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NP-hardness of the Sparsest $k$-Subgraph Problem in Chordal Graphs*

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Abstract
Given a simple undirected graph $G = (V, E)$ and an integer $k \leq |V|$, the Sparsest $k$-Subgraph problem asks for a set of $k$ vertices which induce the minimum number of edges. Whereas its special case Independent Set and many other optimization problems become polynomial-time solvable in chordal graphs, we show that Sparsest $k$-Subgraph remains NP-hard in this graph class.

1 Introduction and Preliminaries

In this report we study the following decision problem:
Sparsest $k$-Subgraph
- Input: a simple undirected graph $G = (V, E)$, $k \in \mathbb{N}$, $C \in \mathbb{N}$
- Question: is there a subset $S \subseteq V$ such that $|S| = k$ and $E(S) \leq C$? Where $E(S)$ is the number of edges induced by $S$.

As a generalization of the classical Independent Set problem (for which we have $C = 0$ in the input), Sparsest $k$-Subgraph is NP-hard [7] and even not approximable unless $P = NP$. Moreover, it is W[1]-hard (parameterized by $k$) [6].

Its maximization version, namely the $k$-Densest Subgraph (or the $k$-Cluster problem), has been extensively studied in the last three decades: in [5], the authors show that $k$-Densest Subgraph is NP-hard in bipartite, comparability and chordal graphs, and is polynomial-time solvable in trees, cographs, bounded treewidth graphs and split graphs. The question of the complexity status of $k$-Densest Subgraph in interval graphs (and even in proper interval graphs) is stated by the authors as an open problem, and is still not answered yet. In addition, [4] shows that both Sparsest $k$-Subgraph and $k$-Densest Subgraph are polynomial time solvable in bounded cliquewidth graphs. Notice that several exact or approximation algorithm exists for $k$-Densest Subgraph in subclasses of perfect graphs: among others, constant approximation algorithms are known for chordal graphs [10], bipartite permutation graphs [3] and PTAS are known for interval graphs [11] and for chordal graphs having a special clique tree [9]. Unfortunately, most of these results seem useless for Sparsest $k$-Subgraph, as we apparently need to complement the input graph to apply them. Nevertheless we can deduce that Sparsest $k$-Subgraph remains NP-hard in co-chordal (which is a subclass of perfect graphs) and is polynomial-time solvable in split graphs.

On the other side, its dual version, namely the Maximum Partial Vertex Cover problem, for which we are looking for $k$ vertices in the input graph which cover the maximum number of edges, is polynomial-time solvable in line graphs [2], and remains NP-hard in bipartite graphs [1,8].

In this report we study the complexity status of Sparsest $k$-Subgraph in chordal graphs.

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Whereas the independent set problem is polynomial-time solvable in perfect graphs (and thus in chordal graphs), we show that \textsc{sparsest $k$-subgraph} remains $\text{NP}$-hard in chordal graphs. Obviously, the same result holds for the maximum partial vertex cover problem.

The two following definitions of chordal graphs are equivalent:
- A graph is chordal if there it does not contain any cycle of length four or more as an induced subgraph.
- A vertex $v$ of $G$ is called simplicial if its neighbourhood $N(v)$ is a clique. The ordering $v_1, \ldots, v_n$ of the vertices of $G$ is a simplicial elimination scheme if for all $i$, $v_i$ is simplicial in $G[v_1, \ldots, v_n]$. A graph is chordal if it has a simplicial elimination scheme.

## 2 The Main Result

### 2.1 Idea of the Proof

The following $\text{NP}$-hardness proof is a reduction from the $k$-clique problem in general graphs. Roughly speaking, given an input instance $G = (V, E)$ together with $k \in \mathbb{N}$, we construct the split graph of adjacencies of $G$, i.e. we build a clique on a set $A$ representing the vertices of $G$, and an independent on a set $F$ representing the edges of $G$, connecting $A$ and $F$ with respect to the adjacencies of the graph. Then, we duplicate each vertex of $A$ $n$ times, creating thus a clique of size $n^2$. On the other hand, we replace each vertex of the independent set by a gadget. If $G$ contains a clique of size $k$, that is a set of $k$ vertices inducing $\binom{k}{2}$ edges, then the solution will take vertices not corresponding to vertices of the clique. Hence, there will be $\binom{k}{2}$ gadgets not adjacent to the solution. Finally, we will force the solution to take the same number of vertices among each gadget.

### 2.2 $\text{NP}$-hardness

**Theorem 1.** \textsc{sparsest $k$-subgraph} remains $\text{NP}$-hard in chordal graphs.

*Proof.* We reduce from the classical $k$-clique problem in general graphs. Let $G = (V, E)$ and $k \in \mathbb{N}$. We note $|V| = n$, $V = \{v_1, \ldots, v_n\}$, $|E| = m$ and $T = n(n - k)$. In the following we will define $G' = (V', E')$ together with $k', C' \in \mathbb{N}$ such that:
- $G', k', C'$ can be constructed in polynomial time
- $G'$ is a chordal graph.
- $G$ contains a clique of size $k$ if and only if one can find $k'$ vertices in $G'$ which induce $C'$ edges or less.

**The construction**

$V'$ is composed of two parts $A$ and $F$.
- We first define a clique over $A = \{a_{ij}^t : i, j \in \{1, \ldots, n\}\}$. Thus, $A$ is a clique of size $n^2$. Moreover, for all $j \in \{1, \ldots, n\}$, we note $A_j = \{a_{1j}^t, \ldots, a_{nj}^t\}$.
- For all $e \in E$, we construct a graph with $F_e$ as vertex set, where $F_e$ is composed of three sets of $T$ vertices: $X_e = \{x_{e1}^t, \ldots, x_{eT}^t\}$, $Y_e = \{y_{e1}^t, \ldots, y_{eT}^t\}$ and $Z_e = \{z_{e1}^t, \ldots, z_{eT}^t\}$. The set $X^e$ induces a stable set, $Z^e$ induces a clique, and $Y^e$ contains a clique of size $T - 1$ on vertices $\{y_{e2}^t, \ldots, y_{eT}^t\}$ (thus, $y_{e1}^t$ is not connected to the other vertices of $Y^e$). Then, for all $j \in \{1, \ldots, T\}$, $x_{e-j}^t$ is connected to $y_{e-j}^t$, and $y_{e-j}^t$ is connected to all vertices of $Z^e$. An example of such a gadget is represented in Figure 1. We define $F = \bigcup_{e \in E} F_e$. 

- For all \( e = \{v_p, v_q\} \in E \), all vertices of \( Z^e \) are connected to \( \{a^e_j : j \in \{1, ..., n\}\} \) and \( \{a^e_j : j \in \{1, ..., n\}\} \).
- We define \( k' = m' + T \) and \( C' = m'(T) + (T) + (m - \binom{k}{2}) \).

**Lemma 1.** \( G' \) is a chordal graph

**Proof.** We have the following simplicial elimination scheme:
- For all \( e \in E \), we can remove \( X_e \) since for all \( j \in \{1, ..., T\} \), \( x^e_j \) is only connected to \( y^e_j \).
- For all \( e \in E \), we can remove \( Y_e \). Indeed the remaining neighbourhood of \( y^e_1 \) is \( Z^e \) which is a clique. And the remaining neighbourhood of \( y^e_j \) with \( j \geq 2 \) is a subset \( Y_e \cup Z_e \backslash \{y^e_1\} \) which induces a clique.
- For all \( e \in E \), we can remove \( Z_e \) since the remaining neighbourhood of \( z^e_j \) is a subset of \( Z_e \) and vertices of \( A \) which induce a clique.
- The remaining vertices induces a clique on \( A \) and thus be eliminated.

Now we prove that \( G \) contains a clique of size \( k \) if and only if \( G' \) contains \( k' \) vertices inducing at most \( C' \) edges.

**G contains a k-clique \( \Rightarrow \) G' contains k' vertices inducing at most C' edges.**

Let us suppose that \( K \subseteq V \) is a clique of size \( k \) in \( G \). Without loss of generality we suppose \( K = \{v_1, ..., v_k\} \). Moreover, we note \( E_0 = \{\{v_p, v_q\} \in E \text{ such that } v_p, v_q \in K\} \) and \( E_1 = \{\{v_p, v_q\} \in E \text{ such that } v_p \notin K \text{ or } v_q \notin K\} \). We construct \( K' \subseteq V' \) as follows:
- For all \( i \in \{k + 1, ..., n\} \) and all \( j = \{1, ..., n\} \), we add \( a^i_j \) to \( K' \).
- For all \( e \in E \), we add all vertices of \( X_e \) to \( K' \).
For all $e \in E_0$, we add all vertices of $Z_e$ to $K'$.
For all $e \in E_1$, we add all vertices of $Y_e$ to $K'$.

One can verify that $K'$ is a set of $k' = 2mT + T$ vertices inducing exactly $C' = m(k') + (\ell_1 + (m - (\ell_1))$ edges. Indeed, we picked $T = n(n - k)$ vertices from $A$ which is a clique and thus induce $\binom{\ell_1}{2}$ edges. Then, for all $e \in E$, we picked $2T$ vertices, which induce $\binom{\ell_2}{2}$ edges if $e \in E_0$, and $((\ell_1 + 1) + (\ell_2 - (\ell_1))$ edges if $e \in E_1$. Since $|E_0| = \binom{\ell_1}{2}$ (and thus $|E_1| = m - \binom{\ell_1}{2}$), we have the desired number of edges.

**G contains a $k$-clique $\iff G'$ contains $k'$ vertices inducing at most $C'$ edges.**

Suppose now that $K'$ is a set of $k'$ vertices of $G'$ which induces at most $C'$ edges. We redefine the sets $E_0$ and $E_1$ as follows: $E_0 = \{(v_p, v_q) \in E \text{ such that for all } j \in \{1, ..., n\} \text{ we have } a_{pj} \notin K' \text{ and } a_{qj} \notin K'\}$, and $E_1 = E \setminus E_0$.

For all $R \subseteq V'$, let $tr(R) = K' \cap R$ be the trace of $K'$ on $R$, and for all $v \in V'$, let $\mu(v) = |tr(N(v))|$ be the number of neighbors of $v$ belonging to $K'$.

Let $u \in K'$ and $v \in V' \setminus K'$. We say that $(K' \setminus \{u\}) \cup \{v\}$ is a safe replacement if and only if we have $\mu(v) \leq \mu(u)$ if $\{u, v\} \notin E'$ and $\mu(v) - 1 \leq \mu(u)$ if $\{u, v\} \in E'$. For sake of readability, we will keep and update the definitions of $E_0$ and $E_1$ when replacing vertices of $A$ (e.g. if we remove a vertex $u \in A$ from $K'$ and that there exists $e \in E_1$ such that vertices of $Z_e$ were only adjacent to $u$ among vertices of $A$, then $e$ now belongs to $E_0$).

The proof consists in replacing some vertices of $K'$ by other vertices not in $K'$ without increasing the number of induced edges, in order to obtain a solution that has the same structure as previously. We call such a replacement a safe modification or a safe replacement.

The core of the proof is based on the three following lemmas.

**Lemma 2.** Without loss of generality (and optimality of $K'$), we can suppose that for all $e \in E$ we have $X_e \subseteq K'$.

**Proof.** Let $S = \bigcup_{e \in E} X_e$. Since we have $k' > |S|$, there always exists $u \in K' \setminus S$. Suppose that there exists $e \in E$ and $i \in \{1, ..., T\}$ such that $x^e_i \notin K'$. If $y^e_i \notin K'$, then we have $\mu(x_i^e) = 0$ and we can thus safely replace any other vertex of $K' \setminus S$ by $x_i^e$. Now, if $y_i^e \in K'$, then $\mu(x_i^e) = 1$. Since $\{x_i^e, y_i^e\} \in E'$, $(K' \setminus \{y_i^e\}) \cup \{x_i^e\}$ is a safe replacement. \hfill $\Box$

**Lemma 3.** $K'$ can be safely modified such that one of the two following holds:

Case A1: for all $e \in E_0$ we have $tr(Z_e) = Z_e$.

Case A2: for all $e \in E_0$ we have $tr(Y_e) = \emptyset$.

**Proof.** Let us first restructure each gadget of $E_0$ separately. For all $e \in E_0$ such that $tr(Y_e) \neq \emptyset$ and $tr(Z_e) \neq Z_e$, let $j_0 = \max\{j \in \{1, ..., T\} : y^e_j \in tr(Y_e)\}$ and let $j_1$ be such that $z^e_j \notin tr(Z_e)$. Recall that Lemma 2 ensures that $x^e_{j_0}$ is in $K'$. If $j_0 \neq 1$, then $\mu(y^e_{j_0}) = y + z + 1$, where $y = |N(y^e_{j_0}) \cap tr(Y_e)|$ and $z = |N(y^e_{j_0}) \cap tr(Z_e)|$. On the other side, we have $\mu(z^e_{j_1}) \leq y + z + 1$ (more precisely, $\mu(z^e_{j_1}) = y + z + 1$ if $y^e_{j_1} \in K'$, and $\mu(z^e_{j_1}) = y + z$ if $y^e_{j_1} \notin K'$). Roughly speaking, this switch ensures that we necessarily “lose” the edge due to the vertex of $X'$ and we gain at most one edge due to $y^e_{j_1}$. Hence $\mu(z^e_{j_1}) \leq \mu(y^e_{j_0})$ and $(K' \setminus \{y^e_{j_0}\}) \cup \{z^e_{j_1}\}$ is a safe replacement. If $j_0 = 1$, then it means that $tr(Y_e) = \{y^e_1\}$. Suppose that there exists $j_1$ such that $z^e_{j_1} \notin tr(Z_e)$. We have $\mu(y^e_{j_1}) = z + 1$ where $z = |N(y^e_{j_1}) \cap tr(Z_e)|$, and $\mu(z^e_{j_1}) = z + 1$.

Here again $(K' \setminus \{y^e_{j_1}\}) \cup \{z^e_{j_1}\}$ is a safe replacement. After all these replacements, given any $e \in E_0$, $tr(Y_e) \neq \emptyset$ implies that $tr(Z_e) = Z_e$.

Then, we proceed to replacements between gadgets $F_e$, $e \in E_0$. If one can find $a, b \in E_0$ such that $tr(Y_a) \neq \emptyset$ and $tr(Z_b) \neq Z_b$, then let $j_0$ be such that $y^a_{j_0} \in tr(Y_a)$ and let $j_1$ be such that $z^b_{j_1} \notin tr(Z_b)$. We have $\mu(y^a_{j_0}) \geq T + 1$ and $\mu(z^b_{j_1}) \leq T - 1$. Thus, $(K' \setminus \{y^a_{j_0}\}) \cup \{z^b_{j_1}\}$ is a safe replacement.

Theses replacements end either when all the $Y_e$ are empty for all $e \in E_0$ or when all the $Z_e$ are full for all $e \in E_0$, which achieves the proof of Lemma 3. \hfill $\Box$
Lemma 4. $K'$ can be safely modified such that one of the two following holds:

Case B1: for all $e \in E_1$ we have $tr(Y_0) = Y_0$.

Case B2: for all $e \in E_1$ we have $tr(Z_0) = \emptyset$.

Proof. The proof is roughly based on the fact that replacing a vertex of $Z_0$ by a vertex of $Y_0$ permits to "lose" at least one edge with vertices $A$ and "gain" one edge with a vertex of $X_0$. Let us formally prove Lemma 4. Similarly to the proof of Lemma 3, we first restructure each gadget of $E_0$ separately: for all $e \in E_0$ such that $tr(Z_0) \neq \emptyset$ and $tr(Y_0) \neq Y_0$, let $j_0 = \max \{j \in \{1,...,T\} : y_j \notin K'\}$ and let $j_1$ be such that $z_{j_1} \in tr(Z_0)$. Recall that by definition of $E_1$, there exists $i,j \in \{1,...,n\}$ such that $z_{j_1}$ is adjacent to $a_i$. We have $\mu(z_{j_1}) \geq y + z + 1$, where $y = |N(z_{j_1}) \cap Y_0|$ and $z = |N(z_{j_1}) \cap Z_0|$. On the other side, we have $\mu(y_{j_0}) \leq z + y + 2$ (indeed, $|N(y_{j_0}) \cap Z_0| = z + 1$, $|N(y_{j_0}) \cap Y_0| \leq y$ and $|N(y_{j_0}) \cap X_0| = 1$).

Since $\{y_{j_0}, z_{j_1}\} \in E'$, it holds that $(K' \setminus \{z_{j_1}\}) \cup \{y_{j_0}\}$ is a safe replacement. After all these replacements, given any $e \in E_1$, $tr(Z_0) \neq \emptyset$ implies that $tr(Y_0) = Y_0$.

We now proceed to replacements between gadgets $F_e, e \in E_1$. If one can find $a,b \in E_1$ such that $tr(Z_0) \neq \emptyset$ and $tr(Y_0) \neq Y_0$, let $j_0$ be such that $y_{j_0} \notin tr(Y_0)$ and let $j_1$ be such that $z_{j_1} \in tr(Z_0)$. We have $\mu(y_{j_0}) \geq T + 1$ and $\mu(y_{j_0}) \leq T - 1$. Thus $(K' \setminus \{z_{j_1}\}) \cup \{y_{j_0}\}$ is a safe replacement. \hfill \Box

Let us now define for each case and each $e \in E$ the set of vertices $D_e \subseteq Y_e \cup Z_e$ that have to be replaced (see Figure 2):

- case A1: for all $e \in E_0$, $D_e = Y_e \cap K'$
- case A2: for all $e \in E_0$, $D_e = Z_e \setminus K'$
- case B1: for all $e \in E_1$, $D_e = Z_e \cap K'$
- case B2: for all $e \in E_1$, $D_e = Y_e \setminus K'$

Notice that if $D_e = \emptyset$ for all $e \in E_0$ (resp. $e \in E_1$), then cases A1 and A2 (resp. B1 and B2) collapse. If such a case happen for all $e \in E$, we can immediately conclude, as shown by the following lemma:

Lemma 5. If $D_e = \emptyset$ for all $e \in E$, then $G$ contains a clique of size $k$.

Proof. By construction, we have $|tr(A)| = T$ and $|tr(F_e)| = 2T$ for all $e \in E$. Thus, $\text{cost}^*(tr(A)) = \binom{\ell}{2}$ and $\text{cost}^*(tr(F_e)) = \binom{\ell}{2} + 1$ if $Y_e \subseteq K'$, and $\text{cost}^*(tr(F_e)) = \binom{\ell}{2}$ if $Z_e \subseteq K'$. By construction, $Y_e \subseteq K'$ if and only if $e \in E_1$. Thus, since $\text{cost}^*(K') \leq \frac{\ell}{2} + m(\frac{\ell}{2}) + m - \frac{\ell}{2}$, we must have $|E_1| \leq m - \frac{\ell}{2}$ which is equivalent to $|E_0| \geq \frac{\ell}{2}$. Hence, there exists at most $\left\lfloor \frac{|A| - T}{n} \right\rfloor = k$ vertices in $G$ inducing at least $\binom{\ell}{2}$ edges, i.e. $G$ contains a clique of size $k$. \hfill \Box

We now have to analyse the four cases of Lemma 3 and 4 (see Figure 2).

Case A1 and B1

To summarize the situation, the solution $K'$ can be partitionned in $K'_A = K' \cap A$, and $K'_{B'} = K' \setminus K'_{A}$, the vertices selected in the gadgets. Let $\Delta_0 = \sum_{e \in E_0} |D_e|$ be the number of extra vertices allocated in all the gadgets $F_e, e \in E_0$, and $\Delta_1 = \sum_{e \in E_1} |D_e|$ be the number of extra vertices allocated in all the gadgets $F_e, e \in E_1$. Let $\Delta = \Delta_0 + \Delta_1$. Notice that we have $|K'_{A} = T - \Delta$, as a "regular" solution that does not select any extra vertex in a gadget has to pick $T$ vertices in $A$. Moreover,

- vertices of $K'$ selected in gadgets of $E_0$ are not adjacent to $K'_{A}$ (by definition of $E_0$)
- each gadget of $E_0$ induces at least $\binom{\ell}{2}$ edges (as we are in case A1)
- each gadget of $E_1$ induces at least $\binom{\ell}{2} + 1$ edges (as we are in case B1)
- each of the $\Delta_0$ vertices is adjacent to at least $T$ vertices in $K'$ (such a vertex is in a set $Y_e$, and thus is connected to the $T$ vertices of $Z_e$)
- each of the $\Delta_1$ vertices is adjacent to at least $T + 1$ vertices in $K'$ (such a vertex is in a set $Z_e$, and thus is connected to at least 1 vertex of $K'_A$ and to the $T$ vertices of $Y_e$)

Let us now lower bound the total cost of $K'$. We have

$$
cost^*(K') \geq |E_0| \left( \frac{T}{2} + |E_1| \left( \frac{T}{2} + 1 \right) + \Delta_0 T + \Delta_1 (T+1) + \left( \frac{T - \Delta}{2} \right) \right)
$$

$$
\geq |E_0| \left( \frac{T}{2} + |E_1| \left( \frac{T}{2} + 1 \right) + \Delta T + \left( \frac{T - \Delta}{2} \right) \right)
$$

$$
\geq |E_0| \left( \frac{T}{2} + |E_1| \left( \frac{T}{2} + 1 \right) + \left( \frac{T}{2} \right) + \frac{\Delta^2}{2} \right)
$$

Notice that in a bad structured solution, a large $\Delta$ allows to select only a few vertices in $A$ ($T - \Delta$ instead of $T$), and thus to have many gadgets (more than $\binom{\Delta}{2}$) in $E_0$. Let us now consider the contrapositive, i.e. we consider that $G$ does not contain a $k$-clique, and show that $K'$ induces more than $C'$ edges.

Let $q$ and $r$ such that $\Delta = qn + r$, $r < n$. Let us upper bound $|E_0|$. As there is $T - \Delta$ vertices in $A$, the number of empty "columns" (column $u$ is empty iff none of the $a_u^e$ is selected) is at most $n - \frac{T - \Delta}{2} \leq k + q$.

As $G$ does not contain a $k$-clique, the $k + q$ vertices corresponding to these $k + q$ columns cannot induce a clique of size $k + q$, and thus $|E_0| < \binom{k+q}{2}$. Thus, we get

$$
cost^*(K') > \left( \frac{k + q}{2} \right) \left( \frac{T}{2} \right) + (m - \left( \frac{k + q}{2} \right)) \left( \frac{T}{2} + 1 \right) + \left( \frac{T}{2} \right) + \frac{\Delta^2}{2}
$$

$$
= C' - \left( \frac{q}{2} \right) + kq + \frac{\Delta^2}{2}
$$

Thus, as $\frac{\Delta^2}{2} > \left( \frac{q}{2} \right) + kq$, we get the desired inequality.

**Case A2 and B2**

Let $\Delta_0 = \sum_{e \in E_0} |D_e|$, $\Delta_1 = \sum_{e \in E_1} |D_e|$ and $\Delta = \Delta_0 + \Delta_1$ (recall that in this case, $D_e \not\subseteq K'$ for all $e \in E$). Here again we suppose $\Delta > 0$. Let us notice that for all $u \in tr(A)$, $\mu(u) \geq T$.

On the other hand, for all $e \in E$ such that there exists $v \in D_e$, we have $\mu(v) \leq T$ (remark that if $e \in E_1$, then $D_e \subseteq Y_e$, and if $e \in E_0$, then $v$ is not adjacent to $tr(A)$ by definition of $E_0$). Thus $(K' \backslash \{u\}) \cup \{v\}$ is a safe replacement. Since before this replacement we had
\[ tr(A) = T + \Delta, \] it is clear that we can repeat this replacement (i.e. \( K' \setminus \{u\} \cup \{v\} \)) where \( u \in tr(A) \) and \( v \in D_c \) for some \( e \in E \) \( \Delta \) times safely. At this point, the updated value of \( \Delta \) is 0, i.e. \( D_c = \emptyset \) for all \( e \in E \). By Lemma 5, we must have a clique of size \( k \) in \( G \).

**Case A2 and B1**

If there exists \( e \in E_0 \) such that there exists \( u \in D_e \), then \( \mu(u) < T \). If such a vertex exists, then either \( |tr(A)| > T \) or there exists \( e' \in E_1 \) such that there exists \( v \in D_{e'} \). In the first case for all \( x \in tr(A) \) we have \( \mu(x) \geq T \), and \((K' \setminus \{x\}) \cup \{u\}\) is a safe replacement. In the second case we have \( \mu(v) > T \) and here again \((K' \setminus \{v\}) \cup \{u\}\) is a safe replacement. After these replacements we must have \( D_c = \emptyset \) for all \( e \in E_0 \), and we can apply the same arguments as for case A1 and B1.

**Case A1 and B2**

If there exists \( e \in E_1 \) such that there exists \( u \in D_e \), then \( \mu(u) < T \). If such a vertex exists, then either \( |tr(A)| > T \) or there exists \( e' \in E_0 \) such that there exists \( v \in D_{e'} \). In the first case for all \( x \in tr(A) \) we have \( \mu(x) \geq T \), and \((K' \setminus \{x\}) \cup \{u\}\) is a safe replacement. In the second case we have \( \mu(v) > T \) and here again \((K' \setminus \{v\}) \cup \{u\}\) is a safe replacement. After these replacements we must have \( D_c = \emptyset \) for all \( e \in E_1 \), and we can apply the same arguments as for case A1 and B1.

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