and  $w^-$  are disjoint sets,  $a_i^+ \neq a_j^-$ . In particular, we can find in  $C[a_i^+, a_j^-]$  two consecutive vertices  $x$  and  $y$  such that  $x \in w^-$  and  $y \in w^+$ . The subgraph  $D' \cup wy \cup C[y, x] \cup xw$  of  $D$  contains at least one handle of  $TH$  which is not in  $D'$  (indeed  $C[a_i, a_j]$  contains exactly one handle of  $TH$ ), in particular it satisfies the condition I). The conditions II) and III) are also easily verified.

So,  $V = V'$  and hence,  $D'$  is a  $p$ -handle which spans  $D$ , with  $p \leq l \leq \alpha$ .  $\square$



## 4 The main theorem. The algorithmic aspect.

Finally, we prove Theorem 1, which is an easy corollary of the previous result.

**Proof of Theorem 1.** Let us fix a vertex  $w_0$  of  $D$ . According to Theorem 4, we can cover  $\{w_0\} \cup N_D^+(w_0) \cup N_D^-(w_0)$  by a  $k_1$ -handle  $H_1$  with  $k_1 \leq \alpha(D[N_D^+(w_0) \cup N_D^-(w_0)])$ . We contract this  $k_1$ -handle to form a digraph  $D_1$  and call  $w_1$  the contracted vertex. We again apply Theorem 4, and cover  $\{w_1\} \cup N_{D_1}^+(w_1) \cup N_{D_1}^-(w_1)$  by a  $k_2$ -handle  $H_2$  with  $k_2 \leq \alpha(D_1[N_{D_1}^+(w_1) \cup N_{D_1}^-(w_1)])$ .

Perform these contractions until only one vertex  $w_p$  remains. For  $l = 1, \dots, p$ , we denote by  $V_l$  the set of vertices of  $D$  contracted to  $w_l$  and which were not contracted to  $w_{l-1}$ , observe that the stability of  $D[V_l]$ , denoted by  $\alpha_l$ , is greater or equal to  $k_l$ . Moreover, if an arc of  $D$  has its endvertices in  $V_i$  and  $V_j$ , we clearly have  $|i - j| \leq 1$ . Consequently,  $1 + \alpha_2 + \alpha_4 + \dots \leq \alpha(D)$  and  $\alpha_1 + \alpha_3 + \alpha_5 + \dots \leq \alpha(D)$ . Now, let  $D'_p := \{w_p\}$  and, starting with  $j := p$ , inductively replace  $w_j$  in  $D'_j$  by the  $k_j$ -handle  $H_j$  to form the digraph  $D'_{j-1}$ . The spanning subgraph  $D'_0$  of  $D$  is a  $k$ -handle where  $k$  is the sum of the  $k_j$ . Moreover,  $k \leq \alpha_1 + \alpha_2 + \dots + \alpha_p \leq 2\alpha - 1$ .  $\square$

To conclude this paper, we invite the reader to check that an algorithm can easily be derived from our proof. The calculation of a completion in which every arc is necessary can be done in polynomial time. The reduction of a tree-handle system can be performed in  $O(|V|)$ , and the path-exchange of Theorem 3 can be calculated in  $O(|E|)$ . From this, the calculation of a  $(2\alpha - 1)$ -handle which spans  $D$  can be done in polynomial time. Although the calculation of the minimum  $k$  for which a strong digraph  $D$  admits a spanning  $k$ -handle cannot be approximated up to any fixed factor (we leave this as an exercise for the reader), the best known bound (see [12]) is the following: there exists an algorithm which calculates a spanning  $k$ -handle of a digraph  $D$  where  $(n + k - 1)/(n + l - 1) \leq 3/2$ , where  $l$  is the minimum value for an  $l$ -handle spanning  $D$ . Our approach gives a better bound for dense graphs, that is when  $\alpha < n/4$ .

## References

- [1] J. Bang-Jensen and G. Gutin, Digraphs. Theory, algorithms and applications, *Springer Monographs in Mathematics*, Springer-Verlag London, 2001.
- [2] J.A. Bondy, Basic graph theory: paths and circuits. *Handbook of combinatorics*, Vol. 1, 2, 3–110, Elsevier, Amsterdam, 1995.
- [3] J.A. Bondy, A short proof of the Chen-Manalastas theorem, *Discrete Maths*, **146** (1995), 289–292.
- [4] J.A. Bondy, Personal communication.
- [5] C.C. Chen and P. Manalastas, Every finite strongly connected digraph of stability 2 has a Hamiltonian path, *Discrete Maths*, **44** (1983), 243–250.
- [6] T. Gallai, Problem 15, in: M. Fiedler, ed., *Theory of Graphs and its Applications* (Czech. Acad. Sci. Prague, 1964) 161.
- [7] G. Gutin, M. Tewes and A. Yeo, Longest paths in strong spanning oriented subgraphs of strong semicomplete multipartite digraphs, *Discrete Math.*, **222** (2000), 269–274.
- [8] S. Khuller, B. Raghavachari and N. Young Approximating the minimum equivalent digraph, *SIAM J. Comput.*, **24** (1995), 859–872.

- [9] S. Thomassé, Covering a strong digraph by  $\alpha - 1$  disjoint paths, a proof of Las Vergnas' conjecture, *J. Combin. Theory Ser. B*, **83** (2001), 331–333.
- [10] S. Thomassé, Spanning a strong digraph by  $\alpha$  cycles, the case  $\alpha = 3$  of Gallai's conjecture, to appear in *Discrete Mathematics*.
- [11] C. Thomassen, Directed cycles with two chords and strong spanning directed subgraphs with few arcs, *J. Combin. Theory Ser. B*, **66** (1996), 24–33.
- [12] A. Vetta, Approximating the minimum strongly connected subgraph via a matching lower bound, preprint.