Orientations of Simplices Determined by Orderings on the Coordinates of their Vertices

Emeric Gioan, Kevin Sol, Gérard Subsol

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Emeric Gioan[1] Kevin Sol[2] Gérard Subsol[1]

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Abstract

Provided n points in an $(n-1)$-dimensional affine space, and one ordering of the points for each coordinate, we address the problem of testing whether these orderings determine if the points are the vertices of a simplex (i.e. are affinely independent), regardless of the real values of the coordinates. We also attempt to determine the orientation of this simplex. In other words, given a matrix whose columns correspond to affine points, we want to know when the sign (or the non-nullity) of its determinant is implied by orderings given to each row for the values of the row. We completely solve the problem in dimensions 2 and 3. We provide a direct combinatorial characterization, along with a formal calculus method. It can also be viewed as a decision algorithm, and is based on testing the existence of a suitable inductive cofactor expansion of the determinant. We conjecture that our method generalizes in higher dimensions. This work aims to be part of a study on how oriented matroids encode shapes of 3-dimensional landmark-based objects. Specifically, applications include the analysis of anatomical data for physical anthropology and clinical research.

Keywords: simplex orientation, determinant sign, chirotope, coordinate ordering, combinatorial algorithm, formal calculus, oriented matroid, 3D model, 3D landmark-based morphometry.

AMS classification: 15A03, 15A15, 15B35, 52C40

1 Introduction

We consider n points in an $(n-1)$-dimensional real affine space. For each of the $n-1$ coordinates, an ordering is given and applied to the n values of the points with respect to this coordinate. We address the problem of testing if these points are the vertices of a simplex (i.e. are affinely independent, i.e. do not belong to a same hyperplane), and of determining the orientation of this simplex, assuming that the coordinates of the points satisfy the given orderings, independently of their real values.

More formally, we consider the following generic matrix (where each e_i is the label of a point, forming

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[1]CNRS, LIRMM, Université de Montpellier, France. *Email:* {lastname}@lirmm.fr

[2]This research was done when Kevin Sol was Ph.D. at the LIRMM (research teams AlGCo and ICAR), Université de Montpellier, France

the set $\mathcal{E}$, and each b_i is the index of a coordinate, forming the set $\mathcal{B}$)

$$M_{\mathcal{E},\mathcal{B}} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_{e_1,b_1} & x_{e_2,b_1} & \dots & x_{e_n,b_1} \\ x_{e_1,b_2} & x_{e_2,b_2} & \dots & x_{e_n,b_2} \\ \vdots & \vdots & & \vdots \\ x_{e_1,b_{n-1}} & x_{e_2,b_{n-1}} & \dots & x_{e_n,b_{n-1}} \end{pmatrix}$$

together with orderings given to the values for each row; we want to know when the sign (or the non-nullity) of the matrix determinant is determined by these orderings only.

Equivalently, we consider the above formal matrix and the affine algebraic variety of $\mathbb{R}^{n\times(n-1)}$ whose equation is $det(M_{\mathcal{E},\mathcal{B}}) = 0$. Then, among (open) regions of $\mathbb{R}^{n\times(n-1)}$ delimited by the hyperplanes $x_{e_i,b_k} = x_{e_j,b_k}$ for all $1 \leq i, j \leq n$ and all $1 \leq k \leq n-1$, we attempt to identify those having a non-empty intersection with this variety (obviously, regions delimited by these hyperplanes are in canonical bijection with coordinate linear orderings).

We completely solve the problem in dimensions 2 (Section 4.1) and 3 (Section 4.2). We provide a direct combinatorial characterization to test if the orientation is determined or not, along with a combinatorial formal calculus method which can also be viewed as a decision algorithm. More precisely, our method is based on testing the existence of a suitable inductive cofactor expansion of the determinant, which allows the computation of the determinant sign using a combinatorial formal calculus. We conjecture that this formal characterization generalizes in higher dimensions (Section 3).

Interestingly, the problem addressed here is formally close to the classical problem of sign nonsingular matrices (SNS), see [3]. However, the two problems are rather separate. Let us explain this briefly. The common feature of the two problems relies in the following situation. Consider a square $n \times n$ matrix N whose entries are signs in $\{+, -\}$. From our setting, such an $n \times n$ matrix N can be obtained naturally from $M_{\mathcal{E},\mathcal{B}}$ and a linear ordering for each row by subtracting a column - say e_i - from every other column, and replacing entries with their signs w.r.t. the linear orderings. Thus, reciprocally, the sign data in such a matrix N is interpreted in our setting as an ordering relation for each row of type: (the set of $-$) $<$ (the set of $+$), corresponding to: (a set $A \subset \mathcal{E}$) $< e_i <$ (a set $B \subset \mathcal{E}$). The question is: assuming real values are assigned to entries of the matrix N, such that these values have the same signs as the signs in the matrix, is this matrix always invertible? To this particular question, the answer is always NO, whatever the signs, unless $n \leq 2$ (see [3, page 108]: an SNS-matrix of order $n \geq 3$ has at least one zero entry; see also Remark 1). However, in more general settings, the answer can be YES. The SNS setting and ours consist in two different variants of the above question. They both yield a non-trivial question and provide interesting classes of sign patterns. In the SNS setting, the variant is to consider the same question with signs in $\{+, -, 0\}$ instead of $\{+, -\}$. In our setting, we specify the question keeping signs in $\{+, -\}$ while restricting the available real values to values satisfying more involved ordering relations for each row (between all elements and not only between two subsets A and B). There seems to be no obvious connection between the two problems. Indeed, the zeros in the SNS setting and the linear orderings in ours place significantly different constraints upon the sets of real values to be tested.

Finally, we point out that this work aims to be part of a general study on how oriented matroids [1] encode shapes of 3-dimensional landmark-based objects. Specifically, applications include the analysis of anatomical data for physical anthropology and clinical research [7][8]. In these applications, we usually study a set of models belonging to a given group (e.g. sets of 3D landmark points located on human or primate skulls) and we search for the significant properties encoded by the combinatorial structure. Our proposed solution allows us to distinguish chirotopes (i.e. simplex orientations) which are determined by the model's "generic" form (e.g. in any skull, the mouth is below the eyes) from those which are specific to anatomical variations. Examples of 3D anatomical data results are presented in Sections 2.2 and 4.3.

2 Preliminaries

We warn the reader that we purposely use rather abstract formalism throughout the paper (formal variables instead of real values, indices within arbitrary ordered sets instead of integers). This will allow us to get simpler and non-ambiguous constructions and definitions.

2.1 Formalism and terminology of the problem

Let us fix an (ordered) set $\mathcal{E} = \{e_1, \ldots, e_n\}$, with size n, of *labels*, and an (ordered) canonical basis $\mathcal{B} = \{b_1, \ldots, b_{n-1}\}$, with size $n-1$, of the $(n-1)$-dimensional real space $\mathbb{R}^{n-1}$. We denote $M_{\mathcal{E},\mathcal{B}}$ - or M for short when the context is clear - the formal matrix

$$M_{\mathcal{E},\mathcal{B}} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_{e_1,b_1} & x_{e_2,b_1} & \cdots & x_{e_n,b_1} \\ x_{e_1,b_2} & x_{e_2,b_2} & \cdots & x_{e_n,b_2} \\ \vdots & \vdots & & \vdots \\ x_{e_1,b_{n-1}} & x_{e_2,b_{n-1}} & \ldots & x_{e_n,b_{n-1}} \end{pmatrix}$$

whose entry in column i and row $j+1$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, is the formal variable x_{e_i,b_j}. The determinant $det(M_{\mathcal{E},\mathcal{B}})$ of this formal matrix is a multivariate polynomial in these formal variables, and the main object studied in this paper.

Let $\mathcal{P}$ be a set of n points, labeled by $\mathcal{E}$, in $\mathbb{R}^{n-1}$ considered as an affine space. We denote $M_{\mathcal{E},\mathcal{B}}(\mathcal{P})$ - or $M(\mathcal{P})$ for short - the matrix whose columns give the coordinates of points in $\mathcal{P}$ w.r.t. the basis $\mathcal{B}$. This comes down to specifying real values for the formal variables x_{e_i,b_j} in the matrix $M_{\mathcal{E},\mathcal{B}}$ above. For $e \in \mathcal{E}$ and $b \in \mathcal{B}$, we denote $x_{e,b}(\mathcal{P})$ the real value given to the formal variable $x_{e,b}$ in $\mathcal{P}$. We may sometimes denote $x_{e,b}$ for short instead of $x_{e,b}(\mathcal{P})$ when the context is clear. We call *orientation* of $\mathcal{P}$, or *chirotope* of $\mathcal{P}$ in the oriented matroid terminology, the sign of $det(M(\mathcal{P}))$, belonging to the set $\{+, -, 0\}$. It is the sign of the real evaluation of the polynomial $det(M)$ at the real values given by $\mathcal{P}$. This sign is not equal to zero if and only if $\mathcal{P}$ forms a *simplex* (basis of the affine space).

We call *ordering configuration on* $(\mathcal{E}, \mathcal{B})$ - or *configuration* for short - a list $\mathcal{C}$ of $n-1$ orderings $<_{b_1}, \ldots, <_{b_{n-1}}$ on $\mathcal{E}$, with one ordering for each element of $\mathcal{B}$. In general, such an ordering can be any partial ordering. If every ordering $<_b$, $b \in \mathcal{B}$, is linear, then $\mathcal{C}$ is called a *linear ordering configuration*. An element of $\mathcal{E}$ which is the smallest or the greatest in a linear ordering on $\mathcal{E}$ is called *extreme* in this ordering. We call *reversion* of an ordering the ordering obtained by reversing every inequality in this ordering.

Given a configuration $\mathcal{C}$ on $(\mathcal{E}, \mathcal{B})$ and a set of n points $\mathcal{P}$ labeled by $\mathcal{E}$, we say that $\mathcal{P}$ *satisfies* $\mathcal{C}$ if, for all $b \in \mathcal{B}$, the natural order (in the set of real numbers $\mathbb{R}$) of the coordinates b of the points in $\mathcal{P}$ is compatible with the ordering $<_b$ of $\mathcal{C}$, that is precisely :

$$\forall b \in \mathcal{B},\ \forall e, f \in \mathcal{E},\ e <_b f \ \Rightarrow\ x_{e,b}(\mathcal{P}) < x_{f,b}(\mathcal{P}).$$

One may observe that the set of all $\mathcal{P}$ satisfying $\mathcal{C}$ forms a convex polyhedron, or more precisely: a (full dimensional) region of the space $\mathbb{R}^{n\times(n-1)}$, delimited by hyperplanes of equations of type $x_{e,b} = x_{f,b}$ for $b \in \mathcal{B}$ and $e, f \in \mathcal{E}$.

We say that a configuration $\mathcal{C}$ is *fixed* if all the sets of points $\mathcal{P}$ satisfying $\mathcal{C}$ form a simplex and have the same orientation. In this case, the sign of $det(M(\mathcal{P}))$ is the same for all $\mathcal{P}$ satisfying $\mathcal{C}$. Then we call *sign of* $det(M)$ this sign, belonging to $\{\boxed{+}, \boxed{-}\}$ accordingly, and we denote it $\sigma_{\mathcal{C}}(det(M))$. If $\mathcal{C}$ is *non-fixed*, then its *sign* is $\sigma_{\mathcal{C}}(det(M)) = \boxed{\pm}$.

Lemma 1. *The following propositions are equivalent:*

(a) The configuration $\mathcal{C}$ is non-fixed, that is $\sigma_{\mathcal{C}}(det(M)) = \boxed{\pm}$.

(b) There exist two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfying $\mathcal{C}$ and forming simplices that do not have the same orientation, that is $det(M(\mathcal{P}_1)) > 0$ and $det(M(\mathcal{P}_2)) < 0$;

(c) There exists a set of points $\mathcal{P}$ satisfying $\mathcal{C}$ and such that the points of $\mathcal{P}$ belong to one hyperplane, that is $det(M(\mathcal{P})) = 0$.

Proof. By definition we have a) if and only if b) or c) is true. The region of the space $\mathbb{R}^{n\times(n-1)}$ whose elements $\mathcal{P}$ satisfy $\mathcal{C}$ is a convex and, topologically, an open set of points in $\mathbb{R}^{n\times(n-1)}$. So b) implies c) by convexity and continuity of the determinant. Moreover, c) implies b) since, given $\mathcal{P}$ in this region such that $det(M(\mathcal{P})) = 0$, one can add a matrix small enough to $M(\mathcal{P})$ to get $\mathcal{P}'$ in the same region and such that $det(M(\mathcal{P}')) > 0$, or also such that $det(M(\mathcal{P}')) < 0$. □

Two configurations on $(\mathcal{E}, \mathcal{B})$ are called *equivalent* if they are equal up to a permutation of $\mathcal{B}$, a permutation of $\mathcal{E}$ (relabelling), and some reversions of orderings (geometrical symmetries). Note that, in a matricial setting, changing a configuration into an equivalent one comes down to changing the orderings of rows and columns, and to multiplying some rows by -1. Obviously, those operations do not change the non-nullity of the determinant. Hence, two equivalent configurations are fixed or non-fixed simultaneously.

Now, given an ordering configuration $\mathcal{C}$, the aim of the paper is to determine if $\mathcal{C}$ is fixed or non-fixed.

2.2 An application example

Let us consider ten anatomical landmark points in $\mathbb{R}^3$ chosen by experts on the 3D model of a skull from [2], as shown in Figure 1. We choose a canonical basis $(O, \vec{x}, \vec{y}, \vec{z})$ such that the axis $\vec{x}$ goes from the right of the skull to its left, the axis $\vec{y}$ goes from the bottom of the skull to its top, and the axis $\vec{z}$ goes from the front of the skull to its back. This 3D model has the specificity of being a skull, which implies that some coordinate ordering relations are satisfied by those points: for instance the point 9 (right internal ear) will always be on the right, above and behind with respect to point 5 (right part of the chin). Figures 2 and 3 show those points respectively from the front and from the right of the model, with a grid representing those coordinate ordering relations.

For application purpose (e.g. in [7][8]), we are given such models, coming from various individuals (with possible pathologies) and species (e.g. primates and humans), by physical anthropology and clinical research expertswho are interested in mathematically characterizing and classifying them. In this paper, our aim is to detect which configurations are fixed independently of the real values of the landmarks. These particular configurations are interesting to detect: they mean that the corresponding relative positions of points do not depend on some anatomical variabilities (e.g. on being a primate or a human skull), but only on the generic shape of the model (i.e. on being a skull).

The ordering configurations are represented in Figures 2 and 3, with $\mathcal{E}$ being any set of four points, and $\mathcal{B}$ corresponding to the three axis $\{x, y, z\}$. As a preliminary example, let us consider the relations between

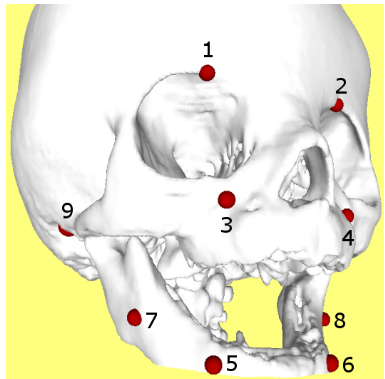


Figure 1: Ten anatomic points on a skull model [2]

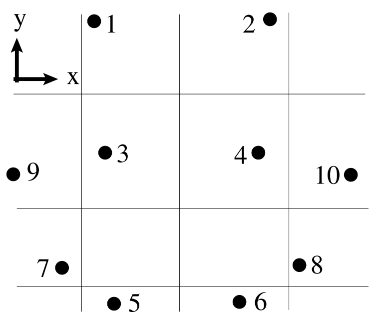


Figure 2: View from the front

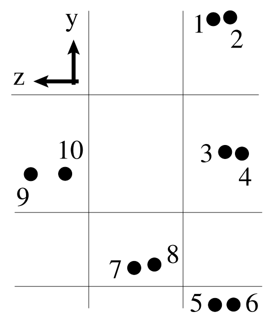


Figure 3: View from the right

points labelled by $\mathcal{E} = \{1, 5, 9, 10\}$; we get the following configuration $\mathcal{C}$, as illustrated in Figures 4, 5 and 6:

$$9 <_x 1 <_x 10 \quad \text{and} \quad 9 <_x 5 <_x 10$$
$$5 <_y 9 <_y 1 \quad \text{and} \quad 5 <_y 10 <_y 1$$
$$1 <_z 9 \quad \text{and} \quad 1 <_z 10 \quad \text{and} \quad 5 <_z 9 \quad \text{and} \quad 5 <_z 10$$

Proving that this configuration is fixed can be seen as a geometry exercise. The sketch is the following. Let us prove that line $(1, 5)$ and line $(9, 10)$ cannot intersect: this implies that the points cannot belong to a same hyperplane, and hence form a simplex with fixed orientation. Consider two planes α_1, α_2 parallel to the directions y, z, two planes β_1, β_2 parallel to the directions x, z, and one plane γ parallel to the directions x, y, consistent with the coordinate orderings, as shown in Figures 4, 5 and 6. These 5 planes separate $\mathbb{R}^3$ in 18 regions. Consider the plane δ containing the intersection of planes $\alpha_1, \beta_2, \gamma$, containing the intersections of planes $\alpha_2, \beta_1, \gamma$, and parallel to the direction z, as shown in Figure 4. Consider a region (among the 18 regions) intersecting δ. Prove that, if line $(1, 5)$ and line $(9, 10)$ both intersect this region, then the two intersections are contained in two distinct parts of this region separated by δ, meaning that the two lines do not intersect. The other cases (other regions) are either symmetric to this one or trivial.

In the rest of the paper, we develop tools to automatically detect fixed configurations, without having to use specific geometric constructions for each configuration as done above. Instead, our approach consists in unifying all configurations under a common combinatorial criterion. We will continue to study this example using this approach in Section 4.3.

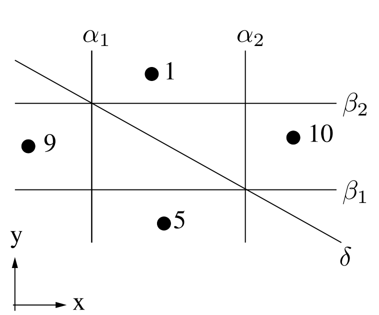


Figure 4: View from the front

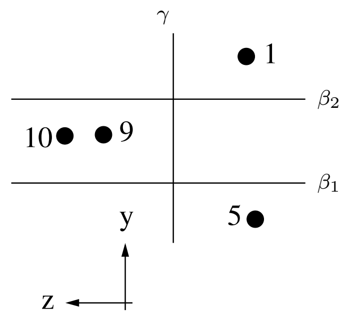


Figure 5: View from the left

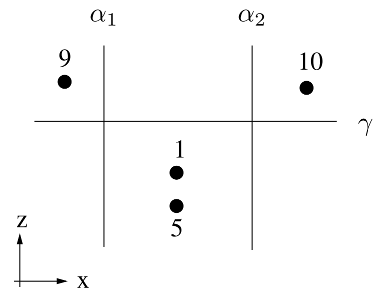


Figure 6: View from below

3 Computable fixity criteria and conjectures

3.1 From partial orderings to linear orderings

We recall that a linear extension of an ordering on a set $\mathcal{E}$ is a linear ordering on $\mathcal{E}$ compatible with this ordering. A *linear extension* of an ordering configuration $\mathcal{C}$ on $(\mathcal{E}, \mathcal{B})$ is a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ obtained by replacing each ordering on $\mathcal{E}$ in C by one of its linear extensions.

Lemma 2. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$. If there exists a set $\mathcal{P}$ of n points satisfying $\mathcal{C}$ and contained in a hyperplane, then there exists a set of n points $\mathcal{P}'$ contained in a hyperplane and a linear extension $\mathcal{C}'$ of $\mathcal{C}$ satisfied by $\mathcal{P}'$.*

Proof. Assume that, for every row of $M(\mathcal{P})$ except the first, all entries in this row are distinct. Then the linear orderings of these real values in each row define a linear ordering configuration $\mathcal{C}'$. This configuration $\mathcal{C}'$ is a linear extension of $\mathcal{C}$ since $\mathcal{P}$ satisfies $\mathcal{C}$. Then $\mathcal{P}' = \mathcal{P}$ and $\mathcal{C}'$ have the required properties.

Assume two columns with labels $e, f \in \mathcal{E}$ of $M(\mathcal{P})$ are equal. Let $v = (v_1, \ldots, v_{n-1})$ be a vector parallel to the (affine) hyperplane containing $\mathcal{P}$. Let $\mathcal{P}'$ be obtained by adding $\varepsilon.(0, v_1, \ldots, v_{n-1})$ to coordinates of the point labelled by f, for some $\varepsilon > 0$. Obviously, the value ε can be chosen small enough in order to have that $\mathcal{P}'$ still satisfies $\mathcal{C}$. By definition of v, $\mathcal{P}'$ is contained in the same hyperplane as $\mathcal{P}$. Iteratively using this construction ultimately yields a set of points $\mathcal{P}'$ satisfying $\mathcal{C}$, contained in the same hyperplane as $\mathcal{P}$ and such that the columns of $M(\mathcal{P}')$ are all distinct.

Assume that in the row $b \in \mathcal{B}$ in $M(\mathcal{P})$, the value in columns labelled by $e, f \in \mathcal{E}$ are the same. Up to transforming $\mathcal{P}$ as above, we assume that those two columns are not equal. Then, there exists a row $b' \in \mathcal{B}$ such that $x_{e,b'} \neq x_{f,b'}$. Let $\mathcal{P}'$ be the set of points whose matrix $M(\mathcal{P}')$ is obtained by adding ε times row b' to row b in $M(\mathcal{P})$, for some $\varepsilon > 0$. Obviously, the value ε can be chosen small enough to have that $\mathcal{P}'$ still satisfies $\mathcal{C}$. Since the determinant of the matrices $M(\mathcal{P})$ and $M(\mathcal{P}')$ are equal, $\mathcal{P}'$ is contained in a hyperplane. Using this construction iteratively ultimately yields a set of points $\mathcal{P}'$ satisfying $\mathcal{C}$, contained in a hyperplane and satisfying the hypothesis presented in the first paragraph of this proof. □

Proposition 1. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$. The configuration $\mathcal{C}$ is non-fixed if and only if there exists a non-fixed linear extension of $\mathcal{C}$. The configuration $\mathcal{C}$ is fixed if and only if every linear extension of $\mathcal{C}$ is fixed.*

Proof. We prove the first assertion in the proposition. The second is obviously equivalent. By Lemma 1, if $\mathcal{C}$ is non-fixed then there exists a set $\mathcal{P}$ of n points satisfying $\mathcal{C}$ such that $det(M(P)) = 0$, that is $\mathcal{P}$ is contained in a hyperplane. Lemma 2 implies that there exists $\mathcal{P}'$ satisfying a linear extension $\mathcal{C}'$ of $\mathcal{C}$ and such that $det(M(\mathcal{P}')) = 0$. Hence $\mathcal{C}'$ is non-fixed. Conversely, let $\mathcal{C}'$ be a non-fixed linear extension of $\mathcal{C}$. By Lemma 1, there exists $\mathcal{P}$ such that $det(M(P)) = 0$ and $\mathcal{P}$ satisfies $\mathcal{C}'$. In particular, $\mathcal{P}$ satisfies $\mathcal{C}$, and hence $\mathcal{C}$ is non-fixed. $\square$

With the above result, we only need to test the fixity of linear ordering configurations in order to deduce the fixity of any configuration. In the following, we will concentrate on linear ordering configurations.

3.2 Formal fixity

Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$. We consider formal expressions of type $x_{e,b} - x_{f,b}$ for $e, f \in \mathcal{E}$, $e \neq f$, and $b \in \mathcal{B}$, which we may sometimes denote $x_{e-f,b}$ for short. Such a formal expression gets a *formal sign w.r.t.* $\mathcal{C}$ denoted $\sigma_{\mathcal{C}}(x_{e,b} - x_{f,b})$ and belonging to $\{\boxed{+}, \boxed{-}\}$, the following way:

$$\sigma_{\mathcal{C}}(x_{e,b} - x_{f,b}) = \boxed{+} \text{ if } f <_b e; \qquad \sigma_{\mathcal{C}}(x_{e,b} - x_{f,b}) = \boxed{-} \text{ if } e <_b f.$$

Recall that the polynomial $det(M_{\mathcal{E},\mathcal{B}})$ is a multivariate polynomial on variables $x_{e,b}$ for $b \in \mathcal{B}$ and $e \in \mathcal{E}$. Assume a particular formal expression of $det(M_{\mathcal{E},\mathcal{B}})$ is a sum of multivariate monomials where each variable is replaced by some $x_{e,b} - x_{f,b}$, for $b \in \mathcal{B}$ and $e, f \in \mathcal{E}$. Various expressions of this type can be obtained by suitable transformations and determinant cofactor expansions from the matrix M, as we will do more precisely below. This particular expression of $det(M_{\mathcal{E},\mathcal{B}})$ gets a *formal sign w.r.t.* $\mathcal{C}$ belonging to $\{\boxed{+}, \boxed{-}, \boxed{?}\}$, by replacing each expression of type $x_{e,b} - x_{f,b}$ with its formal sign $\sigma_{\mathcal{C}}(x_{e,b} - x_{f,b})$ and applying the following formal calculus rules:

$$\boxed{+} \cdot \boxed{+} = \boxed{-} \cdot \boxed{-} = \boxed{+}, \qquad \boxed{+} \cdot \boxed{-} = \boxed{-} \cdot \boxed{+} = \boxed{-},$$

$$\boxed{+} + \boxed{+} = \boxed{+} - \boxed{-} = \boxed{+}, \qquad \boxed{-} + \boxed{-} = \boxed{-} - \boxed{+} = \boxed{-},$$

$$\boxed{+} + \boxed{-} = \boxed{-} + \boxed{+} = \boxed{?},$$

and the result of any operation involving a $\boxed{?}$ term or factor is also $\boxed{?}$.

We say that $\mathcal{C}$ is *formally fixed* if $det(M_{\mathcal{E},\mathcal{B}})$ has such a formal expression whose formal sign is not $\boxed{?}$.

Example. Consider the following matrix $M = M_{\mathcal{E},\mathcal{B}}$ for $\mathcal{E} = \{a, b, c\}$ and $\mathcal{B} = \{1, 2\}$:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ x_{a,1} & x_{b,1} & x_{c,1} \\ x_{a,2} & x_{b,2} & x_{c,2} \end{pmatrix}$$

and consider the configuration $\mathcal{C}$ defined by:

$$\begin{array}{ccccc} a & <_1 & b & <_1 & c \\ b & <_2 & c & <_2 & a \end{array}$$

A formal expression of $det(M)$ is:

$$det(M) = x_{b-a,1} \cdot x_{c-a,2} - x_{b-a,2} \cdot x_{c-a,1}$$

whose formal sign w.r.t. $\mathcal{C}$ is

$$\boxed{+} \cdot \boxed{-} - \boxed{-} \cdot \boxed{+} = \boxed{?}.$$

Another formal expression of $det(M)$ is:

$$det(M) = x_{b-a,1} \cdot x_{c-b,2} - x_{b-a,2} \cdot x_{c-b,1}$$

whose formal sign w.r.t. $\mathcal{C}$ is

$$\boxed{+} \cdot \boxed{+} - \boxed{-} \cdot \boxed{+} = \boxed{+}.$$

This second expression shows that $\mathcal{C}$ is formally fixed.

Observation 1. *If $\mathcal{C}$ is formally fixed, then $\mathcal{C}$ is fixed.*

More precisely, given an expression (as above) whose formal sign w.r.t. $\mathcal{C}$ is $\boxed{+}$ or $\boxed{-}$, the evaluation of this determinant for any set of real values $\mathcal{P}$ satisfying $\mathcal{C}$ necessarily provides a real number whose sign is consistent with the formal sign of this expression. In this case, this resulting sign does not depend on the chosen expression, as long as it is not $\boxed{?}$, and $\sigma_C(det(M))$ equals this sign.

Conversely, one may wonder if for every fixed configuration $\mathcal{C}$ there would exist a suitable expression of the determinant formally showing in the above way that $\mathcal{C}$ is fixed. That is, equivalently, do we have: if every formal expression of $det(M_{\mathcal{E},\mathcal{B}})$ has formal sign $\boxed{?}$, then $\sigma_C(det(M)) = \boxed{\pm}$? We strongly believe in this result, which we state as a conjecture, and which we will prove for $n \leq 4$ (see Theorems 2 and 3).

Conjecture 1. *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$. Then $\mathcal{C}$ is fixed if and only if $\mathcal{C}$ is formally fixed.*

3.3 Formal fixity by expansion

Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$, and $\mathcal{E}' = \mathcal{E} \setminus \{e\}$, $\mathcal{B}' = \mathcal{B} \setminus \{b\}$ for some $e \in \mathcal{E}$, $b \in \mathcal{B}$. We call *configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B}')$* the configuration on $(\mathcal{E}', \mathcal{B}')$ obtained by restricting every ordering $<_{b'}$, $b' \in \mathcal{B}'$, of $\mathcal{C}$ to $\mathcal{E}'$. Moreover, we say that *all the configurations induced by $\mathcal{C}$ on $\mathcal{E}'$ are fixed* if, for every $b \in \mathcal{B}$, the configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B} \setminus \{b\})$ is a fixed configuration. Note that, from a geometrical viewpoint, if $\mathcal{P}$ is a set of points satisfying $\mathcal{C}$, and $\mathcal{P}_e$ is obtained by removing the point with label $e \in \mathcal{E}$ from $\mathcal{P}$, then the projection $\mathcal{P}'$ of $\mathcal{P}_e$ on $\mathcal{B}'$ along b satisfies $\mathcal{C}'$. Indeed, the matrix $M_{\mathcal{E}',\mathcal{B}'}$, resp. $M_{\mathcal{E}',\mathcal{B}'}(\mathcal{P}')$, is obtained by removing the column corresponding to e and the row corresponding to b from $M_{\mathcal{E},\mathcal{B}}$, resp. $M_{\mathcal{E},\mathcal{B}}(\mathcal{P})$.

As previously, let $M = M_{\mathcal{E},\mathcal{B}}$ with $\mathcal{E} = \{e_1, ..., e_n\}_<$ and $\mathcal{B} = \{b_1, ..., b_{n-1}\}_<$. Let $e_i, e_j \in \mathcal{E}$, with $e_i \neq e_j$. Consider the matrix obtained from M by subtracting the j-th column (corresponding to e_j), from the i-th column (corresponding to e_i), that is:

$$\begin{pmatrix} 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ x_{e_1,b_1} & \dots & x_{e_{i-1},b_1} & x_{e_i,b_1} - x_{e_j,b_1} & x_{e_{i+1},b_1} & \dots & x_{e_n,b_1} \\ x_{e_1,b_2} & \dots & x_{e_{i-1},b_2} & x_{e_i,b_2} - x_{e_j,b_2} & x_{e_{i+1},b_2} & \dots & x_{e_n,b_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{e_1,b_{n-1}} & \dots & x_{e_{i-1},b_{n-1}} & x_{e_i,b_{n-1}} - x_{e_j,b_{n-1}} & x_{e_{i+1},b_{n-1}} & \dots & x_{e_n,b_{n-1}} \end{pmatrix}$$

The determinant of this matrix equals $det(M)$. The cofactor expansion formula for the determinant of this matrix w.r.t. its i-th column yields:

$$det(M_{\mathcal{E},\mathcal{B}}) = \sum_{k=1}^{n-1} (-1)^{i+k+1} \cdot (x_{e_i,b_k} - x_{e_j,b_k}) \cdot det\big(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}} \big)$$

which we call *expression of $det(M)$ by expansion with respect to (e_i, e_j)*.

Then the above particular expression of $det(M)$ gets a *formal sign* w.r.t. $\mathcal{C}$ in the following manner. First, replace each expression of type $x_{e,b} - x_{f,b}$ with its formal sign w.r.t. $\mathcal{C}$ in $\{\boxed{+}, \boxed{-}\}$, and replace each

$det(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}})$, $1 \leq k \leq n-1$, with its sign $\sigma_{\mathcal{C}_k}(det(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}})) \in \{\boxed{+}, \boxed{-}, \boxed{\pm}\}$, where $\mathcal{C}_k$ is the configuration induced by $\mathcal{C}$ on $(\mathcal{E} \setminus \{e_i\}, \mathcal{B} \setminus \{b_k\})$. This leads to the formal expression:

$$\sum_{k=1}^{n-1} (-1)^{i+k+1} \cdot \sigma_{\mathcal{C}}(x_{e_i,b_k} - x_{e_j,b_k}) \cdot \sigma_{\mathcal{C}_k}\Big(det\big(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}} \big)\Big),$$

Then, provide the formal sign of this expression by using the same formal calculus rules as previously, completed with the following one:

$$\boxed{+} \cdot \boxed{\pm} = \boxed{-} \cdot \boxed{\pm} = \boxed{?}.$$

If there exists such an expression of $det(M)$ by expansion whose formal sign is $\boxed{+}$ or $\boxed{-}$, then $\mathcal{C}$ is called *formally fixed by expansion*.

Observation 2. *If $\mathcal{C}$ is formally fixed by expansion, then $\mathcal{C}$ is fixed.*

The above observation is similar to Observation 1: if $\mathcal{C}$ is formally fixed by expansion then $\sigma_{\mathcal{C}}(det(M))$ is given as the formal sign of any expression certifying that $\mathcal{C}$ is formally fixed by expansion. Notice that if $\mathcal{C}$ is formally fixed by expansion then all the configurations $\mathcal{C}_k$ induced by $\mathcal{C}$ are fixed, since we must have $\sigma_{\mathcal{C}_k}(det(M_{\mathcal{E}\setminus\{e_i\},\mathcal{B}\setminus\{b_k\}})) \in \{\boxed{+}, \boxed{-}\}$.

Conjecture 2. *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$. Then $\mathcal{C}$ is fixed if and only if $\mathcal{C}$ is formally fixed by expansion.*

We point out that if Conjecture 1 is true in dimension $n-1$, then Conjecture 2 in dimension n implies Conjecture 1 in dimension n. Indeed, in this case, the fixity of the $(n-1)$-dimensional configurations corresponding to cofactors can be determined using formal expressions.

Remark 1. Assume $n = 4$, consider any 3×3 matrix M' obtained from M by subtracting some columns, and deleting the first row and one column, so that every entry in M' is of type $x_{e,b} - x_{f,b}$ for $e, f \in \mathcal{E}$ and $b \in \mathcal{B}$. We have either $det(M') = det(M)$ or $det(M') = -det(M)$. Then replace in the matrix M' each formal expression $x_{e,b} - x_{f,b}$ with its formal sign $\sigma_{\mathcal{C}}(x_{e,b} - x_{f,b})$ w.r.t. to a given configuration $\mathcal{C}$. We obtain a 3×3 matrix N with entries in $\{\boxed{+}, \boxed{-}\}$. The point of this remark is that formally computing the sign of the determinant of the matrix N, using the same formal rules as above, will always provide the result $\boxed{?}$. The proof of this property is left as an exercise to the reader. In fact, as already noticed in the introduction of the paper, this property generalizes in any dimension, it is known as: an SNS-matrix of order $n \geq 3$ has at least one zero, see [3, page 108]. This shows that a formal matrix M', such as the above one, cannot be used alone to derive a formal expression of the determinant of the original matrix M proving the fixity of a configuration. One would always need to transform submatrices of M, which is what we do implicitly by the inductive use of formal signs of induced configurations in order to determine formal fixity.

Finally, the point of this paper is to deal with the property of being formally fixed by expansion as an inductive criterion for fixity. Next, we will prove Conjecture 2 for $n = 4$, providing at the same time more precise and direct characterizations in this case (see Theorem 3).

3.4 A non-fixity criterion

The following Lemma 3 will be our main tool to prove that a configuration is non-fixed. We point out that, when $n = 4$, the sufficient condition for being non-fixed provided by Lemma 3 turns out to be a necessary and sufficient condition (see Theorem 4). However, the authors feel that this equivalence result is too hazardous to be stated as a general conjecture in dimension n.

Lemma 3. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$. If there exist $e \in \mathcal{E}$ and $b \in \mathcal{B}$ satisfying the following properties:*

- *e is extreme in the ordering $<_b$ of $\mathcal{C}$ and*
- *the configuration $\mathcal{C}'$ induced by $\mathcal{C}$ on $(\mathcal{E} \setminus \{e\}, \mathcal{B} \setminus \{b\})$ is non-fixed,*

then $\mathcal{C}$ is non-fixed.

Proof. To lighten notations, let us denote $\mathcal{E} = \{1, ..., n\}$ and $\mathcal{B} = \{1, ..., n-1\}$. Up to equivalence of configurations, we can assume that $e = 1$, that $b = 1$ and that 1 is minimal in the ordering $<_1$.

The expression of $det(M)$ by expansion with respect to $(1, 2)$ yields:

$$\begin{aligned} det(M_{\mathcal{E},\mathcal{B}}) &= \sum_{k=1}^{n-1} (-1)^k \cdot (x_{1,k} - x_{2,k}) \cdot det\big(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{k\}} \big) \\ &= (x_{2,1} - x_{1,1}) \cdot det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}) + P[x_{i,j}]_{(i,j)\neq(1,1)}. \end{aligned}$$

where $P[x_{i,j}]_{(i,j)\neq(1,1)}$ is a polynomial in the same variables as $M_{\mathcal{E},\mathcal{B}}$ not depending on $x_{1,1}$.

By hypothesis, the configuration $\mathcal{C}'$ is non-fixed, that is $\sigma_{\mathcal{C}'}(det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}})) = \boxed{\pm}$. By Lemma 1, there exist real values $\mathcal{P}'_+$ and $\mathcal{P}'_-$ for the entries of this matrix, that is two sets of $n-1$ points labeled by $\mathcal{E} \setminus \{1\}$ in dimension $n-2$, such that $det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}(\mathcal{P}'_+)) > 0$ and $det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}(\mathcal{P}'_-)) < 0$.

Let us define a set of n points $\mathcal{P}_+$ labeled by $\mathcal{E}$ in dimension $n-1$ the following manner. The formal variables in $M_{\mathcal{E},\mathcal{B}}$ with real values specified by $\mathcal{P}'_+$ get the same values in $\mathcal{P}$. All values not specified by $\mathcal{P}'_+$ except $x_{1,1}$ are fixed arbitrarily but consistently with the orderings in $\mathcal{C}$. The value $x_{1,1}$ is chosen small enough so that $x_{1,1}$ is minimal in $<_1$ and

$$(x_{2,1} - x_{1,1}) \cdot det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}(\mathcal{P}'_+)) > -P[x_{i,j}]_{(i,j)\neq(1,1)}$$

This is possible since $det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}(\mathcal{P}'_+)) > 0$ and the second term of the inequality does not depend on $x_{1,1}$. By this definition, we have obtained $det(M(\mathcal{P}_+)) > 0$.

Similarly, we define $\mathcal{P}_-$ by choosing $x_{1,1}$ small enough so that $x_{1,1}$ is minimal in $<_1$ and

$$(x_{2,1} - x_{1,1}) \cdot det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}(\mathcal{P}'_-)) < -P[x_{i,j}]_{(i,j)\neq(1,1)}$$

This is possible since $det(M_{\mathcal{E}\setminus\{1\},\mathcal{B}\setminus\{1\}}(\mathcal{P}'_-)) < 0$ and the second term of the inequality does not depend on $x_{1,1}$. By this definition, we have obtained $det(M(\mathcal{P}_-)) < 0$.

We have built $\mathcal{P}_+$ and $\mathcal{P}_-$ providing opposite signs to real evaluations of $det(M_{\mathcal{E},\mathcal{B}})$. That is, by Lemma 1, $\mathcal{C}$ is non-fixed. ☐

4 Characterizations in low dimensions

4.1 Results in dimension 2

In this section we fix $n = 3$ and $\mathcal{E} = \{A, B, C\}$. In order to lighten notations of variables $x_{e,b}$ for $e \in \mathcal{E}$ and $b \in \mathcal{B}$, we sooner denote:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix}$$

We also denote $\mathcal{B} = \{x, y\}$ and $<_x, <_y$ the orderings in a configuration.

It is easy to verify that, up to equivalence of configurations, there exist exactly two linear ordering configurations:

$$A <_x B <_x C \qquad\qquad A <_x B <_x C$$
$$A <_y B <_y C \qquad\qquad B <_y C <_y A$$

which correspond to the following respective grid representations:

		C
	B	
A		

A		
		C
	B	

Theorem 1. *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 3$, $\mathcal{E} = \{A, B, C\}$ and $\mathcal{B} = \{x, y\}$. The following properties are equivalent:*

a) $\mathcal{C}$ is non-fixed;

b) the two orderings on $\mathcal{E}$ in $\mathcal{C}$ are either equal or equal to reversions of each other;

c) up to equivalence, $\mathcal{C}$ is equal to

$$A <_x B <_x C$$
$$A <_y B <_y C$$

Proof. The equivalence between b) and c) is straightforward and left to the reader. Let us prove that c) implies a). Let $\mathcal{C}$ be given by condition c). Let us choose $\mathcal{P}$ satisfying $\mathcal{C}$ and $x_A = y_A$, $x_B = y_B$, $x_C = y_C$. We have $det(M(\mathcal{P})) = 0$, hence $\mathcal{C}$ is non-fixed by Lemma 1. In order to prove that a) implies c), we can equally prove that the other possible linear ordering configuration (up to equivalence) is fixed. This result is given by Theorem 2 below. □

Theorem 2. *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 3$, $\mathcal{E} = \{A, B, C\}$ and $\mathcal{B} = \{x, y\}$. The following properties are equivalent:*

a) $\mathcal{C}$ is fixed;

b) $\mathcal{C}$ is formally fixed;

c) up to equivalence, $\mathcal{C}$ is equal to

$$A <_x B <_x C$$
$$B <_y C <_y A$$

Proof. Recall that b) implies a) is always true. Let us prove that c) implies b). We have:

$$
\begin{aligned}
det(M) &= det\begin{pmatrix} 0 & 1 & 0 \\ x_{A-B} & x_B & x_{C-B} \\ y_{A-B} & y_B & y_{C-B} \end{pmatrix} \\
&= (x_A - x_B)\cdot(y_C - y_B) - (y_A - y_B)\cdot(x_C - x_B).
\end{aligned}
$$

The formal sign of this expression of $det(M)$ w.r.t. $\mathcal{C}$ is

$$\boxed{-}\cdot\boxed{+}-\boxed{+}\cdot\boxed{+}=\boxed{-}.$$

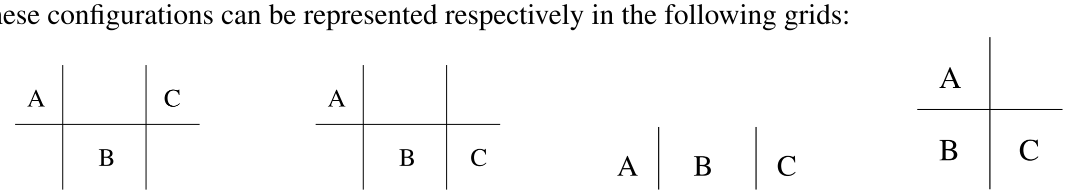
Hence $\mathcal{C}$ is formally fixed.

Finally, to prove that a) implies c), we just need to prove that the other linear ordering configuration (up to equivalence) is non-fixed. This has been already shown in the proof of Theorem 1. □

Now that we have listed fixed and non-fixed linear ordering configurations, we are able to determine all fixed and non-fixed configurations using Proposition 1. Let us omit configurations for which two elements of $\mathcal{E}$ are comparable in no ordering in the configuration, since these configurations are obviously non-fixed. Then there remain four ordering configurations which are not linear (up to equivalence of configurations), as one can easily check:

$A <_x B <_x C$ $B <_y A$ $B <_y C$	$A <_x B <_x C$ $B <_y A$ $C <_y A$	$A <_x B <_x C$	$A <_x C$ $B <_x C$ $B <_y A$ $C <_y A$
fixed	non-fixed (because of $C <_y B <_y A$, and implying the non-fixity of the next ones)	non-fixed	non-fixed

These configurations can be represented respectively in the following grids:



4.2 Results in dimension 3

In this section we fix $n = 4$ and $\mathcal{E} = \{A, B, C, D\}$. In order to lighten notations of variables $x_{e,b}$ for $e \in \mathcal{E}$ and $b \in \mathcal{B}$, we sooner denote:

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_A & x_B & x_C & x_D \\ y_A & y_B & y_C & y_D \\ z_A & z_B & z_C & z_D \end{pmatrix}$$

We also denote $\mathcal{B} = \{x, y, z\}$ and $<_x, <_y, <_z$ the orderings in a configuration.

As noted in Section 3, in order to prove that a configuration $\mathcal{C}$ is formally fixed by expansion, we need to find an element $e \in E$ such that all the configurations induced by $\mathcal{C}$ on $\mathcal{E} \setminus \{e\}$ are fixed. The proposition below characterizes such induced configurations.

Proposition 2. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$, $\mathcal{E} = \{A, B, C, D\}$ and $\mathcal{B} = \{x, y, z\}$. All the configurations induced by $\mathcal{C}$ on $\{A, B, C\}$ are fixed if and only if $\mathcal{C}$ is equivalent to a configuration whose orderings satisfy:*

$$\begin{array}{ccccc} B & <_x & C & <_x & A \\ C & <_y & A & <_y & B \\ A & <_z & B & <_z & C \end{array}$$

Proof. $\Rightarrow$) The configuration induced by $\mathcal{C}$ on $(\{A, B, C\}, \{y, z\})$ is fixed. According to Theorem 2, this implies that the restrictions of $<_y$ and $<_z$ to $\{A, B, C\}$ are not equal nor are reversions of each other. Similarly, the restrictions of $<_x$ and $<_y$, as well as the restriction of $<_x$ and $<_z$, to $\{A, B, C\}$ are not equal nor are reversions of each other. These three properties easily imply the result. We omit the details. $\Leftarrow$) The proof is direct by Theorem 2 using the same criteria as above. □

Let us now state Theorem 3, which is the main theorem of the paper. Its proof is the content of Section 5. This proof will prove Theorem 4 below at the same time.

Theorem 3. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$, $\mathcal{E} = \{A, B, C, D\}$ and $\mathcal{B} = \{x, y, z\}$. The following propositions are equivalent:*

a) $\mathcal{C}$ is fixed;
b) $\mathcal{C}$ is formally fixed;
c) $\mathcal{C}$ is formally fixed by expansion;
d) up to equivalence, $\mathcal{C}$ satisfies:

$$\begin{array}{ccccc} B & <_x & C & <_x & A \\ C & <_y & A & <_y & B \\ A & <_z & B & <_z & C \end{array}$$

and there exists $X \in \{A, B, C\}$ such that either $X <_b D$ for every $b \in \mathcal{B}$, or $D <_b X$ for every $b \in \mathcal{B}$.

Theorem 4. *Let $\mathcal{C}$ be a linear ordering configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$. Then $\mathcal{C}$ is non-fixed if and only if conditions of Lemma 3 are satisfied, that is: there exist $e \in \mathcal{E}$ and $b \in \mathcal{B}$ such that the configuration $\mathcal{C}'$ induced by $\mathcal{C}$ on $(\mathcal{E} \setminus \{e\}, \mathcal{B} \setminus \{b\})$ is non-fixed and e is extreme in the ordering $<_b$ of $\mathcal{C}$.*

Given the integer n and the sets $\mathcal{E}$ and $\mathcal{B}$ as previously, there are $(n!)^{n-1}$ linear ordering configurations on $(\mathcal{E}, \mathcal{B})$. We computed the number of classes of linear ordering configurations up to equivalence from $n = 2$ to $n = 6$, yielding the sequence: 1, 2, 21, 5097, 71965235. Implementation details for this computation (required for the case $n = 6$) are given in [4]. This integer sequence has been added to The On-Line Encyclopedia of Integer Sequences [6]. We have no general formula: we leave as an open question to find one.

Open question 1. *Find a general formula to count, for any n, the number of classes of linear ordering configurations up to equivalence.*

We computed the result provided by Theorem 3 to list the fixed linear ordering configurations when $n = 4$. We found that there are exactly 4 fixed configurations among the 21 linear ordering configurations up to equivalence:

$$\begin{array}{lcl} B <_x C <_x A <_x D & \qquad & B <_x C <_x D <_x A \\ C <_y A <_y B <_y D & & C <_y A <_y B <_y D \\ A <_z B <_z C <_z D & & A <_z B <_z C <_z D \end{array}$$

$$B <_x D <_x C <_x A \qquad B <_x C <_x D <_x A$$
$$C <_y A <_y B <_y D \qquad C <_y D <_y A <_y B$$
$$A <_z B <_z C <_z D \qquad A <_z B <_z C <_z D$$

The interest of the results of this section is that it provides a combinatorial characterization as well as an algorithm capable of deciding if a configuration is fixed or not. We also need to point out that our result statements concern the fixity (or lack thereof) of the considered configuration, but not its exact $\boxed{+}$ or $\boxed{-}$ value. This sign can be easily derived from the construction stating the fixity. This sign can also be obtained by choosing any set of points $\mathcal{P}$ satisfying the configuration and evaluating the sign of the real number $det(M(\mathcal{P}))$. Finally, from the list of fixed linear ordering configurations given above, one may compute the list of all fixed (partial) ordering configurations using Proposition 1. We do not give this list here.

4.3 Example continued

Let us apply the previous results to several configurations in the 3D model shown in Section 2.2. We recall that ordering configurations are represented in Figures 2 and 3, with $\mathcal{E}$ being any set of four points, and $\mathcal{B}$ corresponding to the three axis $\{x, y, z\}$.

Example 1. Fixed linear ordering configurations providing a fixed partial ordering configuration: the configuration on $\mathcal{E} = \{2, 5, 8, 9\}$ is fixed.

This configuration is given by the orderings:

$$9 <_x 5 <_x 2 <_x 8$$
$$5 <_y 8 <_y 9 <_y 2$$
$$2 <_z 8 <_z 9 \quad \text{and} \quad 5 <_z 8 <_z 9$$

Its two linear extensions, respectively $\mathcal{C}_1$ and $\mathcal{C}_2$, are the following:

$$9 <_x 5 <_x 2 <_x 8 \qquad 9 <_x 5 <_x 2 <_x 8$$
$$5 <_y 8 <_y 9 <_y 2 \qquad 5 <_y 8 <_y 9 <_y 2$$
$$2 <_z 5 <_z 8 <_z 9 \qquad 5 <_z 2 <_z 8 <_z 9$$

Let us write these orderings another way:

$$2 <_z 5 <_z 8 <_z 9 \qquad 5 <_z 2 <_z 8 <_z 9$$
$$5 <_y 8 <_y 9 <_y 2 \qquad 5 <_y 8 <_y 9 <_y 2$$
$$9 <_x 5 <_x 2 <_x 8 \qquad 9 <_x 5 <_x 2 <_x 8$$

In this way, we see that, up to a permutation of $\mathcal{B}$ (that is for $\{i, j, k\} = \{x, y, z\}$) and if we choose $A = 9$, $B = 2$, $C = 8$ and $D = 5$, then the orderings in those configurations both satisfy:

$$B <_i C <_i A$$
$$C <_j A <_j B$$
$$A <_k B <_k C$$

as required by Theorem 3. Moreover, for each of these orderings, D is smaller than C (i.e. $5 <_x 8$, $5 <_y 8$, $5 <_z 8$). Therefore, according to Theorem 3, those two configurations are fixed. It follows that $\mathcal{C}$ is fixed by Proposition 1.

Example 2. A non-fixed ordering configuration implied by a non-fixed linear ordering configuration: the configuration on $\mathcal{E} = \{1, 3, 7, 10\}$ is non-fixed.

It is given by the orderings:

$$7 <_x 3 <_x 10 \quad \text{and} \quad 7 <_x 1 <_x 10$$
$$7 <_y 3 <_y 1 \quad \text{and} \quad 7 <_y 10 <_y 1$$
$$1 <_z 7 <_z 10 \quad \text{and} \quad 3 <_z 7 <_z 10$$

One of its linear extensions is $\mathcal{C}'$:

$$7 <_x 3 <_x 1 <_x 10$$
$$7 <_y 10 <_y 3 <_y 1$$
$$3 <_z 1 <_z 7 <_z 10$$

The configuration induced by $\mathcal{C}'$ on $(\{7,3,1\},\{x,y\})$ is

$$7 <_x 3 <_x 1$$
$$7 <_y 3 <_y 1$$

which is non-fixed by Theorem 1. Since 10 is extreme in the ordering $<_z$ of configuration $\mathcal{C}'$, $\mathcal{C}'$ is non-fixed by Lemma 3, and so is $\mathcal{C}$ by Proposition 1.

Example 3. We leave as an exercise to check, using Proposition 1 and Theorem 3, that the configuration on $\{1,5,9,10\}$ from Section 2.2 is fixed. There are four linear extensions to consider, up to symmetries.

Let us conclude by considering the entire Section 2.2 example with ten points in $\mathbb{R}^3$. Since there is one configuration for each set of 4 points, there are $\binom{4}{10} = 210$ configurations to study. We wrote a program to test the fixity of these configurations. For each configuration $\mathcal{C}$ the program lists all the linear extensions of $\mathcal{C}$ and computes if each linear extension is fixed or not, based on the results given in Section 4.2. Then, Proposition 1 allows us to conclude. Finally, we find 20 fixed configurations among the 210 configurations. This highlights the significant role of these 20 configurations for 3D skull shape generic characterization, and their non-significant role for the sake of 3D skull shape comparison.

5 Proofs of Theorem 3 and Theorem 4

To prove Theorem 3, we study separately: first, the configurations on $(\mathcal{E},\mathcal{B})$ with $n = 4$ for which there exists a triplet of points $\mathcal{E}' \subseteq \mathcal{E}$ such that all the configurations induced by $\mathcal{C}$ on $\mathcal{E}'$ are fixed (characterized by Proposition 2); and second, the other configurations. The fixed configurations in the first case will be identified. Then every other configuration in the first case and every configuration in the second case will be proved to be non-fixed, always using Lemma 3. Hence, Theorem 4 will be proved in the meantime.

5.1 If all the configurations induced on some triplet are fixed

Recall that Proposition 2 characterizes configurations for which all the configurations induced on some given triplet are fixed.

Proposition 3. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E},\mathcal{B})$ with $n = 4$, $\mathcal{E} = \{A,B,C,D\}$ and $B = \{x,y,z\}$ such that:*

$$\begin{array}{ccccc} B & <_x & C & <_x & A \\ C & <_y & A & <_y & B \\ A & <_z & B & <_z & C \end{array}$$

Then $\mathcal{C}$ is formally fixed by expansion if and only if there exists $X \in \{A, B, C\}$ such that either $X <_b D$ for every $b \in \mathcal{B}$, or $D <_b X$ for every $b \in \mathcal{B}$.

Proof. $\Leftarrow$) Let us denote $\mathcal{E}' = \{A, B, C\}$ and $\mathcal{C}_b$ the configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B} \setminus \{b\})$ for $b \in \mathcal{B}$. The expansion of $det(M)$ w.r.t. (D, X) yields:

$$det(M) = x_{D-X} \cdot det(M_{\mathcal{E}',\mathcal{B}\setminus\{x\}}) - y_{D-X} \cdot det(M_{\mathcal{E}',\mathcal{B}\setminus\{y\}}) + z_{D-X} \cdot det(M_{\mathcal{E}',\mathcal{B}\setminus\{z\}}).$$

We have

$$\begin{aligned} det(M_{\mathcal{E}',\mathcal{B}\setminus\{x\}}) &= det\begin{pmatrix} 1 & 1 & 1 \\ y_A & y_B & y_C \\ z_A & z_B & z_C \end{pmatrix} \\ &= det\begin{pmatrix} y_{B-A} & y_{C-A} \\ z_{B-A} & z_{C-A} \end{pmatrix} \\ &= (y_B - y_A) \cdot (z_C - z_A) - (z_B - z_A) \cdot (y_C - y_A). \end{aligned}$$

whose formal sign w.r.t. $\mathcal{C}_x$ is:

$$(\boxed{+} \cdot \boxed{+}) - (\boxed{+} \cdot \boxed{-}) = \boxed{+}.$$

Similarly, we have

$$det(M_{\mathcal{E}',\mathcal{B}\setminus\{z\}}) = det\begin{pmatrix} x_{B-A} & x_{C-A} \\ y_{B-A} & y_{C-A} \end{pmatrix} = (x_B - x_A) \cdot (y_C - y_A) - (y_B - y_A) \cdot (x_C - x_A)$$

whose formal sign w.r.t. $\mathcal{C}_z$ is

$$(\boxed{-} \cdot \boxed{-}) - (\boxed{+} \cdot \boxed{-}) = \boxed{+}.$$

And we have

$$det(M_{\mathcal{E}',\mathcal{B}\setminus\{y\}}) = det\begin{pmatrix} x_{B-A} & x_{C-B} \\ z_{B-A} & z_{C-B} \end{pmatrix} = (x_B - x_A) \cdot (z_C - z_B) - (z_B - z_A) \cdot (x_C - x_B)$$

whose formal sign w.r.t. $\mathcal{C}_y$ is

$$(\boxed{-} \cdot \boxed{+}) - (\boxed{+} \cdot \boxed{+}) = \boxed{-}.$$

Now, if the formal signs of x_{D-X}, y_{D-X} and z_{D-X} are all positive (resp. negative), then the formal sign of the above expression of $det(M)$ w.r.t. $\mathcal{C}$ is $\boxed{+}$ (respectively $\boxed{-}$), which proves that $\mathcal{C}$ is formally fixed by expansion.

$\Rightarrow$) We will prove the contrapositive: we assume that there exists no $X \in \{A, B, C\}$ such that $X <_b D$ for all $b \in \mathcal{B}$ or such that $D <_b X$ for all $b \in \mathcal{B}$, and we want to prove that $\mathcal{C}$ is non-fixed. Equivalently, we assume that, for every $X \in \{A, B, C\}$, there exist two orderings in $\mathcal{C}$ such that X is smaller than D in an ordering and D is smaller than X in the other ordering. Let us consider two cases. Observe that we will always use Lemma 3 to prove that $\mathcal{C}$ is non-fixed.

Case 1: there exist two orderings $<_i$ and $<_j$ in $\mathcal{C}$, for $i, j \in \mathcal{B}$, such that, for every $X \in \{A, B, C\}$, we have either $X <_i D$ and $D <_j X$, or $X <_j D$ and $D <_i X$. With the restrictions of $\mathcal{C}$ to $\{A, B, C\}$ given in the hypothesis of the proposition, it is easy to check that only 3 configurations satisfy this assumption, up to equivalence (i.e. up to some permutations of $\mathcal{B}$ and $\mathcal{E}$, and up to ordering reversions).

$$\begin{array}{ccccccc} B & <_x & D & <_x & C & <_x & A \\ C & <_y & A & <_y & D & <_y & B \\ & A & <_z & B & <_z & C & \end{array} \qquad \begin{array}{ccccccc} D & <_x & B & <_x & C & <_x & A \\ C & <_y & A & <_y & B & <_y & D \\ & A & <_z & B & <_z & C & \end{array}$$

$$\begin{array}{ccccccc} D & <_x & B & <_x & C & <_x & A \\ & C & <_y & A & <_y & B & \\ A & <_z & B & <_z & C & <_z & D \end{array}$$

We denote these configurations $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. In these configurations, one ordering is a partial ordering: each of these configurations represents four linear ordering configurations.

Let us prove that $\mathcal{C}_1$ is non-fixed. First, assume that $A <_z D$. The configuration induced by $\mathcal{C}_1$ on $(\{B, C, D\}, \{x, y\})$ is:

$$\begin{array}{ccccc} B & <_x & D & <_x & C \\ C & <_y & D & <_y & B \end{array}$$

Since this configuration is non-fixed by Theorem 1, and since A is minimal in the ordering $<_z$, then $\mathcal{C}_1$ is non-fixed by Lemma 3. Second, similarly, if $D <_z A$, then the configuration induced by $\mathcal{C}_1$ on $(\{A, C, D\}, \{y, z\})$ is non-fixed by Theorem 1:

$$\begin{array}{ccccc} C & <_y & A & <_y & D \\ D & <_z & A & <_z & C \end{array}$$

And B is minimal in $<_x$, so $\mathcal{C}_1$ is non-fixed by Lemma 3.

Now we will show that $\mathcal{C}_2$ is non-fixed. The proof is similar. Assume that $A <_z D$. As before, A is minimal in the ordering $<_z$. The configuration induced by $\mathcal{C}_2$ on $(\{B, C, D\}, \{x, y\})$ is non-fixed, so $\mathcal{C}_2$ is non-fixed by Lemma 3. Assume that $D <_z A$. Then, the configuration induced by $\mathcal{C}_2$ on $(\{A, B, D\}, \{x, y\})$ is non-fixed. Since C is maximal in the ordering $<_z$, we find that $\mathcal{C}_2$ is non-fixed by Lemma 3.

We will use the same method to prove that $\mathcal{C}_3$ is non-fixed. First, assume that $C <_y D$. The configuration induced by $\mathcal{C}_3$ on $(\{A, B, D\}, \{x, z\})$ is non-fixed. Since C is minimal in the ordering $<_y$, then $\mathcal{C}_3$ is non-fixed by Lemma 3. Secondly, if $D <_y C$ then the configuration induced by $\mathcal{C}_3$ on $(\{B, C, D\}, \{y, z\})$ is non-fixed. Since A is maximal in the ordering $<_x$, then $\mathcal{C}_3$ is non-fixed by Lemma 3.

Case 2: the assumption of case 1 does not hold. It is easy to check that, up to equivalence of configurations, there are two other configurations such that X is smaller than D in an ordering and D is smaller than X in another ordering:

$$\begin{array}{ccccccc} D & <_x & B & <_x & C & <_x & A \\ C & <_y & A & <_y & D & <_y & B \\ A & <_z & B & <_z & D & <_z & C \end{array} \qquad \begin{array}{ccccccc} B & <_x & D & <_x & C & <_x & A \\ C & <_y & D & <_y & A & <_y & B \\ A & <_z & D & <_z & B & <_z & C \end{array}$$

We denote these configurations $\mathcal{C}_4$ and $\mathcal{C}_5$.

We first prove that $\mathcal{C}_4$ is non-fixed. The configuration induced by $\mathcal{C}_4$ on $(\{B, C, D\}, \{y, z\})$ is non-fixed (again by Theorem 1), and A is maximal in the ordering $<_x$. So the configuration $\mathcal{C}_4$ is non-fixed by Lemma 3.

Similarly, the configuration induced by $\mathcal{C}_5$ on $(\{B, C, D\}, \{x, y\})$ is non-fixed. Since A is minimal in the ordering $<_z$, $\mathcal{C}_5$ is non-fixed by Lemma 3. $\square$

5.2 If for every triplet there is at least one non-fixed induced configuration

We call *triplet* a set of three elements. For short, we may denote ABC the triplet $\{A, B, C\}$. Consider two linear orderings $<_i$ and $<_j$ on a same set containing the triplet $\{A, B, C\}$ and assume, without loss of generality, that $A <_i B <_i C$. We say that $<_i$ and $<_j$ are *equal*, resp. *reversed, on the triplet* $\{A, B, C\}$, if $A <_j B <_j C$, resp. $C <_j B <_j A$. Recall that, in this entire section, according to Theorem 1, $<_i$ and $<_j$ are equal or reversed on $\{A, B, C\}$ if and only if the configuration formed by $<_i$ and $<_j$ on $(\{A, B, C\}, \{i, j\})$ is non-fixed. In what follows, we will often consider two such orderings equal or reversed on such a triplet. For short, we may also say that the triplet $\{A, B, C\}$ is *conformal* w.r.t. the two orderings $<_i$ and $<_j$.

Lemma 4. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$, $\mathcal{E} = \{A, B, C, D\}$ and $\mathcal{B} = \{x, y, z\}$. Assume that $A <_x B <_x C <_x D$. Consider the three pairs of triplets $\{\{A, B, C\}, \{B, C, D\}\}$, $\{\{A, B, C\}, \{A, C, D\}\}$ and $\{\{A, B, D\}, \{B, C, D\}\}$. If, for at least one of these pairs, the two orderings $<_x$ and $<_y$ are equal or reversed on both triplets of the pair, then $<_x$ and $<_y$ are equal or reversed on $\{A, B, C, D\}$, that is: either $A <_y B <_y C <_y D$ or $D <_y C <_y B <_y A$.*

Proof. First, we prove that if two orderings $<_i$ and $<_j$ are equal or reversed on two triplets of $\mathcal{E}$, then they are either equal on the two triplets, or reversed on the two triplets. These two triplets have two elements in common. We denote these triplets XYZ and XYW. Up to relabelling, we can assume that $X <_i Y$. Assume for a contradiction that $<_i$ and $<_j$ are equal on XYZ and reversed on XYW. The triplet XYZ shows that $X <_j Y$, whereas XYW shows that $Y <_j X$, which is a contradiction.

Let us turn back to the pairs of triplets $\{ABC, BCD\}$, $\{ABC, ACD\}$ and $\{ABD, BCD\}$. Let us first consider the pair $\{ABC, BCD\}$. Assume that $<_x$ and $<_y$ are equal on ABC and BCD. We prove that $<_x$ and $<_y$ are equal on $\mathcal{E}$. We have $A <_x B <_x C$ by the lemma's hypothesis, and $<_x$ and $<_y$ equal on ABC by assumption, so we have $A <_y B <_y C$. We have $C <_x D$ by hypothesis, and $<_x$ and $<_y$ equal on BCD by assumption, so we have $C <_y D$. Hence, the ordering $<_y$ is $A <_y B <_y C <_y D$. If we consider the pair $\{ABC, ACD\}$, then we obtain the same result with the same proof. Consider the pair $\{ABD, BCD\}$, and assume that $<_x$ and $<_y$ are equal on ABD and BCD. Then the ordering $<_y$ restricted to ABD is $A <_y B <_y D$. The orderings $<_x$ and $<_y$ are also equal on BCD. So we have $B <_y C <_y D$. This proves that we have $A <_y B <_y C <_y D$.

Now, assume that there is a pair of triplets such that $<_x$ and $<_y$ are reversed on the two triplets of the pair. Let us denote $<_{opp(y)}$ the reversion of the ordering $<_y$. Then $<_x$ and $<_{opp(y)}$ are equal on the two triplets of the pair. From the result above, we get $A <_{opp(y)} B <_{opp(y)} C <_{opp(y)} D$, that is $D <_y C <_y B <_y A$. □

Proposition 4. *Let $\mathcal{C}$ be a configuration on $(\mathcal{E}, \mathcal{B})$ with $n = 4$. If for every triplet $\mathcal{E}' \subseteq \mathcal{E}$ there exists $b \in \mathcal{B}$ such that the configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B} \setminus \{b\})$ is non-fixed, then $\mathcal{C}$ is non-fixed.*

Proof. We consider separate cases. To this aim, for each ordering $<_b$ in $\mathcal{C}$, we count the number of non-fixed configurations induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B} \setminus \{b\})$ with $\mathcal{E}' = \mathcal{E} \setminus \{e\}$ for some $e \in \mathcal{E}$. By Theorem 1, it amounts to counting, for each pair of orderings in $\mathcal{C}$ (i.e. for each $\mathcal{B} \setminus \{b\}$)), the number of triplets $\mathcal{E}'$ in $\{ABC, ABD, ACD, BCD\}$ which are conformal w.r.t. this pair of orderings. Without loss of generality (up to a permutation of $\mathcal{B}$), we can assume that the pair or orderings $\{<_x, <_y\}$ maximizes this number of triplets. Also, without loss of generality (up to a permutation of $\mathcal{E}$), we can assume that $A <_x B <_x C <_x D$. The different cases correspond to the number of triplets of $\mathcal{E}$ which are conformal w.r.t. $<_x$ and $<_y$.

Case 1: four triplets of $\mathcal{E}$ are conformal w.r.t. $<_x$ and $<_y$.

All the triplets of $\mathcal{E}$ are conformal w.r.t. $<_x$ and $<_y$. Therefore, we have $A <_y B <_y C <_y D$ or $D <_y C <_y B <_y A$ by Lemma 4. Let $X \in \mathcal{E}$ be an extreme element in the ordering $<_z$. The triplet $\mathcal{E}' = \{A, B, C, D\} \setminus \{X\}$ is conformal w.r.t. $<_x$ and $<_y$. That is, using Theorem 1: the configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \{x, y\})$ is non-fixed. Hence, $\mathcal{C}$ is non-fixed by Lemma 3.

Case 2: three triplets of $\mathcal{E}$ are conformal w.r.t. $<_x$ and $<_y$.

We prove that this case is not possible. Otherwise, the set of three triplets contains one of the pairs $\{ABC, BCD\}$, $\{ABC, ACD\}$, $\{ABD, BCD\}$. Then we have $A <_y B <_y C <_y D$ or $D <_y C <_y B <_y A$ by Lemma 4. Therefore, the four triplets are conformal w.r.t. $<_x$ and $<_y$.

Case 3: two triplets of $\mathcal{E}$ are conformal w.r.t. $<_x$ and $<_y$.

This is the most critical case. First, we build a bipartite graph G between the four triplets of $\mathcal{E}$ and the three orderings of $\mathcal{C}$. Let us denote $T_1,\ T_2,\ T_3,\ T_4$ those triplets, and $O_1,\ O_2,\ O_3$ those orderings (we will study later in our case analysis which label corresponds to which triplet/ordering). Thus, the edges are of the form (T, O) with T in $\{T_1, \cdots, T_4\}$ and O in $\{O_1, O_2, O_3\}$. The graph G is defined the following way: if a triplet T is conformal w.r.t. the two orderings O_a and O_b, for $a, b \in \{1, 2, 3\}$, then both (T, O_a) and (T, O_b) are edges of G; and every edge (T, O_a) in G means that there exists O_b such that T is conformal w.r.t. O_a and O_b.

Let us state several useful claims.

1. *We have: (T, O_a) and (T, O_b) are edges of G if and only if T is conformal w.r.t. the two orderings O_a and O_b.* Indeed, if (T, O_a) and (T, O_b) are edges, but T is not conformal w.r.t. the two orderings O_a and O_b, then there exists an edge (T, O_c) in G, with T conformal w.r.t. the two orderings O_a and O_c, and T conformal w.r.t. O_b and O_c. This directly implies that the orderings O_a and O_b are equal or reversed on T, that is T is conformal w.r.t. O_a and O_b.

2. *We have: the two vertices $<_x$ and $<_y$ of G share exactly two neighbors.* Indeed, exactly two triplets are conformal w.r.t. $<_x$ and $<_y$ by assumption.

3. *We have: if $<_x$ and another vertex in $\{O_1, O_2, O_3\}$ are adjacent in G to the same two vertices in $\{T_1, \cdots, T_4\}$, then these two vertices in $\{T_1, \cdots, T_4\}$ form one of these pairs: $\{ABC, ABD\}$, or $\{ABD, ACD\}$, or $\{ACD, BCD\}$.* Indeed, if two triplets of $\mathcal{E}$ are conformal w.r.t. the same two orderings $<_x$ and $<_a$ in $\mathcal{C}$, for $a \in \{y, z\}$, then these two triplets are not $\{ABC, BCD\}$, nor $\{ABC, ACD\}$, nor $\{ABD, BCD\}$: otherwise, by Lemma 4, the four triplets of $\mathcal{E}$ would be conformal w.r.t. the two orderings, which would contradict the maximality property of $<_x$ and $<_y$ and the assumption of Case 3.

4. *We have: in the graph G, each vertex in $\{T_1, \cdots, T_4\}$ is not isolated, and is adjacent to at least two vertices in $\{O_1, O_2, O_3\}$.* Indeed, by hypothesis of the proposition, for every triplet T of $\mathcal{E}$, there exists $b \in \mathcal{B}$ such that the configuration induced by $\mathcal{C}$ on $(T, \mathcal{B} \setminus \{b\})$ is non fixed, that is, as in Theorem 1, such that T is conformal w.r.t. the two orderings in $\mathcal{C}$ different from $<_b$.

5. *We have: there are at least eight edges in G.* This is obtained directly from Claim 4 above.

6. *We have: all the vertices in $\{O_1, O_2, O_3\}$ have degree at least two.* Indeed, assume that a vertex in $\{O_1, O_2, O_3\}$ has degree zero or one, then, among the two other vertices in $\{O_1, O_2, O_3\}$, one has degree four and the other has a degree at least equal to three. Hence, those two orderings are adjacent to three common triplets. We get a contradiction with the definition of $<_x$ and $<_y$, because this pair has been assumed to maximize the number of triplets conformal w.r.t. it (and we assumed that this number was equal to two).

By Claim 5 and Claim 6, we have that: either there is at least one vertex of G in $\{O_1, O_2, O_3\}$ with degree 4, or there are at least two vertices of G in $\{O_1, O_2, O_3\}$ with degree 3. Figures 7 and 8 show respectively the two bipartite graphs with a minimal number of edges and satisfying one of these two properties, up to permutations of $\{T_1, ..., T_4\}$ and $\{O_1, O_2, O_3\}$. So we now assume, without loss of generality, that removing edges from G leads to one the two graphs depicted in Figures 7 and 8.

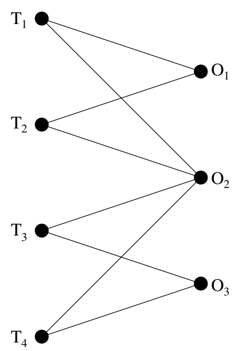


Figure 7: G minimal with one vertex having degree 4

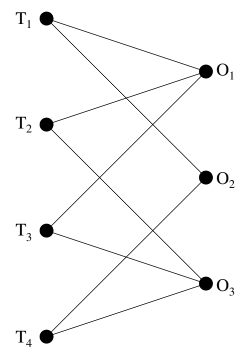


Figure 8: G minimal with two vertices having degree 3

Let us consider the two cases separately, according to these two properties.

- G has at least one vertex with degree 4 among $\{O_1, O_2, O_3\}$. See Figure 7.

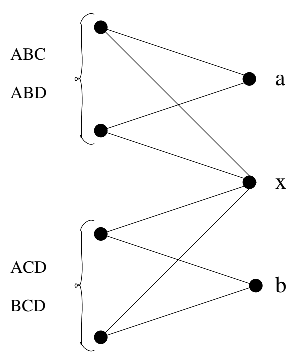


Figure 9: O_2 is $<_x$

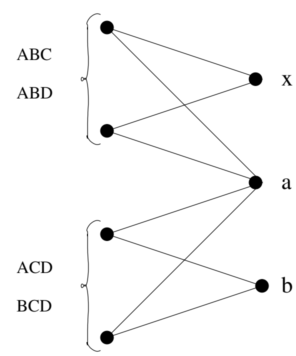


Figure 10: O_1 is $<_x$

First, assume that the ordering $<_x$ is the vertex O_2 with degree four. In this case, the two triplets T_1 and T_2 are adjacent to the same two orderings O_1 and $<_x$. By Claim 3 above, this implies that T_1 and T_2 form one of these pairs: $\{ABC, ABD\}$, $\{ABD, ACD\}$, $\{ACD, BCD\}$. The same result holds for the two triplets T_3 and T_4 adjacent to O_3 and $<_x$. Since these four triplets are distinct, $\{T_1, T_2\}$ and $\{T_3, T_4\}$ are not equal to $\{ABD, ACD\}$. Without loss of generality, we can assume that $\{T_1, T_2\} = \{ABC, ABD\}$ and $\{T_3, T_4\} = \{ACD, BCD\}$.

Let us denote $a, b \in \{y, z\}$ such that $<_a$ is O_1, and $<_b$ is O_3. This case is illustrated in Figure 9. In this case, ABC is adjacent in G to $<_x$ and $<_a$. Hence, based on Claim 1, the configuration induced by $\mathcal{C}$ on $(ABC, \{x, a\})$ is non-fixed. On the other hand, ACD is adjacent to $<_x$ and $<_b$ in G, meaning that $<_x$ and $<_b$ are equal or reversed on ACD. Since D is extreme in $<_x$, then D is extreme in $<_b$ restricted to ACD. Similarly, BCD is adjacent to $<_x$ and $<_b$ in G implies that D is extreme in $<_b$ restricted to BCD. Hence D is extreme in $<_b$. So we find that $\mathcal{C}$ is non-fixed by Lemma 3.

Second, assume that the ordering $<_x$ is the vertex O_1 or O_3. Without loss of generality, since O_1 and O_3 play symmetric roles in G, we can assume that $<_x$ is O_1. By Claim 3, this implies that T_1 and T_2 form one of these pairs: $\{ABC, ABD\}$, $\{ABD, ACD\}$, $\{ACD, BCD\}$.

We will prove that $\{T_1, T_2\}$ is not equal to $\{ABD, ACD\}$. Assume, for a contradiction, that $\{T_1, T_2\}$ is equal to $\{ABD, ACD\}$. Let $a, b \in \{y, z\}$ such that O_2 is $<_a$ and O_3 is $<_b$. Then, $(ABD, <_x)$ and $(ABD, <_a)$ are edges of G. By Claim 1, ABD is conformal w.r.t. $<_x$ and $<_a$. Since the ordering $<_x$ restricted to ABD is $A <_x B <_x D$, the ordering $<_a$ restricted to ABD is either $A <_a B <_a D$ or $D <_a B <_a A$. Similarly, with the triplet ACD, we have either $A <_a C <_a D$ or $D <_a C <_a A$. If $A <_a B <_a C <_a D$ or $D <_a C <_a B <_a A$, then the four triplets are conformal w.r.t. $<_x$ and $<_a$, which is a contradiction with the choice of $<_x$ and $<_y$ (they maximize the number of triplets conformal with a pair of orderings), and the assumption of Case 3 (this maximal number equals two). Therefore, we have either $A <_a C <_a B <_a D$ or $D <_a B <_a C <_a A$. On the other hand, the pair $\{T_3, T_4\}$ is equal to $\{ABC, BCD\}$. Since ABC is conformal w.r.t. $<_a$ and $<_b$, the ordering $<_b$ restricted to ABC is either $A <_b C <_b B$ or $B <_b C <_b A$. Similarly, with the triplet BCD we have either $C <_b B <_b D$ or $D <_b B <_b C$. So we have either $A <_b C <_b B <_b D$ or $D <_b B <_b C <_b A$. Hence, the four triplets are conformal w.r.t. $<_a$ and $<_b$, which is, similarly to above, a contradiction with the maximality property of $<_x$ and $<_y$. So $\{T_1, T_2\}$ is not equal to $\{ABD, ACD\}$.

If $\{T_1, T_2\}$ is equal to $\{ABC, ABD\}$. Figure 10 illustrates this case. We observe that BCD is conformal w.r.t. $<_y$ and $<_z$, meaning that the configuration induced by $\mathcal{C}$ on $(BCD, \{y, z\})$ is non-fixed. Since we have $A <_x B <_x C <_x D$, A is minimal in $<_x$. Hence, the configuration $\mathcal{C}$ is non-fixed by Lemma 3. If $\{T_1, T_2\}$ is equal to $\{ACD, BCD\}$. We observe that ABC is conformal w.r.t. $<_y$ and $<_z$, meaning that the configuration induced by $\mathcal{C}$ on $(ABC, \{y, z\})$ is non-fixed. Since we have $A <_x B <_x C <_x D$, D is maximal in $<_x$. Hence, the configuration $\mathcal{C}$ is non-fixed by Lemma 3.

- The graph G has at least 2 vertices with degree 3 among $\{O_1, O_2, O_3\}$. See Figure 8.

  By Claim 2, $<_x$ and $<_y$ have exactly two common neighbors. We first prove that, without loss of generality, we can assume that O_1 is $<_x$, and O_3 is $<_y$. First, assume that O_2 is $<_x$ or $<_y$ and O_2 has degree two. Then either O_1 or O_3 has degree four (in order to share two neighbors with O_2), implying that O_1 and O_3 have three common neighbors, which is a contradiction with the maximality property of $<_x$ and $<_y$ and the assumption of Case 3. So, if O_2 has degree two, then we necessarily have $\{<_x, <_y\} = \{O_1, O_3\}$. Now, if O_2 has degree at least three, then we can exchange O_2 with O_1 or O_3 (and some triplets accordingly) in order to have $\{<_x, <_y\} = \{O_1, O_3\}$. Finally, we have $\{<_x, <_y\} = \{O_1, O_3\}$, and O_1 and O_3 play symmetric roles in G, so we can choose to have $O_1 = <_x$ and $O_3 = <_y$.

  By Claim 3, there are 2 cases: either the vertices T_2 and T_3 form the pair $\{ACD, BCD\}$, or they form a pair among $\{ABC, ABD\}$ and $\{ABD, ACD\}$.

  In the first case, we have that ACD and BCD are conformal w.r.t. $<_x$ and $<_y$. Figure 11 shows this case. Hence, ABC is conformal w.r.t. $<_z$ and a second ordering $<_a$ for $\{a, b\} = \{x, y\}$. So the configuration induced by $\mathcal{C}$ on $(ABC, \{z, a\})$ is non-fixed, by Theorem 1. Assume $b = x$. Since D is extreme in $<_x$, then $\mathcal{C}$ is non-fixed by Lemma 3. Now, assume that $b = y$. Since D is extremal in $<_x$, and ACD is conformal w.r.t. $<_x$ and $<_y$, then D is also extreme in $<_y$ restricted to ACD. Similarly, with the triplet BCD, we get that D is extreme in the ordering $<_y$ restricted to BCD and

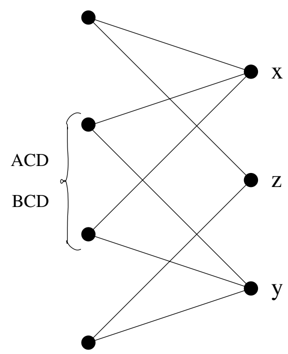


Figure 11: $\{T_2, T_3\} = \{ACD, BCD\}$

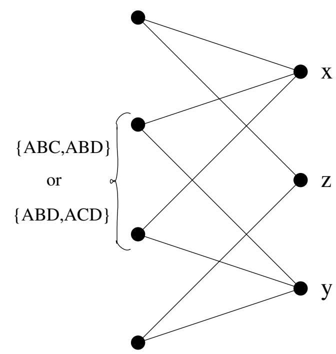


Figure 12: $\{T_2, T_3\} \neq \{ACD, BCD\}$

hence extreme in $<_y$. So $\mathcal{C}$ is non-fixed by Lemma 3.

In the second case, the pair of triplets conformal w.r.t. $<_x$ and $<_y$ is either $\{ABC, ABD\}$ or $\{ABD, ACD\}$. Figure 12 illustrates this case. So BCD is conformal w.r.t. $<_z$ and a second ordering $<_a$, for $\{a, b\} = \{x, y\}$. The configuration induced by $\mathcal{C}$ on $(BCD, \{z, a\})$ is non-fixed, by Theorem 1. Assume $b = x$. Since A is extreme in $<_x$, $\mathcal{C}$ is non-fixed by Lemma 3. Now, assume $b = y$. Similarly to our previous demonstration, since A is extreme in $<_x$, we get that A is extreme in $<_y$ when $\{ABC, ABD\}$ are both conformal w.r.t. $<_x$ and $<_y$, as well as when $\{ABD, ACD\}$ are both conformal w.r.t. $<_x$ and $<_y$. So $\mathcal{C}$ is non-fixed by Lemma 3.

Case 4: zero or one triplets of $\mathcal{E}$ are conformal w.r.t. $<_x$ and $<_y$. We will prove that this case is not possible. Since there is no triplet of $\mathcal{E}$ such that all the configurations induced by $\mathcal{C}$ on this triplet are fixed, each triplet is conformal w.r.t. at least two orderings of $\mathcal{C}$. There are three pairs of orderings and four triplets, so there is at least one pair of orderings such that two triplets are conformal w.r.t. this pair of orderings. □

5.3 Final proofs

Proof of Theorem 3. Recall that b) implies a) is given by Observation 1. We have c) implies b) as noted below Conjecture 2: if a configuration induced on a triplet is fixed then it is formally fixed by Theorem 2; as a consequence, formal fixity by expansion for $n = 4$ implies formal fixity for $n = 4$. We have d) implies c) by Proposition 3. Lastly, to prove that a) implies d), assume d) is false. If $\mathcal{C}$ is equivalent to a configuration whose orderings satisfy

$$\begin{array}{ccccc} B & <_x & C & <_x & A \\ C & <_y & A & <_y & B \\ A & <_z & B & <_z & C \end{array}$$

then it is non-fixed by Propositions 3. If $\mathcal{C}$ is not equivalent to such a configuration, then, by Proposition 2 and a permutation of $\mathcal{E}$, we have that: for every triplet $\mathcal{E}'$ of $\mathcal{E}$ there exists $b \in \mathcal{B}$ such that the configuration induced by $\mathcal{C}$ on $(\mathcal{E}', \mathcal{B} \setminus \{b\})$ is non-fixed. Then $\mathcal{C}$ is non-fixed by Proposition 4. So we find that a) is false, meaning that a) implies d). □

Proof of Theorem 4. This proof follows from the proofs of Propositions 3 and 4 which enumerate, by means of several cases, all possible non-fixed configurations up to equivalence. In every case, the fact that a configuration is non-fixed is proved using Lemma 3, up to equivalence. Hence, every non-fixed configuration is equivalent to a configuration satisfying the hypothesis of this lemma. Since permutations of $\mathcal{B}$ or $\mathcal{E}$ and ordering reversions obviously do not change this property, we get that every non-fixed configuration satisfies the hypothesis of this lemma. $\square$

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