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# 2-distance coloring of sparse graphs<sup>★</sup>

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## Abstract

A 2-distance coloring of a graph is a coloring of the vertices such that two vertices at distance at most 2 receive distinct colors. We prove that every graph with maximum degree  $\Delta$  at least 4 and maximum average degree less than  $\frac{7}{3}$  admits a 2-distance  $(\Delta + 1)$ -coloring. This result is tight. This improves previous known results of Dolama and Sopena.

*Keywords:* 2-distance coloring; square coloring; maximum average degree.

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## 1 Introduction

All the graphs we consider here are simple, finite and undirected. Let  $G = (V, E)$  be a graph. For any subgraph  $H$  of  $G$ , we denote  $V(H)$  and  $E(H)$  the vertices and edges of  $H$ . For any vertex  $v \in V$ , the *degree* of  $v$  in  $G$ , denoted  $d(v)$ , is the number of neighbors of  $v$  in  $G$ . The *maximum degree* of  $G$ , denoted  $\Delta(G)$ , is  $\max_{v \in V} d(v)$ . The *maximum average degree* of  $G$ , denoted  $\text{mad}(G)$ , is the maximum for every subgraph  $H$  of  $G$  of  $\frac{2|E(H)|}{|V(H)|}$ . A *2-distance coloring*

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of a graph  $G$  is a coloring of the vertices of  $G$  such that two vertices that are adjacent or have a common neighbor receive distinct colors. This is equivalent to a proper vertex-coloring of the square of  $G$ . We define  $\chi^2(G)$  as the smallest  $k$  such that  $G$  admits a 2-distance  $k$ -coloring. Note that any graph  $G$  satisfies  $\chi^2(G) \geq \Delta(G) + 1$ . The *girth*  $g(G)$  is the length of a shortest cycle in  $G$ . Two vertices  $x$  and  $y$  are *p-linked* if there exists a path  $x-v_1 \cdots -v_p-y$  such that vertices  $v_1, \dots, v_p$  have degree 2, and  $v_1 \cdots -v_p$  is called a *branch* of  $x$  (or  $y$ ).

Borodin, Ivanova and Neustroeva [1] studied sparse planar graphs, and prove the following result:

**Theorem 1.1** ([1]) *Every planar graph  $G$  with  $g(G) \geq 15$  and  $\Delta(G) \geq 4$  admits a 2-distance  $(\Delta(G) + 1)$ -coloring.*

Note that this result was later extended to list-coloring [2].

Dolama and Sopena [3] proved a more general result than Theorem 1.1, which is not restricted to planar graphs anymore. Theorem 1.2 however presents a slight loss in quality compared to Theorem 1.1: since for any planar graph  $G$ ,  $(\text{mad}(G) - 2)(g(G) - 2) < 4$ , Theorem 1.2 implies only that Theorem 1.1 holds for  $g(G) \geq 16$ .

**Theorem 1.2** ([3]) *Every graph  $G$  with  $\text{mad}(G) < \frac{16}{7}$  and  $\Delta(G) \geq 4$  admits a 2-distance  $(\Delta(G) + 1)$ -coloring.*

We aim at making the upper bound on the maximum average degree optimal, and prove the following.

**Theorem 1.3** *Every graph  $G$  with  $\text{mad}(G) < \frac{7}{3}$  and  $\Delta(G) \geq 4$  admits a 2-distance  $(\Delta(G) + 1)$ -coloring.*

The bound we obtain is optimal. Indeed, as pointed out by Montassier [6], there is a graph  $G$  with  $\text{mad}(G) = \frac{7}{3}$ ,  $\Delta(G) = 4$  and  $\chi^2(G) = 6$  (see Figure 1).

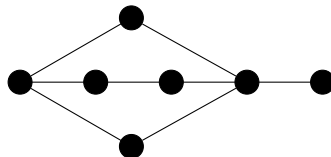


Fig. 1. A graph  $G$  with  $\text{mad}(G) = \frac{7}{3}$ ,  $\Delta(G) = 4$  and  $\chi^2(G) = 6$ .

When restricted to planar graphs, Theorem 1.3 is an improvement of Theorem 1.1 as it implies that Theorem 1.1 holds with  $g(G) \geq 14$ . It is not comparable to the more general result in [2], since we are not considering list-coloring.

We are going to use a discharging method to prove Theorem 1.3. We will prove that there are some configurations a minimal counter-example cannot contain, and, then use discharging rules to show that this graph does not exist.

## 2 Proof

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label ' $i$ ' means "exactly  $i$  neighbors", the label ' $i^+$ ' (resp. ' $i^-$ ') means that it has at least (resp. at most)  $i$  neighbors. Note that the white vertices may coincide with other vertices. The label ' $T(v, a)$ ' inside a vertex  $v$  means that  $T(v, a)$  exists, as defined below.

A configuration  $T(v, a_4)$  (see Figure 2), is inductively defined as a vertex  $v$  of degree 4 with neighbors  $a_1, a_2, a_3, a_4$ , where for  $i \in \{1, 2, 3\}$ , vertex  $v$  is 2-linked by a path  $v-a_i-b_i-w_i$  either to a vertex  $w_i$  of degree at most 3 or to a configuration  $T(w_i, b_i)$ .

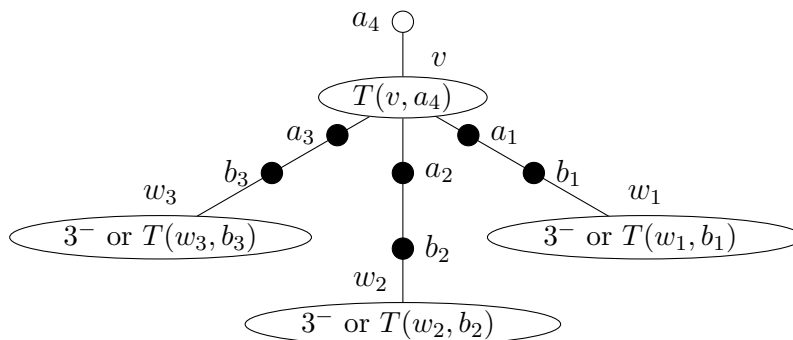


Fig. 2. A  $T(v, a_4)$ .

Now we define configurations  $(C_1)$  to  $(C_5)$  (see Figure 3).

- $(C_1)$  is a vertex of degree 0 or 1.
- $(C_2)$  is a vertex 3-linked to a vertex not of maximal degree.
- $(C_3)$  is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree at most 3.
- $(C_4)$  is a vertex  $u$  of degree at most 3 that is 2-linked by a path  $u-y-x-v$  to a vertex  $v$  such that  $T(v, x)$  exists.
- $(C_5)$  is a vertex  $u$  of degree 3 that is 2-linked to two vertices, and 1-linked by a path  $u-x-v$  to a vertex  $v$  such that  $T(v, x)$  exists.

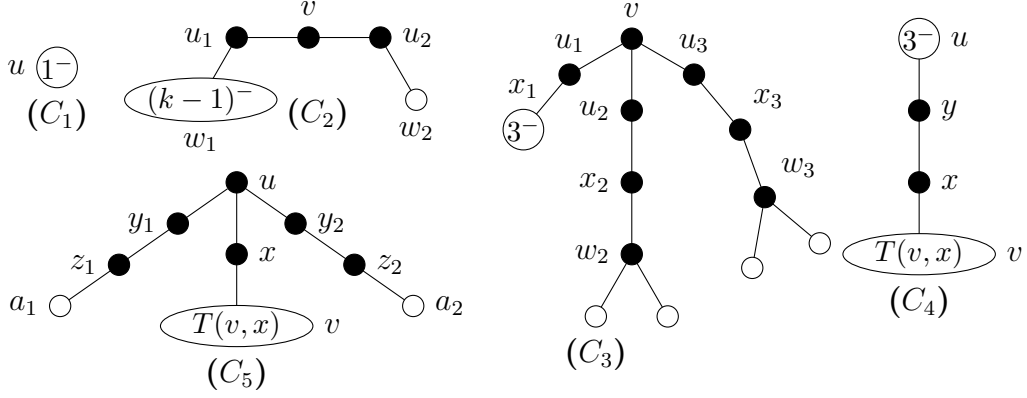


Fig. 3. Forbidden configurations.

In the following lemma, we actually use  $k$  instead of  $\Delta(G)$  in order to ensure that any subgraph of  $G$  admits a  $(k + 1)$ -coloring even though  $\Delta$  can decrease.

A graph is *minimal* for a property if it satisfies this property but none of its subgraphs does.

**Lemma 2.1** *Let  $k \geq 4$  and  $G$  such that  $\Delta(G) \leq k$  and  $G$  admits no 2-distance  $(k + 1)$ -coloring, and  $G$  is minimal for this property. Then  $G$  does not contain any of Configurations  $(C_1)$  to  $(C_5)$ .*

The following lemma will ensure that the discharging rules we introduce later are well-defined.

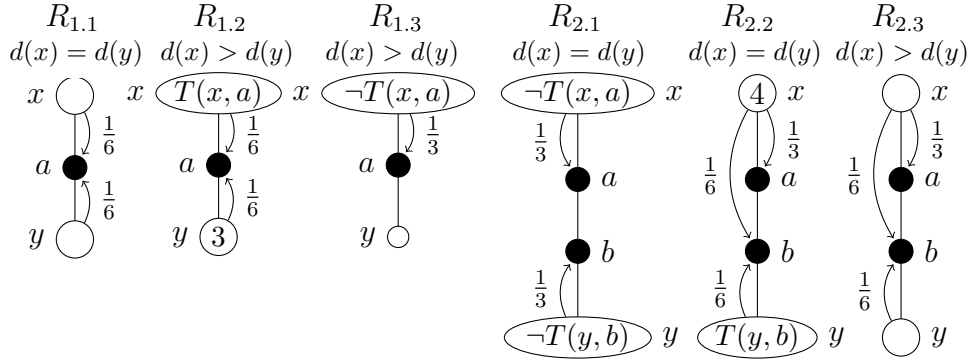
**Lemma 2.2** *In a graph  $G$  where  $(C_4)$  is forbidden, and  $x$  and  $y$  are two vertices of degree 4 that are 2-linked by a path  $x-a-b-y$ , at most one of  $T(x, a)$  and  $T(y, b)$  exists.*

We design discharging rules  $R_1, R_2, R_3$  (see Figure 4). We use them in the proof of Lemma 2.3, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least  $\frac{7}{3}$ . For any two vertices  $x$  and  $y$  of degree at least 3, with  $d(x) \geq d(y)$ ,

- Rule  $R_1$  is when  $x$  and  $y$  are 1-linked by a path  $x - a - y$ .
  - $(R_{1.1})$  If  $d(x) = d(y)$ , then both  $x$  and  $y$  give  $\frac{1}{6}$  to  $a$ .
  - $(R_{1.2})$  If  $d(x) > d(y)$  and  $T(x, a)$  exists, then both  $x$  and  $y$  give  $\frac{1}{6}$  to  $a$ .
  - $(R_{1.3})$  If  $d(x) > d(y)$  and  $T(x, a)$  does not exist, then  $x$  gives  $\frac{1}{3}$  to  $a$ .
- Rule  $R_2$  is when  $x$  and  $y$  are 2-linked by a path  $x - a - b - y$ .
  - $(R_{2.1})$  If  $d(x) = d(y)$  and neither  $T(x, a)$  nor  $T(y, b)$  exist, then  $x$  (resp.  $y$ ) gives  $\frac{1}{3}$  to  $a$  (resp.  $b$ ).

- ( $R_{2.2}$ ) If  $d(x) = d(y)$  and  $T(y, b)$  exists, then  $x$  gives  $\frac{1}{3}$  to  $a$  and both  $x$  and  $y$  give  $\frac{1}{6}$  to  $b$ .
- ( $R_{2.3}$ ) If  $d(x) > d(y)$ , then  $x$  gives  $\frac{1}{3}$  to  $a$  and both  $x$  and  $y$  give  $\frac{1}{6}$  to  $b$ .
- Rule  $R_3$  is when  $x$  and  $y$ , both of degree at least 4, are 3-linked by a path  $x - a - b - c - y$ . Then  $x$  gives  $\frac{1}{3}$  to  $a$  and  $\frac{1}{6}$  to  $b$ , and symmetrically for  $y$ .

**Rule 1:**  $x$  and  $y$  are 1-linked    **Rule 2:**  $x$  and  $y$  are 2-linked



**Rule 3:**  $x$  and  $y$  are 3-linked.

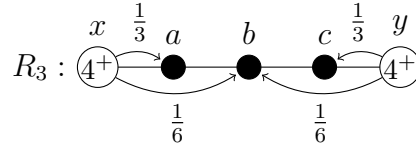


Fig. 4. Discharging rules  $R_1$ ,  $R_2$ ,  $R_3$ .

We use these discharging rules to prove the following lemma:

**Lemma 2.3** *A graph  $G$  that does not contain Configurations  $(C_1)$  to  $(C_5)$  verifies  $\text{mad}(G) \geq \frac{7}{3}$ .*

*Proof of Theorem 1.3*

We prove a stronger version of Theorem 1.3 by contradiction. For  $k \geq 4$ , let  $G$  be a minimal graph such that  $\Delta(G) \leq k$ ,  $\text{mad}(G) < \frac{7}{3}$  and  $G$  does not admit a  $(k+1)$ -coloring. Graph  $G$  is also a minimal graph such that  $\Delta(G) \leq k$  and  $G$  does not admit a  $(k+1)$ -coloring (all its proper subgraphs verify  $\Delta \leq k$  and  $\text{mad} < \frac{7}{3}$ , so they admit a  $(k+1)$ -coloring). By Lemma 2.1, graph  $G$  cannot contain  $(C_1)$  to  $(C_5)$ . Lemma 2.3 implies that  $\text{mad}(G) \geq \frac{7}{3}$ . Contradiction.  $\square$

### 3 Conclusion

We actually proved a slightly stronger result than Theorem 1.3. However, the addition, namely that every graph  $G$  with  $\text{mad}(G) < \frac{7}{3}$  and  $\Delta(G) \leq 3$  admits a 2-distance 5-coloring, can be derived from a result of Dvořák, Škrekovski and Tancer [4].

Note that the proof of Theorem 1.3 also provides an  $O(|V|^3)$  algorithm to find a 2-distance coloring of a graph  $G$  with  $\Delta(G) + 1$  colors if  $G$  verifies the hypothesis of Theorem 1.3: indeed Lemma 2.3 proves that every graph  $G$  with  $\text{mad}(G) < \frac{7}{3}$  contains  $(C_1)$ ,  $(C_2)$ , ... or  $(C_5)$ . Consequently, we can find a  $(C_i)$  in  $G$ , remove the corresponding vertices, and extend the coloring to the initial graph using the proof of Lemma 2.1.

As it was conjectured by Kostochka and Woodall [5] that 2-distance list-coloring requires exactly as many colors as 2-distance coloring, future work could aim at extending Theorem 1.3 to list-coloring.

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