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# *NP*-hardness of the SPARSEST $k$ -SUBGRAPH Problem in Chordal Graphs<sup>\*</sup>

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**Abstract** Given a simple undirected graph  $G = (V, E)$  and an integer  $k \leq |V|$ , the SPARSEST  $k$ -SUBGRAPH problem asks for a set of  $k$  vertices which induce the minimum number of edges. Whereas its special case INDEPENDENT SET and many other optimization problems become polynomial-time solvable in chordal graphs, we show that SPARSEST  $k$ -SUBGRAPH remains *NP*-hard in this graph class.

## 1 Introduction and Preliminaries

In this report we study the following decision problem:

**SPARSEST  $k$ -SUBGRAPH**

- Input: a simple undirected graph  $G = (V, E)$ ,  $k \in \mathbb{N}$ ,  $C \in \mathbb{N}$
- Question: is there a subset  $S \subseteq V$  such that  $|S| = k$  and  $E(S) \leq C$ ? Where  $E(S)$  is the number of edges induced by  $S$ .

As a generalization of the classical INDEPENDENT SET problem (for which we have  $C = 0$  in the input), SPARSEST  $k$ -SUBGRAPH is *NP*-hard [7] and even not approximable unless  $P = NP$ . Moreover, it is  $W[1]$ -hard (parameterized by  $k$ ) [6].

Its maximization version, namely the  $k$ -DENSEST SUBGRAPH (or the  $k$ -CLUSTER problem), has been extensively studied in the last three decades: in [5], the authors show that  $k$ -DENSEST SUBGRAPH is  $\mathcal{NP}$ -hard in bipartite, comparability and chordal graphs, and is polynomial-time solvable in trees, cographs, bounded treewidth graphs and split graphs. The question of the complexity status of  $k$ -DENSEST SUBGRAPH in interval graphs (and even in proper interval graphs) is stated by the authors as an open problem, and is still not answered yet. In addition, [4] shows that both SPARSEST  $k$ -SUBGRAPH and  $k$ -DENSEST SUBGRAPH are polynomial time solvable in bounded cliquewidth graphs. Notice that several exact or approximation algorithm exists for  $k$ -DENSEST SUBGRAPH in subclasses of perfect graphs: among others, constant approximation algorithms are known for chordal graphs [10], bipartite permutation graphs [3] and *PTAS* are known for interval graphs [11] and for chordal graphs having a special clique tree [9]. Unfortunately, most of these results seem useless for SPARSEST  $k$ -SUBGRAPH, as we apparently need to complement the input graph to apply them. Nevertheless we can deduce that SPARSEST  $k$ -SUBGRAPH remains *NP*-hard in co-chordal (which is a subclass of perfect graphs) and is polynomial-time solvable in split graphs.

On the other side, its dual version, namely the MAXIMUM PARTIAL VERTEX COVER problem, for which we are looking for  $k$  vertices in the input graph which *cover* the maximum number of edges, is polynomial-time solvable in line graphs [2], and remains *NP*-hard in bipartite graphs [1,8].

In this report we study the complexity status of SPARSEST  $k$ -SUBGRAPH in chordal graphs.

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Whereas the INDEPENDENT SET problem is polynomial-time solvable in perfect graphs (and thus in chordal graphs), we show that SPARSEST  $k$ -SUBGRAPH remains  $NP$ -hard in chordal graphs. Obviously, the same result holds for the MAXIMUM PARTIAL VERTEX COVER problem.

The two following definitions of chordal graphs are equivalent:

- A graph is chordal if there it does not contain any cycle of length four or more as an induced subgraph.
- A vertex  $v$  of  $G$  is called simplicial if its neighbourhood  $N(v)$  is a clique. The ordering  $v_1, \dots, v_n$  of the vertices of  $G$  is a simplicial elimination scheme if for all  $i$ ,  $v_i$  is simplicial in  $G[v_i, \dots, v_n]$ . A graph is chordal if it has a simplicial elimination scheme.

## 2 The Main Result

### 2.1 Idea of the Proof

The following  $\mathcal{NP}$ -hardness proof is a reduction from the  $k$ -CLIQUE problem in general graphs. Roughly speaking, given an input instance  $G = (V, E)$  together with  $k \in \mathbb{N}$ , we construct the split graph of adjacencies of  $G$ , *i.e.* we build a clique on a set  $A$  representing the vertices of  $G$ , and an independent on a set  $F$  representing the edges of  $G$ , connecting  $A$  and  $F$  with respect to the adjacencies of the graph. Then, we duplicate each vertex of  $A$   $n$  times, creating thus a clique of size  $n^2$ . On the other hand, we replace each vertex of the independent set by a gadget. If  $G$  contains a clique of size  $k$ , that is a set of  $k$  vertices inducing  $\binom{k}{2}$  edges, then the solution will take vertices *not* corresponding to vertices of the clique. Hence, there will be  $\binom{k}{2}$  gadgets *not* adjacent to the solution. Finally, we will force the solution to take the same number of vertices among each gadget.

### 2.2 $\mathcal{NP}$ -hardness

**Theorem 1.** SPARSEST  $k$ -SUBGRAPH remains  $\mathcal{NP}$ -hard in chordal graphs.

*Proof.* We reduce from the classical  $k$ -CLIQUE problem in general graphs. Let  $G = (V, E)$  and  $k \in \mathbb{N}$ . We note  $|V| = n$ ,  $V = \{v_1, \dots, v_n\}$ ,  $|E| = m$  and  $T = n(n - k)$ . In the following we will define  $G' = (V', E')$  together with  $k', C' \in \mathbb{N}$  such that:

- $G', k', C'$  can be constructed in polynomial time
- $G'$  is a chordal graph
- $G$  contains a clique of size  $k$  if and only if one can find  $k'$  vertices in  $G'$  which induce  $C'$  edges or less.

#### The construction

$V'$  is composed of two parts  $A$  and  $F$ .

- We first define a clique over  $A = \{a_i^j : i, j \in \{1, \dots, n\}\}$ . Thus,  $A$  is a clique of size  $n^2$ . Moreover, for all  $j \in \{1, \dots, n\}$ , we note  $A_j = \{a_1^j, \dots, a_n^j\}$ .
- For all  $e \in E$ , we construct a graph with  $F_e$  as vertex set, where  $F_e$  is composed of three sets of  $T$  vertices:  $X_e = \{x_1^e, \dots, x_T^e\}$ ,  $Y_e = \{y_1^e, \dots, y_T^e\}$  and  $Z_e = \{z_1^e, \dots, z_T^e\}$ . The set  $X^e$  induces a stable set,  $Z^e$  induces a clique, and  $Y^e$  contains a clique of size  $T - 1$  on vertices  $\{y_2^e, \dots, y_T^e\}$  (thus,  $y_1^e$  is not connected to the other vertices of  $Y^e$ ). Then, for all  $j \in \{1, \dots, T\}$ ,  $x_j^e$  is connected to  $y_j^e$ , and  $y_j^e$  is connected to all vertices of  $Z^e$ . An example of such a gadget is represented in Figure 1. We define  $F = \bigcup_{e \in E} F_e$ .

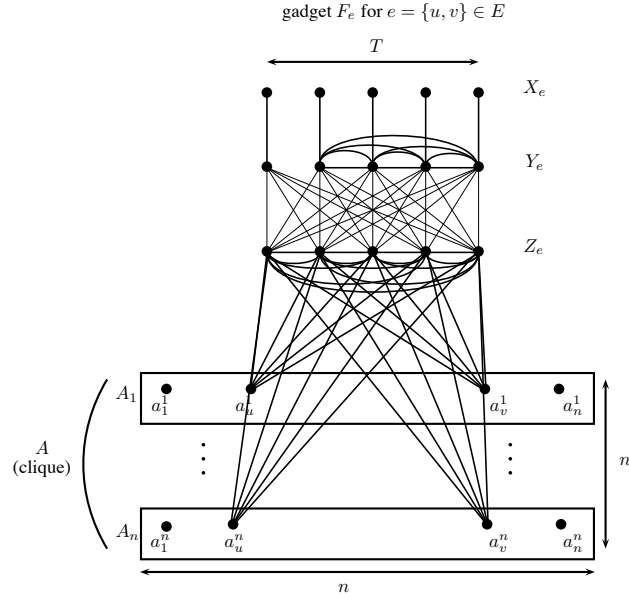


Figure1: Example of a gadget  $F_e$  (with  $T = 5$ ) and its relations to  $A$ .

- For all  $e = \{v_p, v_q\} \in E$ , all vertices of  $Z^e$  are connected to  $\{a_p^j : j \in \{1, \dots, n\}\}$  and  $\{a_q^j : j \in \{1, \dots, n\}\}$ .
- We define  $k' = m2T + T$  and  $C' = m\binom{T}{2} + \binom{T}{2} + (m - \binom{k}{2})$ .

**Lemma 1.**  $G'$  is a chordal graph

*Proof.* We have the following simplicial elimination scheme:

- For all  $e \in E$ , we can remove  $X^e$  since for all  $j \in \{1, \dots, T\}$ ,  $x_j^e$  is only connected to  $y_j^e$ .
- For all  $e \in E$ , we can remove  $Y^e$ . Indeed the remaining neighbourhood of  $y_1^e$  is  $Z^e$  which is a clique. And the remaining neighbourhood of  $y_j^e$  with  $j \geq 2$  is a subset  $Y^e \cup Z^e \setminus \{y_1^e\}$  which induces a clique.
- For all  $e \in E$ , we can remove  $Z^e$  since the remaining neighbourhood of  $z_j^e$  is a subset of  $Z^e$  and vertices of  $A$  which induce a clique.
- The remaining vertices induces a clique on  $A$  and thus be eliminated.

□

Now we prove that  $G$  contains a clique of size  $k$  if and only if  $G'$  contains  $k'$  vertices inducing at most  $C'$  edges.

**$G$  contains a  $k$ -clique  $\Rightarrow G'$  contains  $k'$  vertices inducing at most  $C'$  edges.**

Let us suppose that  $K \subseteq V$  is a clique of size  $k$  in  $G$ . Without loss of generality we suppose  $K = \{v_1, \dots, v_k\}$ . Moreover, we note  $E_0 = \{\{v_p, v_q\} \in E \text{ such that } v_p, v_q \in K\}$  and  $E_1 = \{\{v_p, v_q\} \in E \text{ such that } v_p \notin K \text{ or } v_q \notin K\}$ . We construct  $K' \subseteq V'$  as follows:

- For all  $i \in \{(k+1), \dots, n\}$  and all  $j = \{1, \dots, n\}$ , we add  $a_i^j$  to  $K'$ .
- For all  $e \in E$ , we add all vertices of  $X_e$  to  $K'$ .

- For all  $e \in E_0$ , we add all vertices of  $Z_e$  to  $K'$ .
- For all  $e \in E_1$ , we add all vertices of  $Y_e$  to  $K'$ .

One can verify that  $K'$  is a set of  $k' = 2mT + T$  vertices inducing exactly  $C' = m\binom{T}{2} + \binom{T}{2} + (m - \binom{k}{2})$  edges. Indeed, we picked  $T = n(n - k)$  vertices from  $A$  which is a clique and thus induce  $\binom{T}{2}$  edges. Then, for all  $e \in E$ , we picked  $2T$  vertices, which induce  $\binom{T}{2}$  edges if  $e \in E_0$ , and  $(\binom{T}{2} + 1)$  edges if  $e \in E_1$ . Since  $|E_0| = \binom{k}{2}$  (and thus  $|E_1| = m - \binom{k}{2}$ ), we have the desired number of edges.

**$G$  contains a  $k$ -clique  $\Leftrightarrow G'$  contains  $k'$  vertices inducing at most  $C'$  edges.**

Suppose now that  $K'$  is a set of  $k'$  vertices of  $G'$  which induces at most  $C'$  edges. We redefine the sets  $E_0$  and  $E_1$  as follows:  $E_0 = \{\{v_p, v_q\} \in E \text{ such that for all } j \in \{1, \dots, n\} \text{ we have } a_p^j \notin K' \text{ and } a_q^j \notin K'\}$ , and  $E_1 = E \setminus E_0$ .

For all  $R \subseteq V'$ , let  $tr(R) = K' \cap R$  be the trace of  $K'$  on  $R$ , and for all  $v \in V'$ , let  $\mu(v) = |tr(N(v))|$  be the number of neighbors of  $v$  belonging to  $K'$ .

Let  $u \in K'$  and  $v \in V' \setminus K'$ . We say that  $(K' \setminus \{u\}) \cup \{v\}$  is a safe replacement if and only if we have  $\mu(v) \leq \mu(u)$  if  $\{u, v\} \notin E'$  and  $\mu(v) - 1 \leq \mu(u)$  if  $\{u, v\} \in E'$ . For sake of readability, we will keep and update the definitions of  $E_0$  and  $E_1$  when replacing vertices of  $A$  (e.g. if we remove a vertex  $u \in A$  from  $K'$  and that there exists  $e \in E_1$  such that vertices of  $Z_e$  were only adjacent to  $u$  among all vertices of  $A$ , then  $e$  now belongs to  $E_0$ ).

The proof consists in replacing some vertices of  $K'$  by other vertices not in  $K'$  without increasing the number of induced edges, in order to obtain a solution that has the same structure as previously. We call such a replacement a *safe modification* or a *safe replacement*. The core of the proof is based on the three following lemmas.

**Lemma 2.** *Without loss of generality (and optimality of  $K'$ ), we can suppose that for all  $e \in E$  we have  $X_e \subseteq K'$ .*

*Proof.* Let  $S = \bigcup_{e \in E} X_e$ . Since we have  $k' > |S|$ , there always exists  $u \in K' \setminus S$ . Suppose that there exists  $e \in E$  and  $i \in \{1, \dots, T\}$  such that  $x_i^e \notin K'$ . If  $y_i^e \notin K'$ , then we have  $\mu(x_i^e) = 0$  and we can thus safely replace any other vertex of  $K' \setminus S$  by  $x_i^e$ . Now, if  $y_i^e \in K'$ , then  $\mu(x_i^e) = 1$ . Since  $\{x_i^e, y_i^e\} \in E'$ ,  $(K' \setminus \{y_i^e\}) \cup \{x_i^e\}$  is a safe replacement.  $\square$

**Lemma 3.**  *$K'$  can be safely modified such that one of the two following holds:*

*Case A1: for all  $e \in E_0$  we have  $tr(Z_e) = Z_e$ .*

*Case A2: for all  $e \in E_0$  we have  $tr(Y_e) = \emptyset$ .*

*Proof.* Let us first restructure each gadget of  $E_0$  separately. For all  $e \in E_0$  such that  $tr(Y_e) \neq \emptyset$  and  $tr(Z_e) \neq Z_e$ , let  $j_0 = \max\{j \in \{1, \dots, T\} : y_j^e \in tr(Y_e)\}$  and let  $j_1$  be such that  $z_{j_1}^e \notin tr(Z_e)$ . Recall that Lemma 2 ensures that  $x_{j_0}^e$  is in  $K'$ . If  $j_0 \neq 1$ , then  $\mu(y_{j_0}^e) = y + z + 1$ , where  $y = |N(y_{j_0}^e) \cap tr(Y_e)|$  and  $z = |N(y_{j_0}^e) \cap tr(Z_e)|$ . On the other side, we have  $\mu(z_{j_1}^e) \leq y + z + 1$  (more precisely,  $\mu(z_{j_1}^e) = y + z + 1$  if  $y_1^e \in K'$ , and  $\mu(z_{j_1}^e) = y + z$  if  $y_1^e \notin K'$ ). Roughly speaking, this switch ensures that we necessarily “loose” the edge due to the vertex of  $X^e$  and we gain at most one edge due to  $y_1^e$ . Hence  $\mu(z_{j_1}^e) \leq \mu(y_{j_0}^e)$  and  $(K' \setminus \{y_{j_0}^e\}) \cup \{z_{j_1}^e\}$  is a safe replacement. If  $j_0 = 1$ , then it means that  $tr(Y_e) = \{y_1^e\}$ . Suppose that there exists  $j_1$  such that  $z_{j_1}^e \notin tr(Z_e)$ . We have  $\mu(y_1^e) = z + 1$  where  $z = |N(y_1^e) \cap tr(Z_e)|$ , and  $\mu(z_{j_1}^e) = z + 1$ . Here again  $(K' \setminus \{y_1^e\}) \cup \{z_{j_1}^e\}$  is a safe replacement. After all these replacements, given any  $e \in E_0$ ,  $tr(Y_e) \neq \emptyset$  implies that  $tr(Z_e) = Z_e$ .

Then, we proceed to replacements between gadgets  $F_e$ ,  $e \in E_0$ . If one can find  $a, b \in E_0$  such that  $tr(Y_a) \neq \emptyset$  and  $tr(Z_b) \neq Z_b$ , then let  $j_0$  be such that  $y_{j_0}^a \in tr(Y_a)$  and let  $j_1$  be such that  $z_{j_1}^b \notin tr(Z_b)$ . We have  $\mu(y_{j_0}^a) \geq T + 1$  and  $\mu(z_{j_1}^b) \leq T - 1$ . Thus,  $(K' \setminus \{y_{j_0}^a\}) \cup \{z_{j_1}^b\}$  is a safe replacement.

Theses replacements end either when all the  $Y_e$  are empty for all  $e \in E_0$  or when all the  $Z_e$  are full for all  $e \in E_0$ , which achieves the proof of Lemma 3.  $\square$

**Lemma 4.**  $K'$  can be safely modified such that one of the two following holds:

Case B1: for all  $e \in E_1$  we have  $tr(Y_e) = Y_e$ .

Case B2: for all  $e \in E_1$  we have  $tr(Z_e) = \emptyset$ .

*Proof.* The proof is roughly based on the fact that replacing a vertex of  $Z_e$  by a vertex of  $Y_e$  permits to “lose” at least one edge with vertices  $A$  and “gain” one edge with a vertex of  $X_e$ . Let us formally prove Lemma 4. Similarly to the proof of Lemma 3, we first restructure each gadget of  $E_1$  separately: for all  $e \in E_1$  such that  $tr(Z_e) \neq \emptyset$  and  $tr(Y_e) \neq Y_e$ , let  $j_0 = \max\{j \in \{1, \dots, T\} : y_j^e \notin K'\}$  and let  $j_1$  be such that  $z_{j_1}^e \in tr(Z_e)$ . Recall that by definition of  $E_1$ , there exists  $i, j \in \{1, \dots, n\}$  such that  $z_{j_1}^e$  is adjacent to  $a_i^j$ . We have  $\mu(z_{j_1}^e) \geq y + z + 1$ , where  $y = |N(z_{j_1}^e) \cap Y_e|$  and  $z = |N(z_{j_1}^e) \cap Z_e|$ . On the other side, we have  $\mu(y_{j_0}^e) \leq z + y + 2$  (indeed,  $|N(y_{j_0}^e) \cap Z_e| = z + 1$ ,  $|N(y_{j_0}^e) \cap Y_e| \leq y$  and  $|N(y_{j_0}^e) \cap X_e| = 1$ ). Since  $\{y_{j_0}^e, z_{j_1}^e\} \in E'$ , it holds that  $(K' \setminus \{z_{j_1}^e\}) \cup \{y_{j_0}^e\}$  is a safe replacement. After all these replacements, given any  $e \in E_1$ ,  $tr(Z_e) \neq \emptyset$  implies that  $tr(Y_e) = Y_e$ .

We now proceed to replacements between gadgets  $F_e$ ,  $e \in E_1$ . If one can find  $a, b \in E_1$  such that  $tr(Z_a) \neq \emptyset$  and  $tr(Y_b) \neq Y_b$ , then let  $j_0$  be such that  $y_{j_0}^b \notin tr(Y_b)$  and let  $j_1$  be such that  $z_{j_1}^a \in tr(Z_a)$ . We have  $\mu(z_{j_1}^a) \geq T + 1$  and  $\mu(y_{j_0}^b) \leq T - 1$ . Thus  $(K' \setminus \{z_{j_1}^a\}) \cup \{y_{j_0}^b\}$  is a safe replacement.  $\square$

Let us now define for each case and each  $e \in E$  the set of vertices  $D_e \subseteq Y_e \cup Z_e$  that have to be replaced (see Figure 2):

- case A1: for all  $e \in E_0$ ,  $D_e = Y_e \cap K'$
- case A2: for all  $e \in E_0$ ,  $D_e = Z_e \setminus K'$
- case B1: for all  $e \in E_1$ ,  $D_e = Z_e \cap K'$
- case B2: for all  $e \in E_1$ ,  $D_e = Y_e \setminus K'$

Notice that if  $D_e = \emptyset$  for all  $e \in E_0$  (resp.  $e \in E_1$ ), then cases A1 and A2 (resp. B1 and B2) collapse. If such a case happen for all  $e \in E$ , we can immediately conclude, as shown by the following lemma:

**Lemma 5.** If  $D_e = \emptyset$  for all  $e \in E$ , then  $G$  contains a clique of size  $k$ .

*Proof.* By construction, we have  $|tr(A)| = T$  and  $|tr(F_e)| = 2T$  for all  $e \in E$ . Thus,  $cost^*(tr(A)) = \binom{T}{2}$  and  $cost^*(tr(F_e)) = \binom{T}{2} + 1$  if  $Y_e \subseteq K'$ , and  $cost^*(tr(F_e)) = \binom{T}{2}$  if  $Z_e \subseteq K'$ . By construction,  $Y_e \subseteq K'$  if and only if  $e \in E_1$ . Thus, since  $cost^*(K') \leq \binom{T}{2} + m \binom{T}{2} + m - \binom{k}{2}$ , we must have  $|E_1| \leq m - \binom{k}{2}$  which is equivalent to  $|E_0| \geq \binom{k}{2}$ . Hence, there exists at most  $\lfloor \frac{|A| - T}{n} \rfloor = k$  vertices in  $G$  inducing at least  $\binom{k}{2}$  edges, *i.e.*  $G$  contains a clique of size  $k$ .  $\square$

We now have to analyse the four cases of Lemma 3 and 4 (see Figure 2).

*Case A1 and B1*

To summarize the situation, the solution  $K'$  can be partitionned in  $K'_A = K' \cap A$ , and  $K'_F = K' \setminus K'_A$ , the vertices selected in the gadgets. Let  $\Delta_0 = \sum_{e \in E_0} |D_e|$  be the number of extra vertices allocated in all the gadgets  $F_e$ ,  $e \in E_0$ , and  $\Delta_1 = \sum_{e \in E_1} |D_e|$  be the number of extra vertices allocated in all the gadgets  $F_e$ ,  $e \in E_1$ . Let  $\Delta = \Delta_0 + \Delta_1$ . Notice that we have  $|K'_A| = T - \Delta$ , as a "regular" solution that does not select any extra vertex in a gadget has to pick  $T$  vertices in  $A$ . Moreover,

- vertices of  $K'$  selected in gadgets of  $E_0$  are not adjacent to  $K'_A$  (by definition of  $E_0$ )
- each gadget of  $E_0$  induces at least  $\binom{T}{2}$  edges (as we are in case A1)
- each gadget of  $E_1$  induces at least  $\binom{T}{2} + 1$  edges (as we are in case B1)

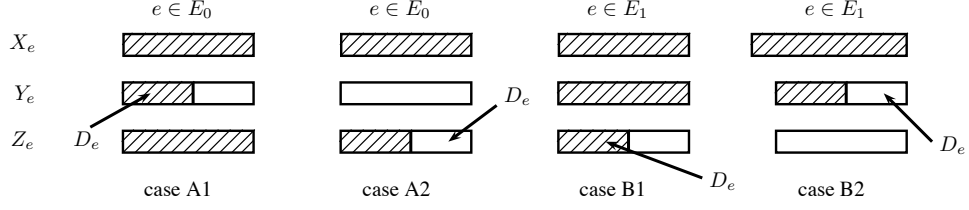


Figure2: Schema of different cases. Shaded rectangles represent part of  $K'$ .

- each of the  $\Delta_0$  vertices is adjacent to at least  $T$  vertices in  $K'$  (such a vertex is in a set  $Y_e$ , and thus is connected to the  $T$  vertices of  $Z_e$ )
- each of the  $\Delta_1$  vertices is adjacent to at least  $T + 1$  vertices in  $K'$  (such a vertex is in a set  $Z_e$ , and thus is connected to at least 1 vertex of  $K'_A$  and to the  $T$  vertices of  $Y_e$ )

Let us now lower bound the total cost of  $K'$ . We have

$$\begin{aligned}
\text{cost}^*(K') &\geq |E_0| \binom{T}{2} + |E_1| \left( \binom{T}{2} + 1 \right) + \Delta_0 T + \Delta_1 (T + 1) + \binom{T - \Delta}{2} \\
&\geq |E_0| \binom{T}{2} + |E_1| \left( \binom{T}{2} + 1 \right) + \Delta T + \binom{T - \Delta}{2} \\
&\geq |E_0| \binom{T}{2} + |E_1| \left( \binom{T}{2} + 1 \right) + \binom{T}{2} + \frac{\Delta^2}{2}
\end{aligned}$$

Notice that in a bad structured solution, a large  $\Delta$  allows to select only a few vertices in  $A$  ( $T - \Delta$  instead of  $T$ ), and thus to have many gadgets (more than  $\binom{k}{2}$ ) in  $E_0$ . Let us now consider the contrapositive, *i.e.* we consider that  $G$  does not contain a  $k$ -clique, and show that  $K'$  induces more than  $C'$  edges.

Let  $q$  and  $r$  such that  $\Delta = qn + r$ ,  $r < n$ . Let us upper bound  $|E_0|$ . As there is  $T - \Delta$  vertices in  $A$ , the number of empty "columns" (column  $u$  is empty iff none of the  $a_u^t$  is selected) is at most  $n - \frac{T - \Delta}{n} \leq k + q$ .

As  $G$  does not contain a  $k$ -clique, the  $k + q$  vertices corresponding to these  $k + q$  columns cannot induce a clique of size  $k + q$ , and thus  $|E_0| < \binom{k + q}{2}$ . Thus, we get

$$\begin{aligned}
\text{cost}^*(K') &> \binom{k + q}{2} \binom{T}{2} + \left( m - \binom{k + q}{2} \right) \left( \binom{T}{2} + 1 \right) + \binom{T}{2} + \frac{\Delta^2}{2} \\
&= C' - \left( \binom{q}{2} + kq \right) + \frac{\Delta^2}{2}
\end{aligned}$$

Thus, as  $\frac{\Delta^2}{2} > \binom{q}{2} + kq$ , we get the desired inequality.

#### Case A2 and B2

Let  $\Delta_0 = \sum_{e \in E_0} |D_e|$ ,  $\Delta_1 = \sum_{e \in E_1} |D_e|$  and  $\Delta = \Delta_0 + \Delta_1$  (recall that in this case,  $D_e \not\subseteq K'$  for all  $e \in E$ ). Here again we suppose  $\Delta > 0$ . Let us notice that for all  $u \in \text{tr}(A)$ ,  $\mu(u) \geq T$ . On the other hand, for all  $e \in E$  such that there exists  $v \in D_e$ , we have  $\mu(v) \leq T$  (remark that if  $e \in E_1$ , then  $D_e \subseteq Y_e$ , and if  $e \in E_0$ , then  $v$  is not adjacent to  $\text{tr}(A)$  by definition of  $E_0$ ). Thus  $(K' \setminus \{u\}) \cup \{v\}$  is a safe replacement. Since before this replacement we had

$tr(A) = T + \Delta$ , it is clear that we can repeat this replacement (*i.e.*  $K' \setminus \{u\} \cup \{v\}$  where  $u \in tr(A)$  and  $v \in D_e$  for some  $e \in E$ )  $\Delta$  times safely. At this point, the updated value of  $\Delta$  is 0, *i.e.*  $D_e = \emptyset$  for all  $e \in E$ . By Lemma 5, we must have a clique of size  $k$  in  $G$ .

#### Case A2 and B1

If there exists  $e \in E_0$  such that there exists  $u \in D_e$ , then  $\mu(u) < T$ . If such a vertex exists, then either  $|tr(A)| > T$  or there exists  $e' \in E_1$  such that there exists  $v \in D_{e'}$ . In the first case for all  $x \in tr(A)$  we have  $\mu(x) \geq T$ , and  $(K' \setminus \{x\}) \cup \{u\}$  is a safe replacement. In the second case we have  $\mu(v) > T$  and here again  $(K' \setminus \{v\}) \cup \{u\}$  is a safe replacement. After these replacements we must have  $D_e = \emptyset$  for all  $e \in E_0$ , and we can apply the same arguments as for case A1 and B1.

#### Case A1 and B2

If there exists  $e \in E_1$  such that there exists  $u \in D_e$ , then  $\mu(u) < T$ . If such a vertex exists, then either  $|tr(A)| > T$  or there exists  $e' \in E_0$  such that there exists  $v \in D_{e'}$ . In the first case for all  $x \in tr(A)$  we have  $\mu(x) \geq T$ , and  $(K' \setminus \{x\}) \cup \{u\}$  is a safe replacement. In the second case we have  $\mu(v) > T$  and here again  $(K' \setminus \{v\}) \cup \{u\}$  is a safe replacement. After these replacements we must have  $D_e = \emptyset$  for all  $e \in E_1$ , and we can apply the same arguments as for case A1 and B1. □

## References

1. N. Apollonio. Private communication, 2012.
2. N. Apollonio and A. Sebő. Minconvex factors of prescribed size in graphs. *SIAM Journal of Discrete Mathematics*, 23(3):1297–1310, 2009.
3. J. Backer and J.M. Keil. Constant factor approximation algorithms for the densest  $k$ -subgraph problem on proper interval graphs and bipartite permutation graphs. *Information Processing Letters*, 110(16):635–638, 2010.
4. H. Broersma, P. A. Golovach, and V. Patel. Tight complexity bounds for FPT subgraph problems parameterized by clique-width. In *Proceedings of the 6th international conference on Parameterized and Exact Computation, IPEC'11*, pages 207–218, Berlin, Heidelberg, 2012. Springer-Verlag.
5. D.G. Corneil and Y. Perl. Clustering and domination in perfect graphs. *Discrete Applied Mathematics*, 9(1):27 – 39, 1984.
6. J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006.
7. M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
8. G. Joret and A. Vetta. Reducing the rank of a matroid. CoRR, abs/1211.4853, 2012.
9. M. Liazi, I. Milis, F. Pascual, and V. Zissimopoulos. The densest  $k$ -subgraph problem on clique graphs. *Journal of Combinatorial Optimization*, 14(4):465–474, 2007.
10. M. Liazi, I. Milis, and V. Zissimopoulos. A constant approximation algorithm for the densest  $k$ -subgraph problem on chordal graphs. *Information Processing Letters*, 108(1):29–32, 2008.
11. T. Nonner. PTAS for densest  $k$ -subgraph in interval graphs. In *Proceedings of the 12th international conference on Algorithms and Data Structures*, pages 631–641. Springer, 2011.