NP-hardness of the Sparsest k-Subgraph Problem in Chordal Graphs

Rémi Watrigant, Marin Bougeret, Rodolphe Giroudeau

$\boldsymbol{NP}$-hardness of the SPARSEST $\boldsymbol{k}$-SUBGRAPH Problem in Chordal Graphs*

R. Watrigant, M. Bougeret, and R. Giroudeau

LIRMM-CNRS-UMR 5506 - 161, rue Ada 34090 Montpellier, France

Abstract Given a simple undirected graph $G = (V, E)$ and an integer $k \leq |V|$, the SPARSEST k-SUBGRAPH problem asks for a set of k vertices which induce the minimum number of edges. Whereas its special case INDEPENDENT SET and many other optimization problems become polynomial-time solvable in chordal graphs, we show that SPARSEST k-SUBGRAPH remains NP-hard in this graph class.

1 Introduction and Preliminaries

In this report we study the following decision problem:
SPARSEST k-SUBGRAPH
- Input: a simple undirected graph $G = (V, E)$, $k \in \mathbb{N}$, $C \in \mathbb{N}$
- Question: is there a subset $S \subseteq V$ such that $|S| = k$ and $E(S) \leq C$? Where $E(S)$ is the number of edges induced by S.

As a generalization of the classical INDEPENDENT SET problem (for which we have $C = 0$ in the input), SPARSEST k-SUBGRAPH is NP-hard [7] and even not approximable unless $P = NP$. Moreover, it is $W[1]$-hard (parameterized by k) [6].
Its maximization version, namely the k-DENSEST SUBGRAPH (or the k-CLUSTER problem), has been extensively studied in the last three decades: in [5], the authors show that k-DENSEST SUBGRAPH is $\mathcal{NP}$-hard in bipartite, comparability and chordal graphs, and is polynomial-time solvable in trees, cographs, bounded treewidth graphs and split graphs. The question of the complexity status of k-DENSEST SUBGRAPH in interval graphs (and even in proper interval graphs) is stated by the authors as an open problem, and is still not answered yet. In addition, [4] shows that both SPARSEST k-SUBGRAPH and k-DENSEST SUBGRAPH are polynomial time solvable in bounded cliquewidth graphs. Notice that several exact or approximation algorithm exists for k-DENSEST SUBGRAPH in subclasses of perfect graphs: among others, constant approximation algorithms are known for chordal graphs [10], bipartite permutation graphs [3] and $PTAS$ are known for interval graphs [11] and for chordal graphs having a special clique tree [9]. Unfortunately, most of these results seem useless for SPARSEST k-SUBGRAPH, as we apparently need to complement the input graph to apply them. Nevertheless we can deduce that SPARSEST k-SUBGRAPH remains NP-hard in co-chordal (which is a subclass of perfect graphs) and is polynomial-time solvable in split graphs.
On the other side, its dual version, namely the MAXIMUM PARTIAL VERTEX COVER problem, for which we are looking for k vertices in the input graph which *cover* the maximum number of edges, is polynomial-time solvable in line graphs [2], and remains NP-hard in bipartite graphs [1,8].
In this report we study the complexity status of SPARSEST k-SUBGRAPH in chordal graphs.

* This work has been funded by grant ANR **2010** BLAN **021902**

Whereas the INDEPENDENT SET problem is polynomial-time solvable in perfect graphs (and thus in chordal graphs), we show thatSPARSEST k-SUBGRAPH remains NP-hard in chordal graphs. Obviously, the same result holds for the MAXIMUM PARTIAL VERTEX COVER problem.

The two following definitions of chordal graphs are equivalent:
- A graph is chordal if there it does not contain any cycle of length four or more as an induced subgraph.
- A vertex v of G is called simplicial if its neighbourhood $N(v)$ is a clique. The ordering $v_1, ..., v_n$ of the vertices of G is a simplicial elimination scheme if for all i, v_i is simplicial in $G[v_i, ..., v_n]$. A graph is chordal if it has a simplicial elimination scheme.

2 The Main Result

2.1 Idea of the Proof

The following $\mathcal{NP}$-hardness proof is a reduction from the k-CLIQUE problem in general graphs. Roughly speaking, given an input instance $G = (V, E)$ together with $k \in \mathbb{N}$, we construct the split graph of adjacencies of G, *i.e.* we build a clique on a set A representing the vertices of G, and an independent on a set F representing the edges of G, connecting A and F with respect to the adjacencies of the graph. Then, we duplicate each vertex of A n times, creating thus a clique of size n^2. On the other hand, we replace each vertex of the independent set by a gadget. If G contains a clique of size k, that is a set of k vertices inducing $\binom{k}{2}$ edges, then the solution will take vertices *not* corresponding to vertices of the clique. Hence, there will be $\binom{k}{2}$ gadgets *not* adjacent to the solution. Finally, we will force the solution to take the same number of vertices among each gadget.

2.2 $\mathcal{NP}$-hardness

Theorem 1. SPARSEST k-SUBGRAPH *remains $\mathcal{NP}$-hard in chordal graphs.*

Proof. We reduce from the classical k-CLIQUE problem in general graphs. Let $G = (V, E)$ and $k \in \mathbb{N}$. We note $|V| = n$, $V = \{v_1, ..., v_n\}$, $|E| = m$ and $T = n(n - k)$. In the following we will define $G' = (V', E')$ together with $k', C' \in \mathbb{N}$ such that:
- G', k', C' can be constructed in polynomial time
- G' is a chordal graph
- G contains a clique of size k if and only if one can find k' vertices in G' which induce C' edges or less.

The construction

V' is composed of two parts A and F.
- We first define a clique over $A = \{a_i^j : i, j \in \{1, ..., n\}\}$. Thus, A is a clique of size n^2. Moreover, for all $j \in \{1, ..., n\}$, we note $A_j = \{a_1^j, ..., a_n^j\}$.
- For all $e \in E$, we construct a graph with F_e as vertex set, where F_e is composed of three sets of T vertices: $X_e = \{x_1^e, ..., x_T^e\}$, $Y_e = \{y_1^e, ..., y_T^e\}$ and $Z_e = \{z_1^e, ..., z_T^e\}$. The set X^e induces a stable set, Z^e induces a clique, and Y^e contains a clique of size $T - 1$ on vertices $\{y_2^e, ..., y_T^e\}$ (thus, y_1^e is not connected to the other vertices of Y^e). Then, for all $j \in \{1, ..., T\}$, x_j^e is connected to y_j^e, and y_j^e is connected to all vertices of Z^e. An example of such a gadget is represented in Figure 1. We define $F = \bigcup_{e \in E} F_e$.

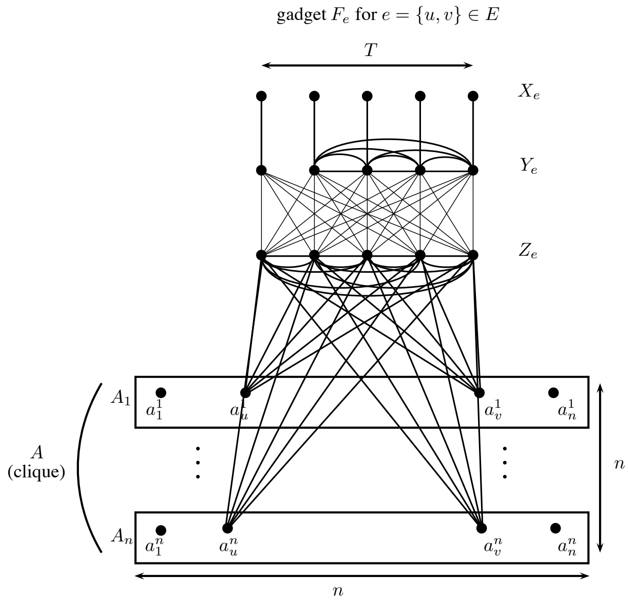

Figure1: Example of a gadget F_e (with $T = 5$) and its relations to A.

- For all $e = \{v_p, v_q\} \in E$, all vertices of Z^e are connected to $\{a_p^j : j \in \{1, ..., n\}\}$ and $\{a_q^j : j \in \{1, ..., n\}\}$.
- We define $k' = m2T + T$ and $C' = m\binom{T}{2} + \binom{T}{2} + (m - \binom{k}{2})$.

Lemma 1. *G' is a chordal graph*

Proof. We have the following simplicial elimination scheme:
- For all $e \in E$, we can remove X^e since for all $j \in \{1, ..., T\}$, x_j^e is only connected to y_j^e.
- For all $e \in E$, we can remove Y^e. Indeed the remaining neighbourhood of y_1^e is Z^e which is a clique. And the remaining neighbourhood of y_j^e with $j \geq 2$ is a subset $Y^e \cup Z^e \setminus \{y_1^e\}$ which induces a clique.
- For all $e \in E$, we can remove Z^e since the remaining neighbourhood of z_j^e is a subset of Z^e and vertices of A which induce a clique.
- The remaining vertices induces a clique on A and thus be eliminated.

□

Now we prove that G contains a clique of size k if and only if G' contains k' vertices inducing at most C' edges.

G contains a k-clique $\Rightarrow$ G' contains k' vertices inducing at most C' edges.

Let us suppose that $K \subseteq V$ is a clique of size k in G. Without loss of generality we suppose $K = \{v_1, ..., v_k\}$. Moreover, we note $E_0 = \{\{v_p, v_q\} \in E$ such that $v_p, v_q \in K\}$ and $E_1 = \{\{v_p, v_q\} \in E$ such that $v_p \notin K$ or $v_q \notin K\}$. We construct $K' \subseteq V'$ as follows:
- For all $i \in \{(k+1), ..., n\}$ and all $j = \{1, ..., n\}$, we add a_i^j to K'.
- For all $e \in E$, we add all vertices of X_e to K'.

- For all $e \in E_0$, we add all vertices of Z_e to K'.
- For all $e \in E_1$, we add all vertices of Y_e to K'.

One can verify that K' is a set of $k' = 2mT + T$ vertices inducing exactly $C' = m\binom{T}{2} + \binom{T}{2} + (m - \binom{k}{2})$ edges. Indeed, we picked $T = n(n-k)$ vertices from A which is a clique and thus induce $\binom{T}{2}$ edges. Then, for all $e \in E$, we picked $2T$ vertices, which induce $\binom{T}{2}$ edges if $e \in E_0$, and $(\binom{T}{2} + 1)$ edges if $e \in E_1$. Since $|E_0| = \binom{k}{2}$ (and thus $|E_1| = m - \binom{k}{2}$), we have the desired number of edges.

G contains a k-clique $\Leftarrow$ G' contains k' vertices inducing at most C' edges.

Suppose now that K' is a set of k' vertices of G' which induces at most C' edges. We redefine the sets E_0 and E_1 as follows: $E_0 = \{\{v_p, v_q\} \in E$ such that for all $j \in \{1, ..., n\}$ we have $a_p^j \notin K'$ and $a_q^j \notin K'\}$, and $E_1 = E \backslash E_0$.
For all $R \subseteq V'$, let $tr(R) = K' \cap R$ be the trace of K' on R, and for all $v \in V'$, let $\mu(v) = |tr(N(v))|$ be the number of neighbors of v belonging to K'.
Let $u \in K'$ and $v \in V' \backslash K'$. We say that $(K' \backslash \{u\}) \cup \{v\}$ is a safe replacement if and only if we have $\mu(v) \leq \mu(u)$ if $\{u, v\} \notin E'$ and $\mu(v) - 1 \leq \mu(u)$ if $\{u, v\} \in E'$. For sake of readability, we will keep and update the definitions of E_0 and E_1 when replacing vertices of A (*e.g.* if we remove a vertex $u \in A$ from K' and that there exists $e \in E_1$ such that vertices of Z_e were only adjacent to u among all vertices of A, then e now belongs to E_0).
The proof consists in replacing some vertices of K' by other vertices not in K' without increasing the number of induced edges, in order to obtain a solution that has the same structure as previously. We call such a replacement a *safe modification* or a *safe replacement*. The core of the proof is based on the three following lemmas.

Lemma 2. *Without loss of generality (and optimality of K'), we can suppose that for all $e \in E$ we have $X_e \subseteq K'$.*

Proof. Let $S = \bigcup_{e \in E} X_e$. Since we have $k' > |S|$, there always exists $u \in K' \backslash S$. Suppose that there exists $e \in E$ and $i \in \{1, ..., T\}$ such that $x_i^e \notin K'$. If $y_i^e \notin K'$, then we have $\mu(x_i^e) = 0$ and we can thus safely replace any other vertex of $K' \backslash S$ by x_i^e. Now, if $y_1^e \in K'$, then $\mu(x_i^e) = 1$. Since $\{x_i^e, y_i^e\} \in E'$, $(K' \backslash \{y_1^e\}) \cup \{x_i^e\}$ is a safe replacement. □

Lemma 3. *K' can be safely modified such that one of the two following holds:*
Case A1: for all $e \in E_0$ we have $tr(Z_e) = Z_e$.
Case A2: for all $e \in E_0$ we have $tr(Y_e) = \emptyset$.

Proof. Let us first restructure each gadget of E_0 separately. For all $e \in E_0$ such that $tr(Y_e) \neq \emptyset$ and $tr(Z_e) \neq Z_e$, let $j_0 = \max\{j \in \{1, ..., T\} : y_j^e \in tr(Y_e)\}$ and let j_1 be such that $z_{j_1}^e \notin tr(Z_e)$. Recall that Lemma 2 ensures that $x_{j_0}^e$ is in K'. If $j_0 \neq 1$, then $\mu(y_{j_0}^e) = y + z + 1$, where $y = |N(y_{j_0}^e) \cap tr(Y_e)|$ and $z = |N(y_{j_0}^e) \cap tr(Z_e)|$. On the other side, we have $\mu(z_{j_1}^e) \leq y + z + 1$ (more precisely, $\mu(z_{j_1}^e) = y + z + 1$ if $y_1^e \in K'$, and $\mu(z_{j_1}^e) = y + z$ if $y_1^e \notin K'$). Roughly speaking, this switch ensures that we necessarily "loose" the edge due to the vertex of X^e and we gain at most one edge due to y_1^e. Hence $\mu(z_{j_1}^e) \leq \mu(y_{j_0}^e)$ and $(K' \backslash \{y_{j_0}^e\}) \cup \{z_{j_1}^e\}$ is a safe replacement. If $j_0 = 1$, then it means that $tr(Y_e) = \{y_1^e\}$. Suppose that there exists j_1 such that $z_{j_1}^e \notin tr(Z_e)$. We have $\mu(y_1^e) = z + 1$ where $z = |N(y_1^e) \cap tr(Z_e)|$, and $\mu(z_{j_1}^e) = z + 1$. Here again $(K' \backslash \{y_1^e\}) \cup \{z_{j_1}^e\}$ is a safe replacement. After all these replacements, given any $e \in E_0$, $tr(Y_e) \neq \emptyset$ implies that $tr(Z_e) = Z_e$.
Then, we proceed to replacements between gadgets F_e, $e \in E_0$. If one can find $a, b \in E_0$ such that $tr(Y_a) \neq \emptyset$ and $tr(Z_b) \neq Z_b$, then let j_0 be such that $y_{j_0}^a \in tr(Y_a)$ and let j_1 be such that $z_{j_1}^b \notin tr(Z_b)$. We have $\mu(y_{j_0}^a) \geq T + 1$ and $\mu(z_{j_1}^b) \leq T - 1$. Thus, $(K' \backslash \{y_{j_0}^a\}) \cup \{z_{j_1}^b\}$ is a safe replacement.
Theses replacements end either when all the Y_e are empty for all $e \in E_0$ or when all the Z_e are full for all $e \in E_0$, which achieves the proof of Lemma 3. □

Lemma 4. *K' can be safely modified such that one of the two following holds:*
Case B1: for all $e \in E_1$ we have $tr(Y_e) = Y_e$.
Case B2: for all $e \in E_1$ we have $tr(Z_e) = \emptyset$.

Proof. The proof is roughly based on the fact that replacing a vertex of Z_e by a vertex of Y_e permits to "loose" at least one edge with vertices A and "gain" one edge with a vertex of X_e. Let us formally prove Lemma 4. Similarly to the proof of Lemma 3, we first restructure each gadget of E_1 separately: for all $e \in E_1$ such that $tr(Z_e) \neq \emptyset$ and $tr(Y_e) \neq Y_e$, let $j_0 = \max\{j \in \{1, ..., T\} : y_j^e \notin K'\}$ and let j_1 be such that $z_{j_1}^e \in tr(Z_e)$. Recall that by definition of E_1, there exists $i, j \in \{1, ..., n\}$ such that $z_{j_1}^e$ is adjacent to a_i^j. We have $\mu(z_{j_1}^e) \geq y + z + 1$, where $y = |N(z_{j_1}^e) \cap Y_e|$ and $z = |N(z_{j_1}^e) \cap Z_e|$. On the other side, we have $\mu(y_{j_0}^e) \leq z + y + 2$ (indeed, $|N(y_{j_0}^e) \cap Z_e| = z + 1$, $|N(y_{j_0}^e) \cap Y_e| \leq y$ and $|N(y_{j_0}^e) \cap X_e| = 1$). Since $\{y_{j_0}^e, z_{j_1}^e\} \in E'$, it holds that $(K' \backslash \{z_{j_1}\}) \cup \{y_{j_0}\}$ is a safe replacement. After all these replacements, given any $e \in E_1$, $tr(Z_e) \neq \emptyset$ implies that $tr(Y_e) = Y_e$.
We now proceed to replacements between gadgets F_e, $e \in E_1$. If one can find $a, b \in E_1$ such that $tr(Z_a) \neq \emptyset$ and $tr(Y_b) \neq Y_b$, then let j_0 be such that $y_{j_0}^b \notin tr(Y_b)$ and let j_1 be such that $z_{j_1}^a \in tr(Z_a)$. We have $\mu(z_{j_1}^a) \geq T + 1$ and $\mu(y_{j_0}^b) \leq T - 1$. Thus $(K' \backslash \{z_{j_1}\}) \cup \{y_{j_1}\}$ is a safe replacement. □

Let us now define for each case and each $e \in E$ the set of vertices $D_e \subseteq Y_e \cup Z_e$ that have to be replaced (see Figure 2):
- case A1: for all $e \in E_0$, $D_e = Y_e \cap K'$
- case A2: for all $e \in E_0$, $D_e = Z_e \setminus K'$
- case B1: for all $e \in E_1$, $D_e = Z_e \cap K'$
- case B2: for all $e \in E_1$, $D_e = Y_e \setminus K'$

Notice that if $D_e = \emptyset$ for all $e \in E_0$ (resp. $e \in E_1$), then cases A1 and A2 (resp. B1 and B2) collapse. If such a case happen for all $e \in E$, we can immediately conclude, as shown by the following lemma:

Lemma 5. *If $D_e = \emptyset$ for all $e \in E$, then G contains a clique of size k.*

Proof. By construction, we have $|tr(A)| = T$ and $|tr(F_e)| = 2T$ for all $e \in E$. Thus, $cost^*(tr(A)) = \binom{T}{2}$ and $cost^*(tr(F_e)) = \binom{T}{2} + 1$ if $Y_e \subseteq K'$, and $cost^*(tr(F_e)) = \binom{T}{2}$ if $Z_e \subseteq K'$. By construction, $Y_e \subseteq K'$ if and only if $e \in E_1$. Thus, since $cost^*(K') \leq \binom{T}{2} + m\binom{T}{2} + m - \binom{k}{2}$, we must have $|E_1| \leq m - \binom{k}{2}$ which is equivalent to $|E_0| \geq \binom{k}{2}$. Hence, there exists at most $\lfloor \frac{|A| - T}{n} \rfloor = k$ vertices in G inducing at least $\binom{k}{2}$ edges, *i.e.* G contains a clique of size k.

□

We now have to analyse the four cases of Lemma 3 and 4 (see Figure 2).

Case A1 and B1

To summarize the situation, the solution K' can be partitionned in $K'_A = K' \cap A$, and $K'_F = K' \setminus K'_A$, the vertices selected in the gadgets. Let $\Delta_0 = \sum_{e \in E_0} |D_e|$ be the number of extra vertices allocated in all the gadgets F_e, $e \in E_0$, and $\Delta_1 = \sum_{e \in E_1} |D_e|$ be the number of extra vertices allocated in all the gadgets F_e, $e \in E_1$. Let $\Delta = \Delta_0 + \Delta_1$. Notice that we have $|K'_A| = T - \Delta$, as a "regular" solution that does not select any extra vertex in a gadget has to pick T vertices in A. Moreover,
- vertices of K' selected in gadgets of E_0 are not adjacent to K'_A (by definition of E_0)
- each gadget of E_0 induces at least $\binom{T}{2}$ edges (as we are in case A1)
- each gadget of E_1 induces at least $\binom{T}{2} + 1$ edges (as we are in case B1)

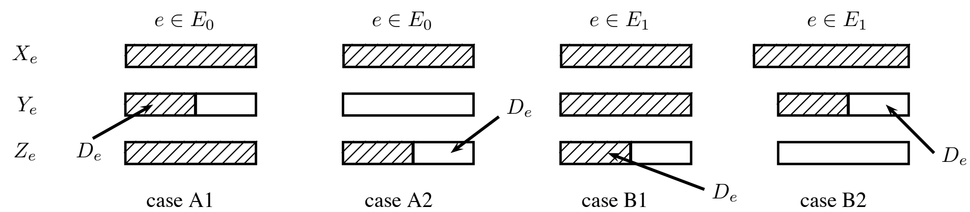

Figure2: Schema of different cases. Shaded rectangles represent part of K'.

- each of the Δ_0 vertices is adjacent to at least T vertices in K' (such a vertex is in a set Y_e, and thus is connected to the T vertcies of Z_e)
- each of the Δ_1 vertices is adjacent to at least $T+1$ vertices in K' (such a vertex is in a set Z_e, and thus is connected to at least 1 vertex of K'_A and to the T vertices of Y_e)

Let us now lower bound the total cost of K'. We have

$$\begin{aligned} cost^*(K') &\geq |E_0|\binom{T}{2} + |E_1|(\binom{T}{2}+1) + \Delta_0 T + \Delta_1(T+1) + \binom{T-\Delta}{2} \\ &\geq |E_0|\binom{T}{2} + |E_1|(\binom{T}{2}+1) + \Delta T + \binom{T-\Delta}{2} \\ &\geq |E_0|\binom{T}{2} + |E_1|(\binom{T}{2}+1) + \binom{T}{2} + \frac{\Delta^2}{2} \end{aligned}$$

Notice that in a bad structured solution, a large Δ allows to select only a few vertices in A ($T-\Delta$ instead of T), and thus to have many gadgets (more than $\binom{k}{2}$) in E_0. Let us now consider the contrapositive, *i.e.* we consider that G does not contain a k-clique, and show that K' induces more than C' edges.

Let q and r such that $\Delta = qn + r$, $r < n$. Let us upper bound $|E_0|$. As there is $T-\Delta$ vertices in A, the number of empty "columns" (column u is empty iff none of the a_u^t is selected) is at most $n - \frac{T-\Delta}{n} \leq k+q$.

As G does not contain a k-clique, the $k+q$ vertices corresponding to these $k+q$ columns cannot induce a clique of size $k+q$, and thus $|E_0| < \binom{k+q}{2}$. Thus, we get

$$\begin{aligned} cost^*(K') &> \binom{k+q}{2}\binom{T}{2} + (m - \binom{k+q}{2})(\binom{T}{2}+1) + \binom{T}{2} + \frac{\Delta^2}{2} \\ &= C' - (\binom{q}{2} + kq) + \frac{\Delta^2}{2} \end{aligned}$$

Thus, as $\frac{\Delta^2}{2} > \binom{q}{2} + kq$, we get the desired inequality.

Case A2 and B2

Let $\Delta_0 = \sum_{e \in E_0} |D_e|$, $\Delta_1 = \sum_{e \in E_1} |D_e|$ and $\Delta = \Delta_0 + \Delta_1$ (recall that in this case, $D_e \not\subset K'$ for all $e \in E$). Here again we suppose $\Delta > 0$. Let us notice that for all $u \in tr(A)$, $\mu(u) \geq T$. On the other hand, for all $e \in E$ such that there exists $v \in D_e$, we have $\mu(v) \leq T$ (remark that if $e \in E_1$, then $D_e \subseteq Y_e$, and if $e \in E_0$, then v is not adjacent to $tr(A)$ by definition of E_0). Thus $(K' \backslash \{u\}) \cup \{v\}$ is a safe replacement. Since before this replacement we had

$tr(A) = T + \Delta$, it is clear that we can repeat this replacement (*i.e.* $K' \setminus \{u\} \cup \{v\}$ where $u \in tr(A)$ and $v \in D_e$ for some $e \in E$) Δ times safely. At this point, the updated value of Δ is 0, *i.e.* $D_e = \emptyset$ for all $e \in E$. By Lemma 5, we must have a clique of size k in G.

Case A2 and B1

If there exists $e \in E_0$ such that there exists $u \in D_e$, then $\mu(u) < T$. If such a vertex exists, then either $|tr(A)| > T$ or there exists $e' \in E_1$ such that there exists $v \in D_{e'}$. In the first case for all $x \in tr(A)$ we have $\mu(x) \geq T$, and $(K' \setminus \{x\}) \cup \{u\}$ is a safe replacement. In the second case we have $\mu(v) > T$ and here again $(K' \setminus \{v\}) \cup \{u\}$ is a safe replacement.
After these replacements we must have $D_e = \emptyset$ for all $e \in E_0$, and we can apply the same arguments as for case A1 and B1.

Case A1 and B2

If there exists $e \in E_1$ such that there exists $u \in D_e$, then $\mu(u) < T$. If such a vertex exists, then either $|tr(A)| > T$ or there exists $e' \in E_0$ such that there exists $v \in D_{e'}$. In the first case for all $x \in tr(A)$ we have $\mu(x) \geq T$, and $(K' \setminus \{x\}) \cup \{u\}$ is a safe replacement. In the second case we have $\mu(v) > T$ and here again $(K' \setminus \{v\}) \cup \{u\}$ is a safe replacement.
After these replacements we must have $D_e = \emptyset$ for all $e \in E_1$, and we can apply the same arguments as for case A1 and B1.

□

References

1. N. Apollonio. Private communication, 2012.
2. N. Apollonio and A. Sebő. Minconvex factors of prescribed size in graphs. SIAM Journal of Discrete Mathematics, 23(3):1297–1310, 2009.
3. J. Backer and J.M. Keil. Constant factor approximation algorithms for the densest k-subgraph problem on proper interval graphs and bipartite permutation graphs. Information Processing Letters, 110(16):635–638, 2010.
4. H. Broersma, P. A. Golovach, and V. Patel. Tight complexity bounds for FPT subgraph problems parameterized by clique-width. In Proceedings of the 6th international conference on Parameterized and Exact Computation, IPEC'11, pages 207–218, Berlin, Heidelberg, 2012. Springer-Verlag.
5. D.G. Corneil and Y. Perl. Clustering and domination in perfect graphs. Discrete Applied Mathematics, 9(1):27 – 39, 1984.
6. J. Flum and M. Grohe. Parameterized Complexity Theory. Springer, 2006.
7. M. R. Garey and D. S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.
8. G. Joret and A. Vetta. Reducing the rank of a matroid. CoRR, abs/1211.4853, 2012.
9. M. Liazi, I. Milis, F. Pascual, and V. Zissimopoulos. The densest k-subgraph problem on clique graphs. Journal of Combinatorial Optimization, 14(4):465–474, 2007.
10. M. Liazi, I. Milis, and V. Zissimopoulos. A constant approximation algorithm for the densest k-subgraph problem on chordal graphs. Information Processing Letters, 108(1):29–32, 2008.
11. T. Nonner. PTAS for densest k-subgraph in interval graphs. In Proceedings of the 12th international conference on Algorithms and Data Structures, pages 631–641. Springer, 2011.