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NP-hardness of the SPARSEST k-SUBGRAPH Problem in Chordal Graphs*

R. Watrigant, M. Bougeret, and R. Giroudeau

LIRMM-CNRS-UMR 5506 - 161, rue Ada 34090 Montpellier, France

Abstract Given a simple undirected graph $G = (V, E)$ and an integer $k \leq |V|$, the SPARSEST k-SUBGRAPH problem asks for a set of k vertices which induce the minimum number of edges. Whereas its special case INDEPENDENT SET and many other optimization problems become polynomial-time solvable in chordal graphs, we show that SPARSEST k-SUBGRAPH remains NP-hard in this graph class.

1 Introduction and Preliminaries

In this report we study the following decision problem:
SPARSEST k-SUBGRAPH

- Input: a simple undirected graph $G = (V, E)$, $k \in \mathbb{N}$, $C \in \mathbb{N}$
- Question: is there a subset $S \subseteq V$ such that $|S| = k$ and $E(S) \leq C$? Where $E(S)$ is the number of edges induced by S.

As a generalization of the classical INDEPENDENT SET problem (for which we have $C = 0$ in the input), SPARSEST k-SUBGRAPH is NP-hard [7] and even not approximable unless $P = NP$. Moreover, it is $W[1]$-hard (parameterized by k) [6].
Its maximization version, namely the k-DENSEST SUBGRAPH (or the k-CLUSTER problem), has been extensively studied in the last three decades: in [5], the authors show that k-DENSEST SUBGRAPH is $\mathcal{NP}$-hard in bipartite, comparability and chordal graphs, and is polynomial-time solvable in trees, cographs, bounded treewidth graphs and split graphs. The question of the complexity status of k-DENSEST SUBGRAPH in interval graphs (and even in proper interval graphs) is stated by the authors as an open problem, and is still not answered yet. In addition, [4] shows that both SPARSEST k-SUBGRAPH and k-DENSEST SUBGRAPH are polynomial time solvable in bounded cliquewidth graphs. Notice that several exact or approximation algorithm exists for k-DENSEST SUBGRAPH in subclasses of perfect graphs: among others, constant approximation algorithms are known for chordal graphs [10], bipartite permutation graphs [3] and $PTAS$ are known for interval graphs [11] and for chordal graphs having a special clique tree [9]. Unfortunately, most of these results seem useless for SPARSEST k-SUBGRAPH, as we apparently need to complement the input graph to apply them. Nevertheless we can deduce that SPARSEST k-SUBGRAPH remains NP-hard in co-chordal (which is a subclass of perfect graphs) and is polynomial-time solvable in split graphs.
On the other side, its dual version, namely the MAXIMUM PARTIAL VERTEX COVER problem, for which we are looking for k vertices in the input graph which *cover* the maximum number of edges, is polynomial-time solvable in line graphs [2], and remains NP-hard in bipartite graphs [1,8].
In this report we study the complexity status of SPARSEST k-SUBGRAPH in chordal graphs.

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Whereas the INDEPENDENT SET problem is polynomial-time solvable in perfect graphs (and thus in chordal graphs), we show thatSPARSEST k-SUBGRAPH remains NP-hard in chordal graphs. Obviously, the same result holds for the MAXIMUM PARTIAL VERTEX COVER problem.

The two following definitions of chordal graphs are equivalent:
- A graph is chordal if there it does not contain any cycle of length four or more as an induced subgraph.
- A vertex v of G is called simplicial if its neighbourhood $N(v)$ is a clique. The ordering $v_1, ..., v_n$ of the vertices of G is a simplicial elimination scheme if for all i, v_i is simplicial in $G[v_i, ..., v_n]$. A graph is chordal if it has a simplicial elimination scheme.

2 The Main Result

2.1 Idea of the Proof

The following $\mathcal{NP}$-hardness proof is a reduction from the k-CLIQUE problem in general graphs. Roughly speaking, given an input instance $G = (V, E)$ together with $k \in \mathbb{N}$, we construct the split graph of adjacencies of G, *i.e.* we build a clique on a set A representing the vertices of G, and an independent on a set F representing the edges of G, connecting A and F with respect to the adjacencies of the graph. Then, we duplicate each vertex of A n times, creating thus a clique of size n^2. On the other hand, we replace each vertex of the independent set by a gadget. If G contains a clique of size k, that is a set of k vertices inducing $\binom{k}{2}$ edges, then the solution will take vertices *not* corresponding to vertices of the clique. Hence, there will be $\binom{k}{2}$ gadgets *not* adjacent to the solution. Finally, we will force the solution to take the same number of vertices among each gadget.

2.2 $\mathcal{NP}$-hardness

Theorem 1. SPARSEST k-SUBGRAPH *remains $\mathcal{NP}$-hard in chordal graphs.*

Proof. We reduce from the classical k-CLIQUE problem in general graphs. Let $G = (V, E)$ and $k \in \mathbb{N}$. We note $|V| = n$, $V = \{v_1, ..., v_n\}$, $|E| = m$ and $T = n(n - k)$. In the following we will define $G' = (V', E')$ together with $k', C' \in \mathbb{N}$ such that:
- G', k', C' can be constructed in polynomial time
- G' is a chordal graph
- G contains a clique of size k if and only if one can find k' vertices in G' which induce C' edges or less.

The construction: V' is composed of two parts A and F.
- We first define a clique over $A = \{a_i^j : i, j \in \{1, ..., n\}\}$. Thus, A is a clique of size n^2. Moreover, for all $j \in \{1, ..., n\}$, we note $A_j = \{a_1^j, ..., a_n^j\}$.
- For all $e \in E$, we construct a graph with F_e as vertex set, where F_e is composed of three sets of T vertices: $X_e = \{x_1^e, ..., x_T^e\}$, $Y_e = \{y_1^e, ..., y_T^e\}$ and $Z_e = \{z_1^e, ..., z_T^e\}$. The set X^e induces a stable set, Z^e induces a clique, and Y^e contains a clique of size $T - 1$ on vertices $\{y_2^e, ..., y_T^e\}$ (thus, y_1^e is not connected to the other vertices of Y^e). Then, for all $j \in \{1, ..., T\}$, x_j^e is connected to y_j^e, and y_j^e is connected to all vertices of Z^e. An example of such a gadget is represented in Figure 1. We define $F = \bigcup_{e \in E} F_e$.
- For all $e = \{v_p, v_q\} \in E$, all vertices of Z^e are connected to $\{a_p^j : j \in \{1, ..., n\}\}$ and $\{a_q^j : j \in \{1, ..., n\}\}$.
- We define $k' = m(2T + 1) + T$ and $C' = m\binom{T}{2} + \binom{T}{2} + (m - \binom{k}{2})$.

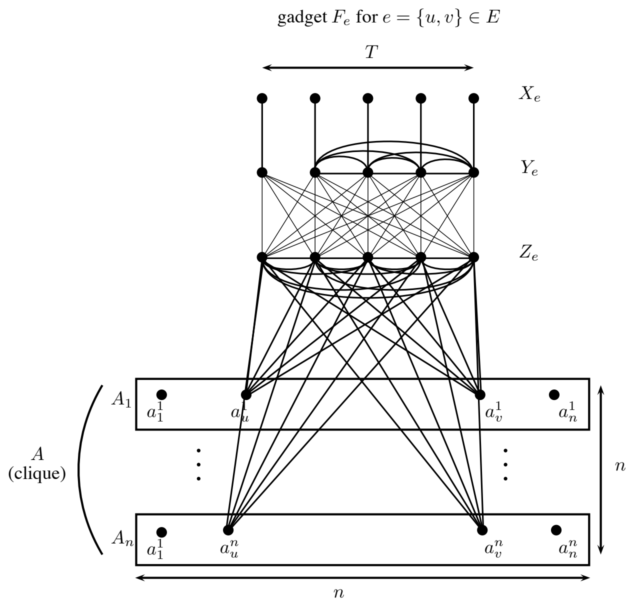

Figure1: Example of a gadget F_e (with $T = 5$) and its relations to A.

The above construction can clearly be computed in polynomial time. The next lemma proves that G' is a chordal graph:

Lemma 1. *G' is a chordal graph.*

Proof. We have the following simplicial elimination scheme:

- For all $e \in E$, we can remove X^e since for all $j \in \{1, ..., T\}$, x_j^e is only connected to y_j^e.
- For all $e \in E$, we can remove Y^e. Indeed the remaining neighbourhood of y_1^e is Z^e which is a clique. And the remaining neighbourhood of y_j^e with $j \geq 2$ is a subset $Y^e \cup Z^e \setminus \{y_1^e\}$ which induces a clique.
- For all $e \in E$, we can remove Z^e since the remaining neighbourhood of z_j^e is a subset of Z^e and vertices of A which induce a clique.
- The remaining vertices induces a clique on A and thus be eliminated.

$\square$

Now we prove that G contains a clique of size k if and only if G' contains k' vertices inducing at most C' edges.

$\underline{\Rightarrow}$ Let us suppose that $K \subseteq V$ is a clique of size k in G. Without loss of generality we suppose $K = \{v_1, ..., v_k\}$. Moreover, we note $E_0 = \{\{v_p, v_q\} \in E$ such that $v_p, v_q \in K\}$ and $E_1 = \{\{v_p, v_q\} \in E$ such that $v_p \notin K$ or $v_q \notin K\}$. We construct $K' \subseteq V'$ as follows:

- For all $i \in \{(k+1), ..., n\}$ and all $j = \{1, ..., n\}$, we add a_i^j to K'.
- For all $e \in E$, we add all vertices of X_e to K'.
- For all $e \in E_0$, we add all vertices of Z_e to K'.
- For all $e \in E_1$, we add all vertices of Y_e to K'.

One can verify that K' is a set of $k' = 2mT + T$ vertices inducing exactly $C' = m\binom{T}{2} + \binom{T}{2} + (m - \binom{k}{2})$ edges. Indeed, we picked $T = n(n-k)$ vertices from A which is a clique and thus induce $\binom{T}{2}$ edges. Then, for all $e \in E$, we picked $2T$ vertices, which induce $\binom{T}{2}$ edges if $e \in E_0$, and $(\binom{T}{2} + 1)$ edges if $e \in E_1$. Since $|E_0| = \binom{k}{2}$ (and thus $|E_1| = m - \binom{k}{2}$), we have the desired number of edges.

$\Leftarrow$ Suppose now that K' is a set of k' vertices of G' which induces at most C' edges. We re-define the sets E_0 and E_1 as follows: $E_0 = \{\{v_p, v_q\} \in E$ such that for all $j \in \{1, ..., n\}$ we have $a_p^j \notin K'$ and $a_q^j \notin K'\}$, and $E_1 = E \backslash E_0$.
Let us now define the notion of safe replacement, and some properties associated to it:

Safe Replacements. Let $u \in K'$ and $v \in V' \backslash K'$. It is clear that $(K' \backslash \{u\}) \cup \{v\}$ is a safe replacement if and only if we have $\mu(v) \leq \mu(u)$ if $\{u, v\} \notin E'$ and $\mu(v) - 1 \leq \mu(u)$ if $\{u, v\} \in E'$. For sake of readability, we will keep and update the definitions of E_0 and E_1 when replacing vertices of A (*e.g.* if we remove a vertex $u \in A$ from K' and that there exists $e \in E_1$ such that vertices of Z_e were only adjacent to u among all vertices of A, then e now belongs to E_0).
For all $R \subseteq V'$, let $tr(R) = K' \cap R$ be the trace of K' on R, and for all $v \in V'$, let $\mu(v) = |tr(N(v))|$ be the number of neighbours of v belonging to K'. The proof consists in replacing some vertices of K' by other vertices not in K' without increasing the number of induced edges, in order to obtain the same solution as previously. We call such a replacement a *safe modification* or a *safe replacement*.
Let us now restructure an optimal solution:

Lemma 2. *Without loss of generality (and optimality of K'), we can suppose that for all $e \in E$ we have $X_e \subseteq K'$.*

Proof. Let $S = \bigcup_{e \in E} X_e$. Since we have $k' > |S|$, there always exists $u \in K' \backslash S$. Suppose that there exists $e \in E$ and $i \in \{1, ..., T\}$ such that $x_i^e \notin K'$. If $y_i^e \notin K'$, then we have $\mu(x_i^e) = 0$ and we can thus safely replace any other vertex of $K' \backslash S$ by x_i^e. Now, if $y_1^e \in K'$, then $\mu(x_i^e) = 1$. Since $\{x_i^e, y_i^e\} \in E'$, we have that $(K' \backslash \{y_1^e\}) \cup \{x_i^e\}$ is a safe replacement. $\square$

In the remaining we suppose $X_e \subseteq K'$ for all $e \in E$.

Lemma 3. *K' can be safely modified such that one of the two following cases holds:*
Case A1: for all $e \in E_0$ we have $tr(Z_e) = Z_e$.
Case A2: for all $e \in E_0$ we have $tr(Y_e) = \emptyset$.

Proof. We first restructure each gadget of E_0 separately: for all $e \in E_0$ such that $tr(Y_e) \neq \emptyset$ and $tr(Z_e) \neq Z_e$, let $j_0 = \max\{j \in \{1, ..., T\} : y_j^e \in tr(Y_e)\}$ and let j_1 be such that $z_{j_1}^e \notin tr(Z_e)$. Recall that Lemma 2 ensures that $x_{j_0}^e$ is in K'. If $j_0 \neq 1$, then $\mu(y_{j_0}^e) = y + z + 1$, where $y = |N(y_{j_0}^e) \cap tr(Y_e)|$ and $z = |N(y_{j_0}^e) \cap tr(Z_e)|$. On the other side, we have $\mu(z_{j_1}^e) \leq y + z + 1$ (more precisely, $\mu(z_{j_1}^e) = y + z + 1$ if $y_1^e \in K'$, and $\mu(z_{j_1}^e) = y + z$ if $y_1^e \notin K'$). Unformally, this switch ensures that we necessarily "loose" the edge due to the vertex of X^e and we gain at most one edge due to y_1^e. Hence $\mu(z_{j_1}^e) \leq \mu(y_{j_0}^e)$ and $(K' \backslash \{y_{j_0}^e\}) \cup \{z_{j_1}^e\}$ is a safe replacement. If $j_0 = 1$, then it means that $tr(Y_e) = \{y_1^e\}$. Suppose that there exists j_1 such that $z_{j_1}^e \notin tr(Z_e)$. We have $\mu(y_1^e) = z + 1$ where $z = |N(y_1^e) \cap tr(Z_e)|$, and $\mu(z_{j_1}^e) = z + 1$. Here again $(K' \backslash \{y_1^e\}) \cup \{z_{j_1}^e\}$ is a safe replacement. After all these replacements, given any $e \in E_0$, $tr(Y_e) \neq \emptyset$ implies that $tr(Z_e) = Z_e$.
Then, we proceed to replacements between gadgets F_e, $e \in E_0$. If one can find $a, b \in E_0$ such that $tr(Y_a) \neq \emptyset$ and $tr(Z_b) \neq Z_b$, then let j_0 be such that $y_{j_0}^a \in tr(Y_a)$ and let j_1 be such that

$z^b_{j_1} \notin tr(Z_b)$. We have $\mu(y^a_{j_0}) \geq T+1$ and $\mu(z^b_{j_1}) \leq T-1$. Thus, $(K' \backslash \{y^a_{j_0}\}) \cup \{z^b_{j_1}\}$ is a safe replacement.
The algorithm ends either when all the Y_e are empty for all$e \in E_0$ or when all the Z_e are full for all $e \in E_0$, which achieves the proof of Lemma 3. □

Lemma 4. *K' can be safely modified such that one of the two following cases holds:*
Case B1: for all $e \in E_1$ we have $tr(Y_e) = Y_e$.
Case B2: for all $e \in E_1$ we have $tr(Z_e) = \emptyset$.

Proof. The proof is roughly based on the fact that replacing a vertex of Z_e by a vertex of Y_e permits to "loose" at least one edge with vertices A and "gain" one edge with a vertex of X_e. Let us formally prove Lemma 4. Similarly to the proof of Lemma 3, we first restructure each gadget of E_1 separately: for all $e \in E_1$ such that $tr(Z_e) \neq \emptyset$ and $tr(Y_e) \neq Y_e$, let $j_0 = \max\{j \in \{1,...,T\} : y^e_j \notin K'\}$ and let j_1 be such that $z^e_{j_1} \in tr(Z_e)$. Recall that by definition of E_1, there exists $i,j \in \{1,...,n\}$ such that $z^e_{j_1}$ is adjacent to a^j_i. We have $\mu(z^e_{j_1}) \geq y+z+1$, where $y = |N(z^e_{j_1}) \cap Y_e|$ and $z = |N(z^e_{j_1}) \cap Z_e|$. On the other side, we have $\mu(y^e_{j_0}) \leq z+y+2$ (indeed, $|N(y^e_{j_0}) \cap Z_e| = z+1$, $|N(y^e_{j_0}) \cap Y_e| \leq y$ and $|N(y^e_{j_0}) \cap X_e| = 1$). Since $\{y^e_{j_0}, z^e_{j_1}\} \in E'$, it holds that $(K' \backslash \{z_{j_1}\}) \cup \{y_{j_0}\}$ is a safe replacement. After all these replacements, given any $e \in E_1$, $tr(Z_e) \neq \emptyset$ implies that $tr(Y_e) = Y_e$.
We now proceed to replacements between gadgets F_e, $e \in E_1$. If one can find $a, b \in E_1$ such that $tr(Z_a) \neq \emptyset$ and $tr(Y_b) \neq Y_b$, then let j_0 be such that $y^b_{j_0} \notin tr(Y_b)$ and let j_1 be such that $z^a_{j_1} \in tr(Z_a)$. We have $\mu(z^a_{j_1}) \geq T+1$ and $\mu(y^b_{j_0}) \leq T-1$. Thus $(K' \backslash \{z_{j_1}\}) \cup \{y_{j_1}\}$ is a safe replacement. □

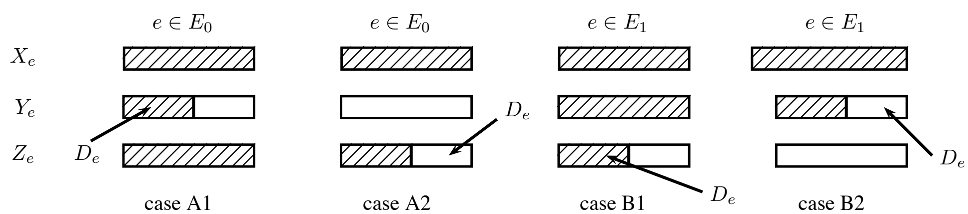

Figure2: Schema of different cases. Shaded rectangles represent part of K'.

Figure 2 summarize the different cases. Let us now define for each case and each $e \in E$ the set of vertices $D_e \subseteq Y_e \cup Z_e$ that have to be replaced (see Figure 2):

- case A1: for all $e \in E_0$, $D_e = Y_e \cap K'$
- case A2: for all $e \in E_0$, $D_e = Z_e \setminus K'$
- case B1: for all $e \in E_1$, $D_e = Z_e \cap K'$
- case B2: for all $e \in E_1$, $D_e = Y_e \setminus K'$

Notice that if $D_e = \emptyset$ for all $e \in E_0$ (resp. $e \in E_1$), then cases A1 and A2 (resp. B1 and B2) collapse. If such a case happen for all $e \in E$, we can immediately conclude, as shown by the following lemma:

Lemma 5. *If $D_e = \emptyset$ for all $e \in E$, then G contains a clique of size k.*

Proof. By construction, we have $|tr(A)| = T$ and $|tr(F_e)| = 2T$ for all $e \in E$. Thus, $E(tr(A)) = \binom{T}{2}$ and $E(tr(F_e)) = \binom{T}{2} + 1$ if $Y_e \subseteq K'$, and $E(tr(F_e)) = \binom{T}{2}$ if $Z_e \subseteq K'$. By construction, $Y_e \subseteq K'$ if and only if $e \in E_1$. Thus, since $E(K') \leq \binom{T}{2} + m\binom{T}{2} + m - \binom{k}{2}$, we must have $|E_1| \leq m - \binom{k}{2}$ which is equivalent to $|E_0| \geq \binom{k}{2}$. Hence, there exists at most $\lfloor \frac{|A|-T}{n} \rfloor = k$ vertices in G inducing at least $\binom{k}{2}$ edges, *i.e.* G contains a clique of size k.

□

We now have to combine the four cases of Lemma 3 and 4.

- Case A1 and B1: let $\Delta_0 = \sum_{e\in E_0} |D_e|$, $\Delta_1 = \sum_{e\in E_1} |D_e|$ and $\Delta = \Delta_0 + \Delta_1$ (recall that in this case, $D_e \subset K'$ for all $e \in E$). If $\Delta = 0$, then by Lemma 5 G contains a clique of size k. Thus we suppose in the following that $\Delta > 0$. It is clear that $|tr(A)| = T - \Delta$. Moreover:

$$E(K') \geq m\binom{T}{2} + \Delta T + |E_1| + \binom{T}{2} - \binom{\Delta}{2} - \Delta(T-\Delta) + \Delta_1 \qquad (1)$$

  Indeed, by construction, for all $e \in E_0$, $tr(F_e)$ contains at least $\binom{T}{2} + |D_e|T$ edges. In particular for all $e \in E_1$, $tr(F_e)$ contains $(\binom{T}{2} + |D_e|T+1)$ edges (which explains the term $|E_1|$). Then, $|tr(A)| = T - \Delta$, which implies that $tr(A)$ induces $\binom{T}{2} - \binom{\Delta}{2} - \Delta(T-\Delta)$ edges. Finally, by definition of E_1, for all $e \in E_1$ and all $j \in \{1, ..., T\}$, z_j^e must be adjacent to some vertex of $tr(A)$, which adds at least Δ_1 edges. Hence,

$$\begin{aligned} E(K') - C' &\geq \Delta T - \binom{\Delta}{2} - \Delta(T-\Delta) + \Delta_1 + |E_1| - m + \binom{k}{2} \\ &= \frac{1}{2}\Delta(\Delta+1) + \Delta_1 + |E_1| + \binom{k}{2} - m \\ &= \frac{1}{2}\Delta(\Delta+1) + \Delta_1 + \binom{k}{2} - |E_0| \quad (\text{since } |E_0| + |E_1| = m) \end{aligned}$$

  Since K' is supposed to be a set of k' vertices inducing at most C' edges, we must have $E(K') - C' \leq 0$, *i.e.* $|E_0| \geq \frac{1}{2}\Delta(\Delta+1) + \Delta_1 + \binom{k}{2}$. Let $\overline{tr(A)} = A \backslash tr(A)$. It is clear that $|\overline{tr(A)}| = kn + \Delta$.
  Recall that for all $e = \{v_p, v_q\} \in E_0$ we have for all $j \in \{1, ..., n\}$ $a_p^j, a_q^j \notin K'$. Thus the number of vertices of G inducing all edges of E_0 is at most $\lfloor \frac{nk+\Delta}{n} \rfloor = k + \lfloor \frac{\Delta}{n} \rfloor$, *i.e.* there exists at most $(k + \lfloor \frac{\Delta}{n} \rfloor)$ vertices in G which induce at least $(\frac{1}{2}\Delta(\Delta+1) + \Delta_1 + \binom{k}{2})$ edges. If $\Delta < n$, then it means that k vertices induce strictly more than $\binom{k}{2}$ edges, which is impossible. If $\Delta \geq n$, then $|E_0| \geq \frac{n(n+1)}{2} + \binom{k}{2} > m$ which is also impossible. Thus it implies that $E(K') - C' > 0$, and K' must induce more than C' edges which contradicts the hypothesis and implies that this case cannot happen.

- Case A2 and B2: let $\Delta_0 = \sum_{e\in E_0} |D_e|$, $\Delta_1 = \sum_{e\in E_1} |D_e|$ and $\Delta = \Delta_0 + \Delta_1$ (recall that in this case, $D_e \not\subset K'$ for all $e \in E$). Here again we suppose $\Delta > 0$. Let us notice that for all $u \in tr(A)$, $\mu(u) \geq T$. On the other hand, for all $e \in E$ such that there exists $v \in D_e$, we have $\mu(v) \leq T$ (remark that if $e \in E_1$, then $D_e \subseteq Y_e$, and if $e \in E_0$, then v is not adjacent to $tr(A)$ by definition of E_0). Thus $(K' \backslash \{u\}) \cup \{v\}$ is a safe replacement. Since before this replacement we had $tr(A) = T + \Delta$, it is clear that we can repeat this replacement (*i.e.* $K' \backslash \{u\} \cup \{v\}$ where $u \in tr(A)$ and $v \in D_e$ for some $e \in E$) Δ times safely. At this point, the updated value of Δ is 0, *i.e.* $D_e = \emptyset$ for all $e \in E$. By Lemma 5, we must have a clique of size k in G.

- Case A2 and B1: if there exists $e \in E_0$ such that there exists $u \in D_e$, then $\mu(u) < T$. If such a vertex exists, then either $|tr(A)| > T$ or there exists $e' \in E_1$ such that there exists $v \in D_{e'}$. In the first case for all $x \in tr(A)$ we have $\mu(x) \geq T$, and $(K' \setminus \{x\}) \cup \{u\}$ is a safe replacement. In the second case we have $\mu(v) > T$ and here again $(K' \setminus \{v\}) \cup \{u\}$ is a safe replacement.
  After these replacements we must have $D_e = \emptyset$ for all $e \in E_0$, and we can apply the same arguments as for case A1 and B1.

- Case A1 and B2: if there exists $e \in E_1$ such that there exists $u \in D_e$, then $\mu(u) < T$. If such a vertex exists, then either $|tr(A)| > T$ or there exists $e' \in E_0$ such that there exists $v \in D_{e'}$. In the first case for all $x \in tr(A)$ we have $\mu(x) \geq T$, and $(K' \setminus \{x\}) \cup \{u\}$ is a safe replacement. In the second case we have $\mu(v) > T$ and here again $(K' \setminus \{v\}) \cup \{u\}$ is a safe replacement.
  After these replacements we must have $D_e = \emptyset$ for all $e \in E_1$, and we can apply the same arguments as for case A1 and B1.

□

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