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# Enumerating the edge-colourings of a regular graph

Stéphane Bessy and Frédéric Havet

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LIRMM, UM2, Montpellier

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CNRS, Sophia-Antipolis

FRANCE

2012 Workshop on Graph Theory and Combinatorics
Deparment of Applied Mathematics, National Sun Yat-sen University,
Kaohsiung, Taiwan.

# Outline


1. Introduction
   - Colorings
   - Algorithmic problems
   - Our results
2. Enumerating the 3-edge colorings of a cubic graph
   - The 3-edge colorings of a 3-regular graph
   - Turning the proof into algorithm
3. Extensions: $k$-edge colorings and the total colorings
   - $k$-edge colorings
   - A more precise bounds for the 3-edge coloring
   - Total coloring
4. Conclusion

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# Vertex Coloring

**Definition ($k$-vertex coloring)**

A *$k$-vertex coloring* of a (non oriented) graph $G = (V, E)$ is a function $c : V \to \{1, \ldots, k\}$ such that $uv \in E$ implies $c(u) \neq c(v)$.

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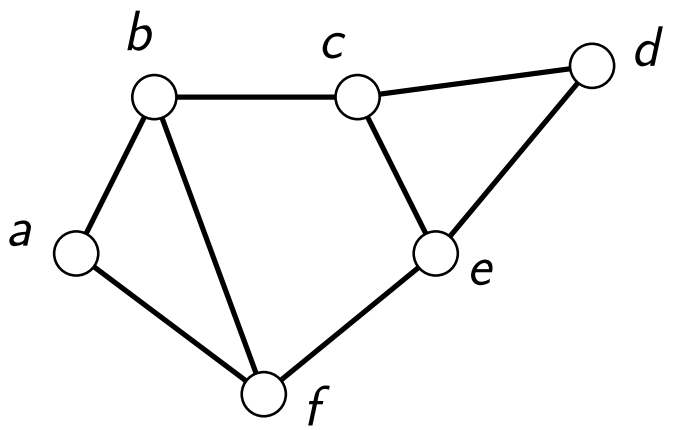

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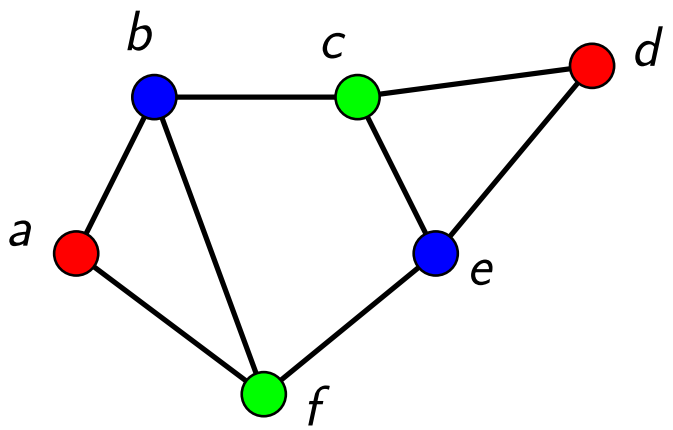

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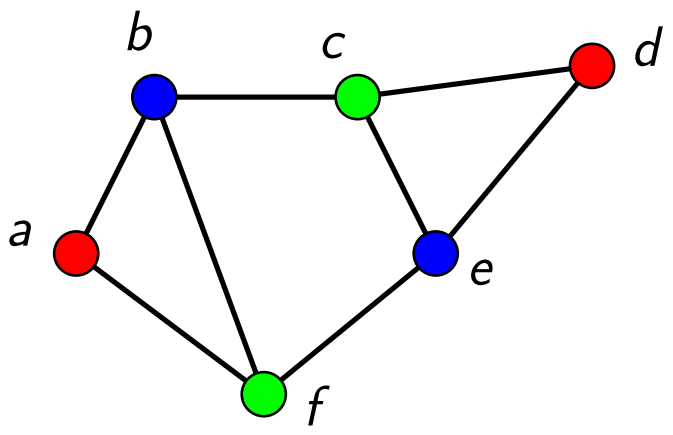


## Definition (chromatic number)

$\chi(G)$ is the minimum $k$ such that $G$ admits a $k$-vertex coloring.

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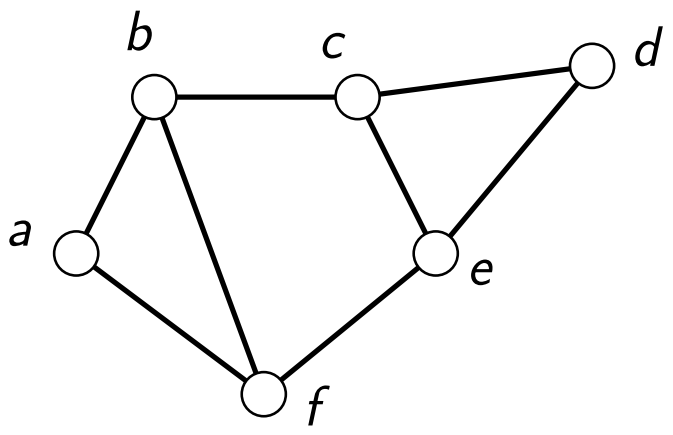

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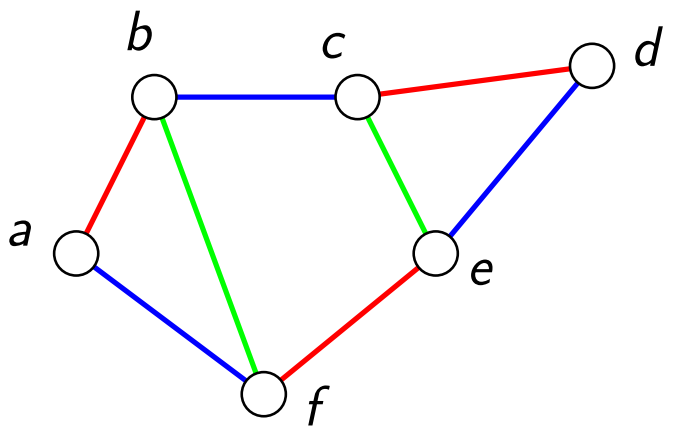

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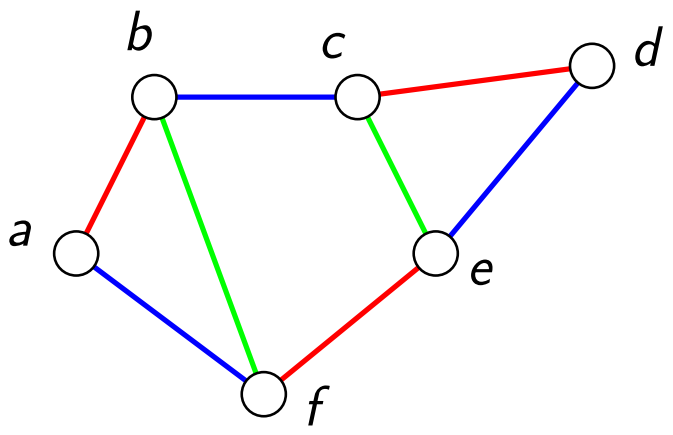
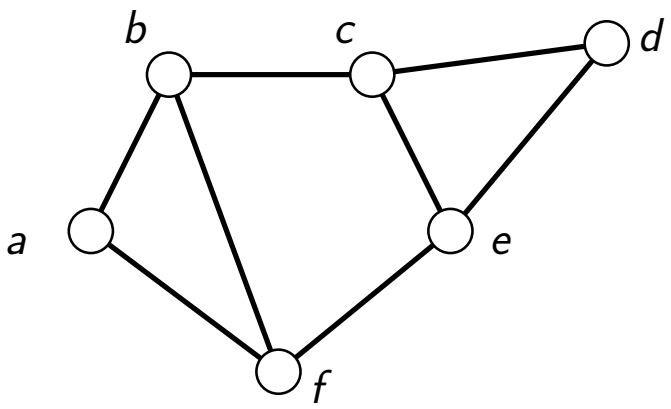



## Definition (chromatic index)

$\chi'(G)$ is the minimum $k$ such that $G$ admits a $k$-edge coloring.

# Edge Coloring

- The *line graph* $L(G)$ of $G$:
  - The vertex set of $L(G)$ is the edge set of $G$
  - Two edges $e$ and $f$ of $G$ are adjacent in $L(G)$ if there are incident in $G$

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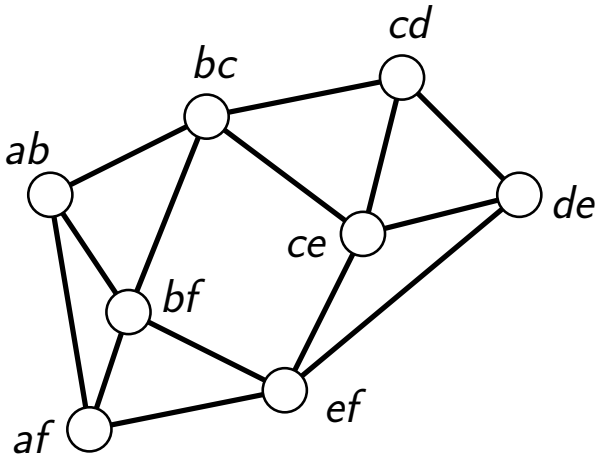


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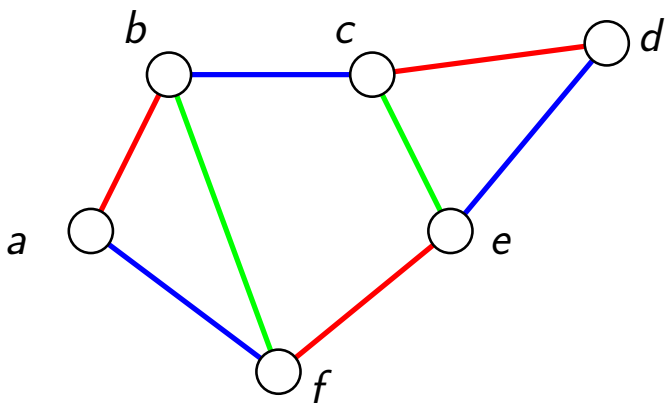


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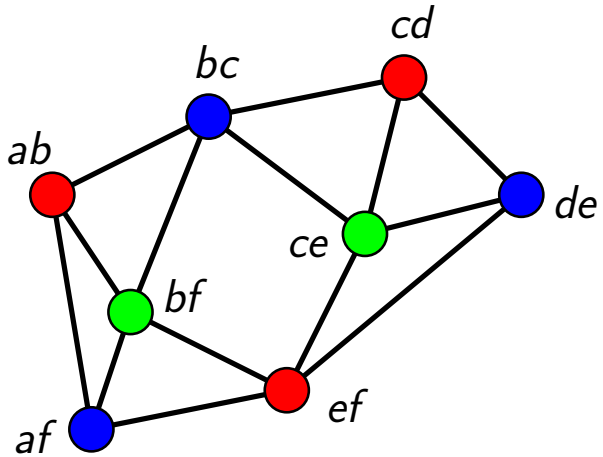


$L(G)$

- $\chi'(G) = \chi(L(G))$

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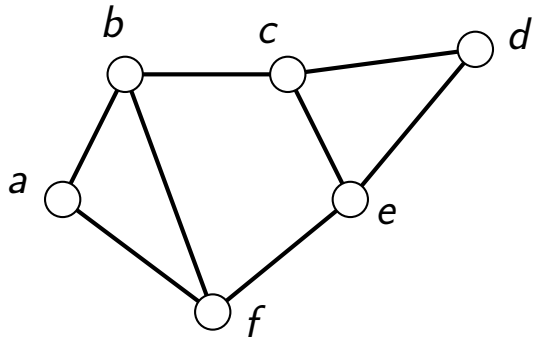

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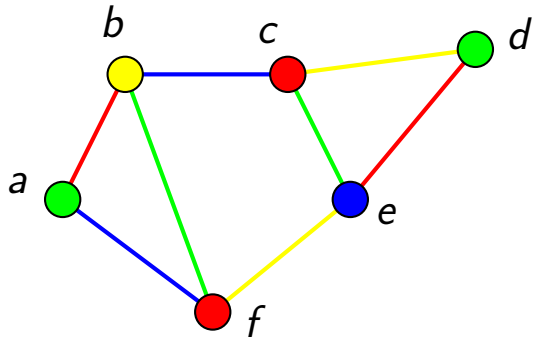

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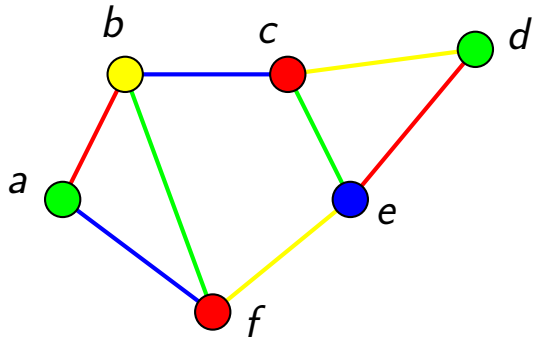


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Let $\Delta(G)$ denotes the maximum degree of $G$.

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- Vizing's Theorem (1964): $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$
- Total coloring conjecture (Behzad, Vizing, ∼1964): $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$

# Algorithmic problems

$k$-VERTEX/EDGE/TOTAL COLORING:

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- *parameterized algorithm*: find an algorithm with running time $f(k).poly(n)$ for some parameter $k$.
- **exact (exponential) algorithms**: find an algorithm with running time $O^\star(c^n)$ $(= O(P(n)c^n))$

# Examples of exact algorithms

ENUM-3-EDGE COLORING:

- *Input:* a graph $G = (V, E)$ with $\Delta(G) \leq 3$.
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Usual methods to improve: **branching algorithms** and dynamic programming.

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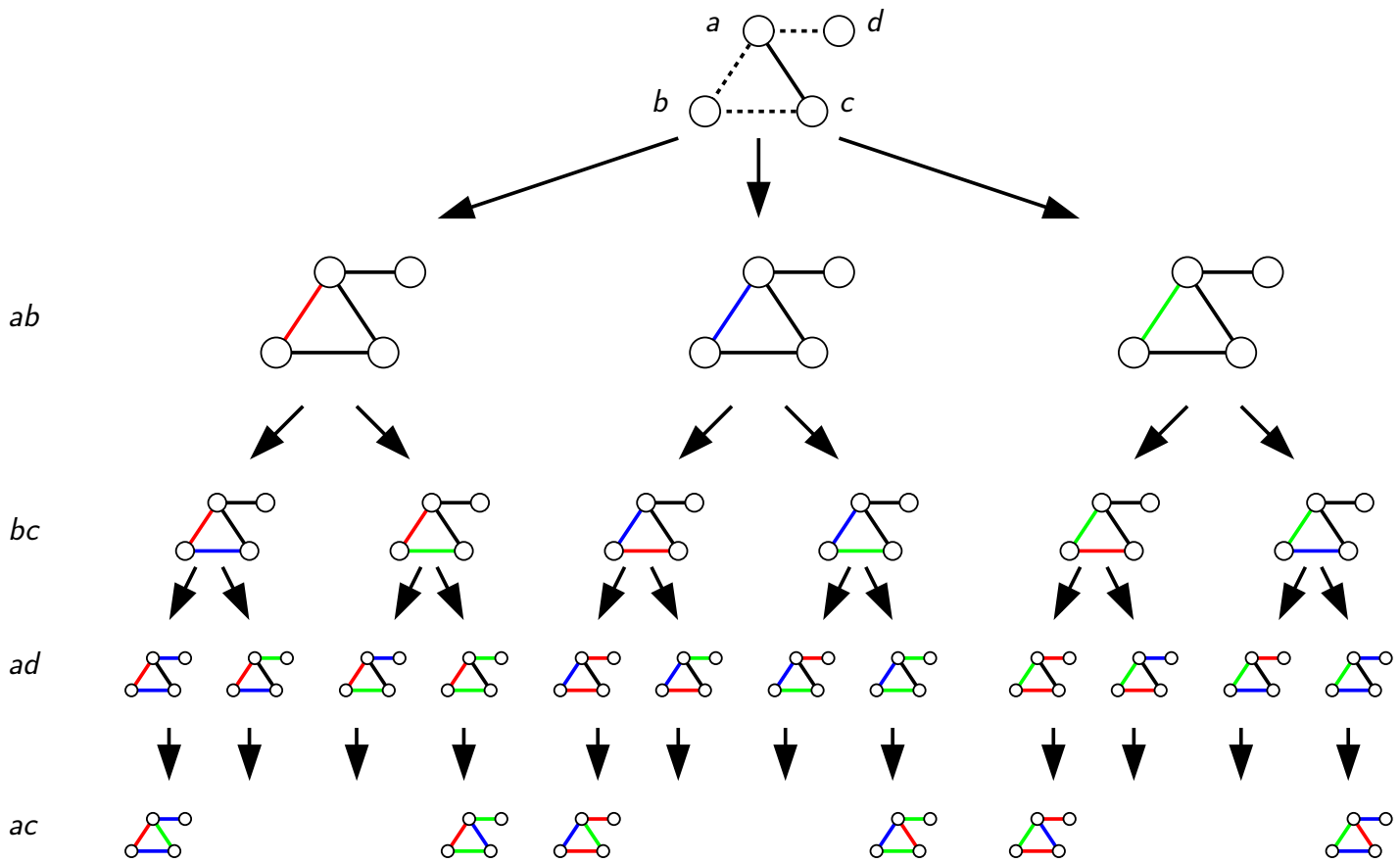

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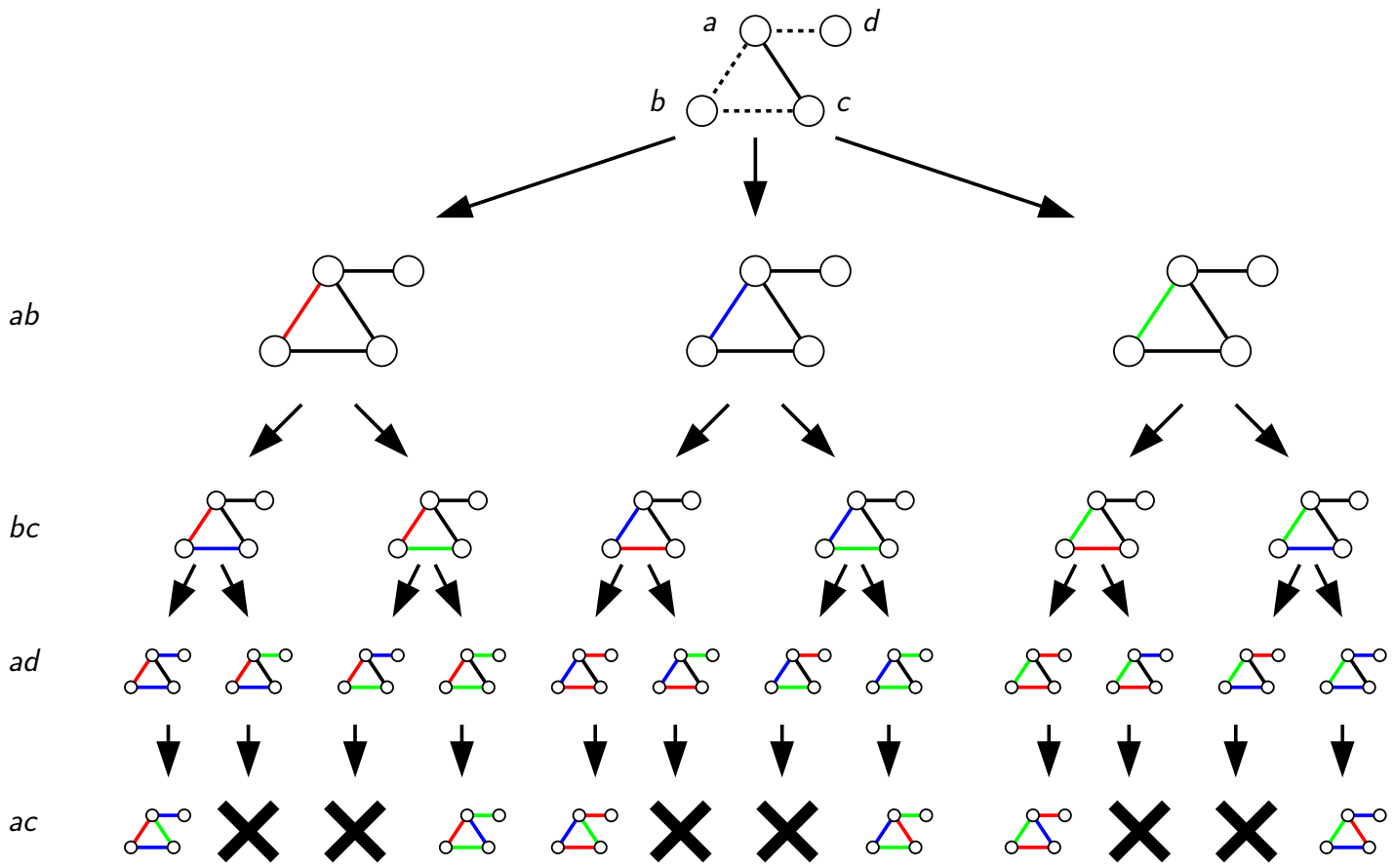

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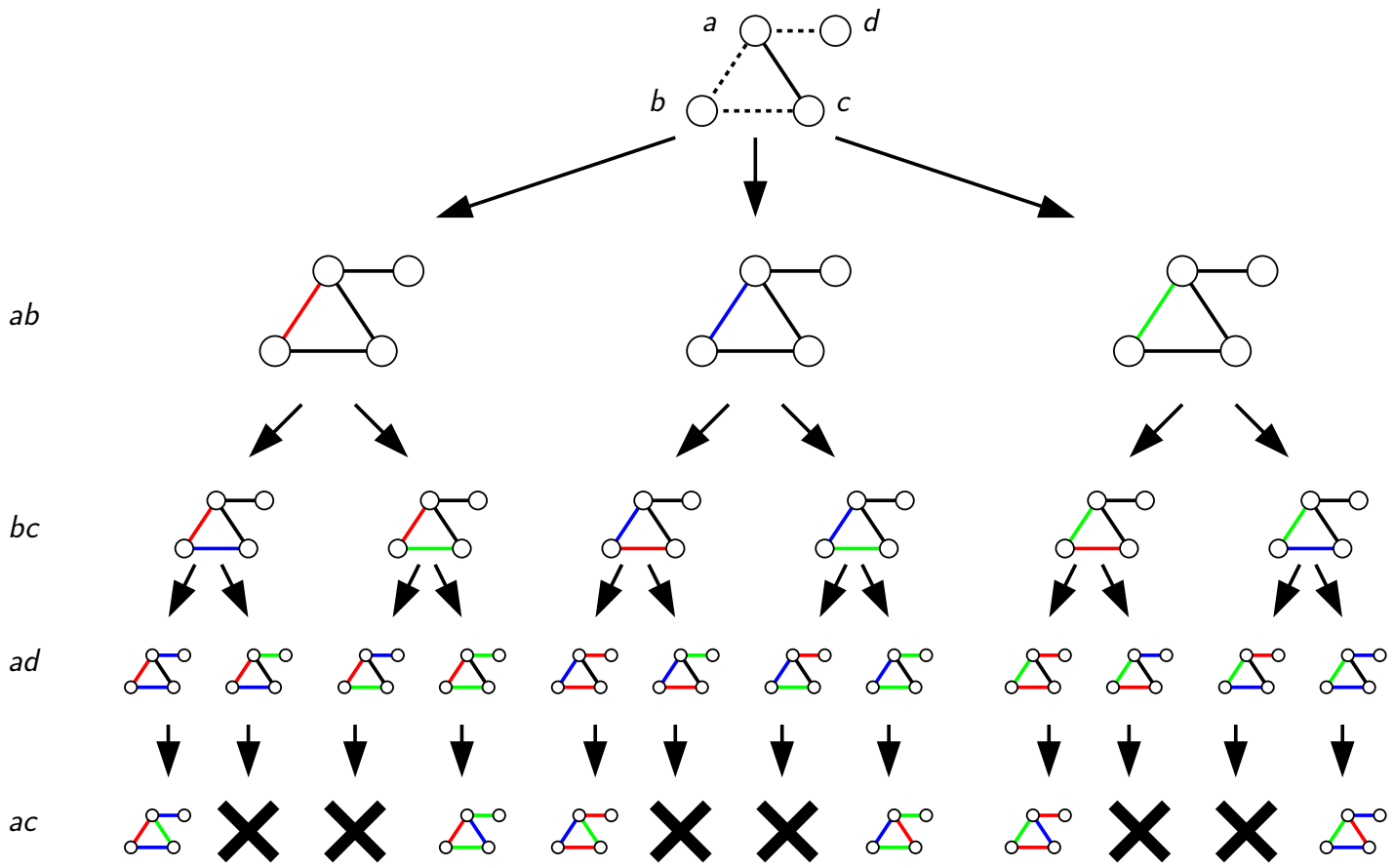


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Polynomial space.

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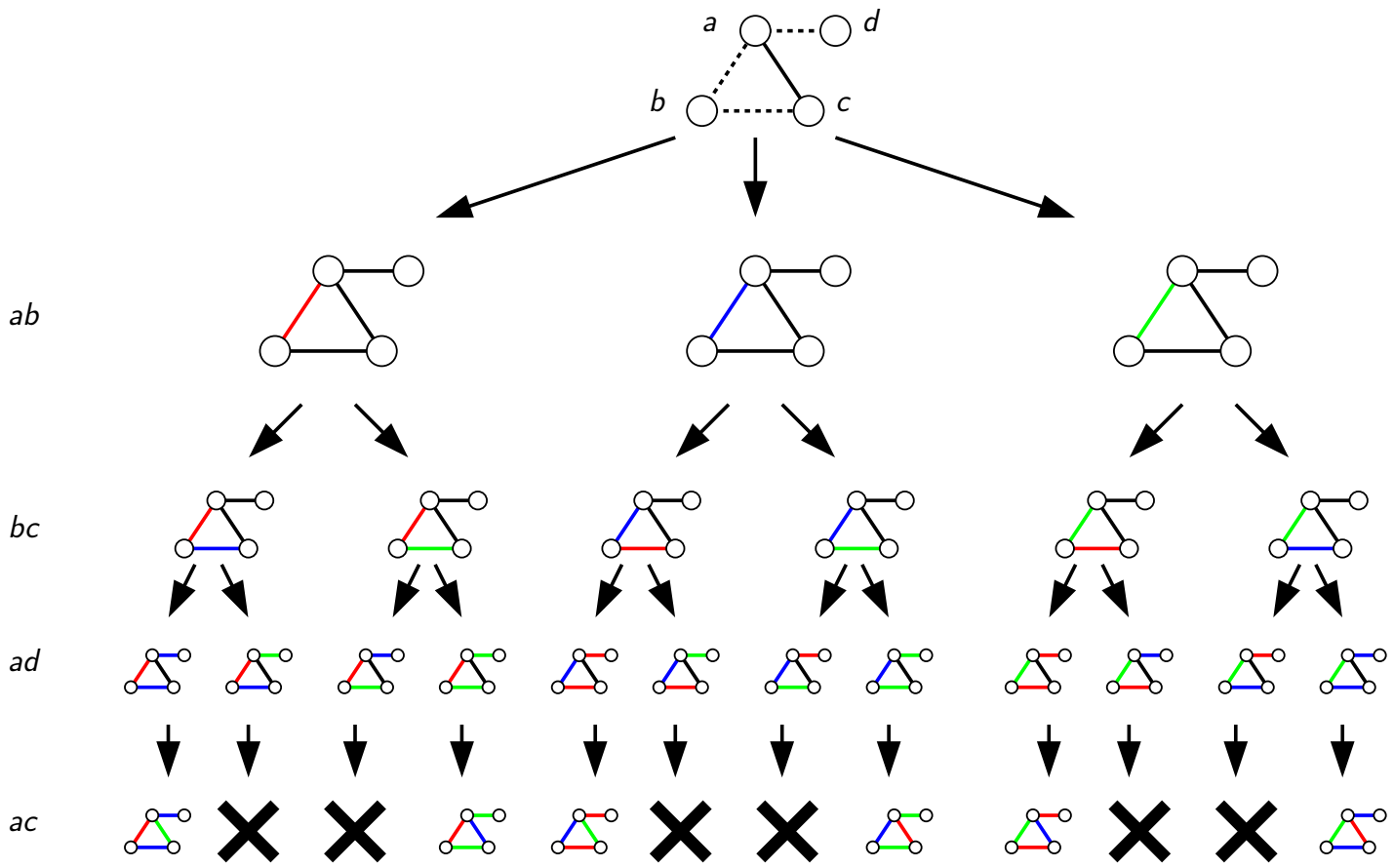


Time analysis: $O^\star(2^m) = O^\star(2^{\frac{3}{2}n}) = O^\star(2.8284^n)$ min. nb of leaves

Polynomial space. min. height

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Here: $\frac{\log 3^{\frac{3}{2}}}{\log 2^{\frac{3}{2}}} = 1.58$

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- MINIMUM VERTEX COLORING in $O^\star(2^n)$ (A. Bjöklund, T. Husfeld and M. Koivisto, 2006)
- ENUM 3-VERTEX COLORING in $O^\star(1.6259^n)$ (F. Fomin, S. Gasper and S. Saurabh, 2007)
- 3-EDGE COLORING in $O^\star(1.344^n)$ (L. Kowalik, 2009)
- ENUM-3-EDGE COLORING in $O^\star(1.5423^n)$ (P.A. Golovatch, D. Kratsch and J.F. Couturier, 2010)
- ENUM-4-TOTAL COLORING in $O^\star(3.0845^n)$ (P.A. Golovatch, D. Kratsch and J.F. Couturier, 2010)

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# Enumerating the 3-edge colorings


1. Introduction
   - Colorings
   - Algorithmic problems
   - Our results
2. Enumerating the 3-edge colorings of a cubic graph
   - The 3-edge colorings of a 3-regular graph
   - Turning the proof into algorithm
3. Extensions: $k$-edge colorings and the total colorings
   - $k$-edge colorings
   - A more precise bounds for the 3-edge coloring
   - Total coloring
4. Conclusion

# Enum-3-edge coloring

We denote by $c_k(G)$ the number of $k$-edge-colorings of a graph $G$.

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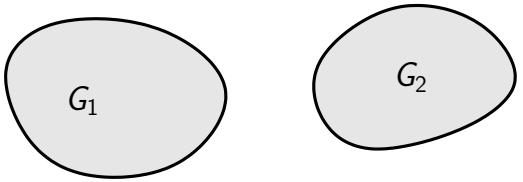


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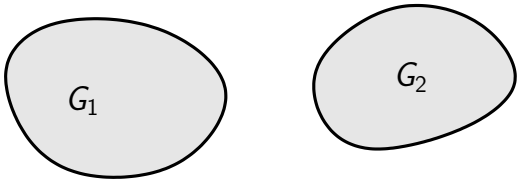
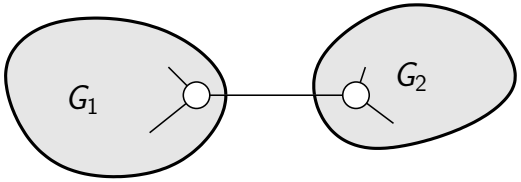



$$c_3(G) = c_3(G_1) \cdot c_3(G_2)$$

We can assume that $G$ is 2-(vertex) connected.




$$c_3(G) = \frac{1}{3} \cdot c_3(G_1) \cdot c_3(G_2)$$

# Number of 3-edge colorings of 3-regular graphs

**Lemma**

*Let $C_n$ be the cycle of length $n$.*

$$c_3(C_n) = \begin{cases} 2^n + 2, & \text{if } n \text{ is even,} \\ 2^n - 2, & \text{if } n \text{ is odd.} \end{cases}$$

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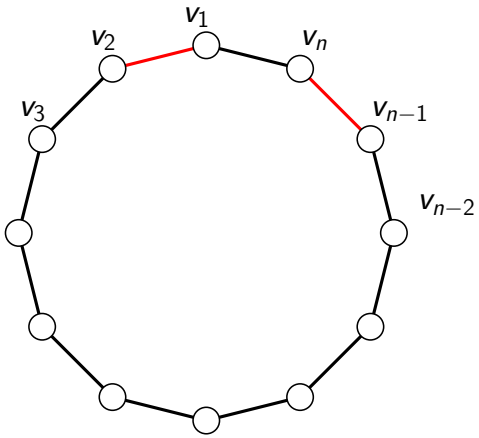


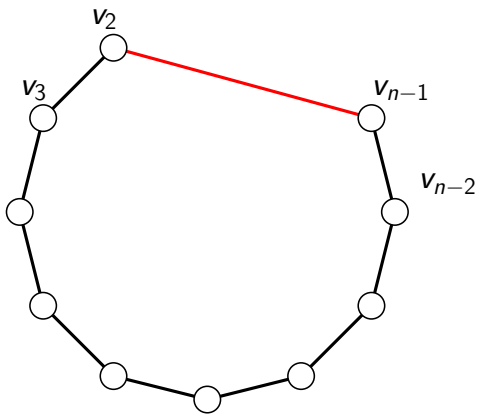


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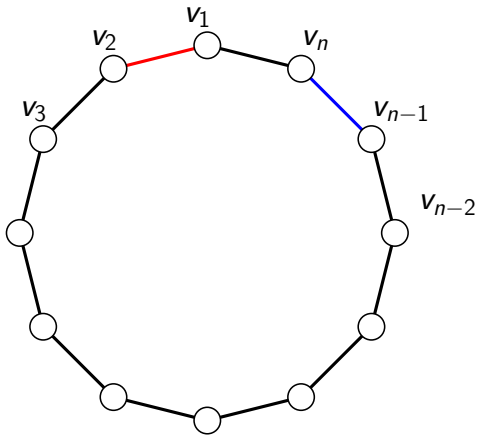


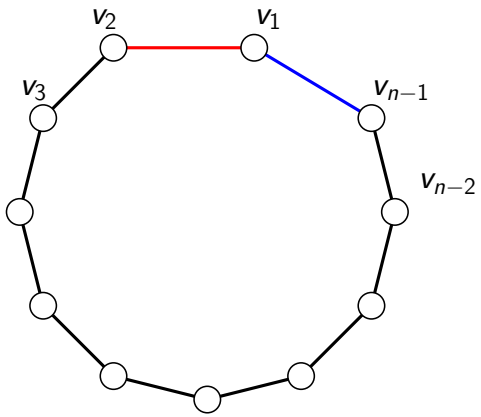


$$c_3(C_n) = 2.c_3(C_{n-2}) + c_3(C_{n-1}) \qquad \square$$

# Number of 3-edge colorings of 3-regular graphs

We denote by $n_i$ the number of vertices of degree $i$ in $G$.

**Theorem**

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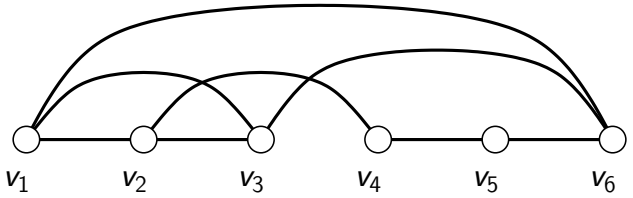
Otherwise, let $v_1$ and $v_n$ be two vertices of degree 3 and consider $v_1, \ldots, v_n$ an *st-ordering* of $G$:

for all $1 < i < n$, $\quad d(v_i)_{\{v_1,\ldots,v_{i-1}\}} \geq 1$ and $d(v_i)_{\{v_{i+1},\ldots,v_n\}} \geq 1$.

(always exists: Lempel et al.,1967)



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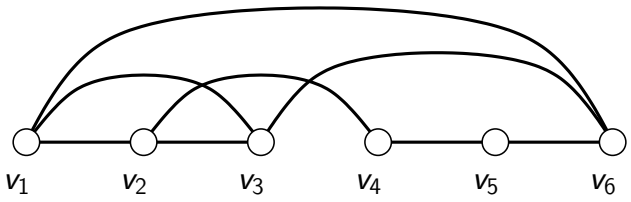


Orient from left to right.
$A^+ = \{v \ : \ d^+(v) = 2 \text{ and } d^-(v) = 1\}$
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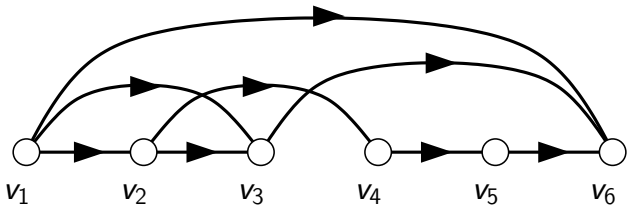


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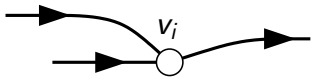
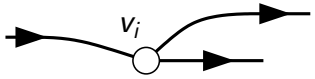
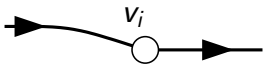
- if $v_i \in A^-$  $c_i \leq c_{i-1}$
- if $v_i \in A^+$  $c_i \leq 2c_{i-1}$
- if $d^+(v_i) = d^-(v_i) = 1$  $c_i \leq 2c_{i-1}$

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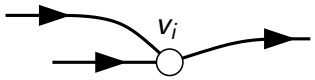
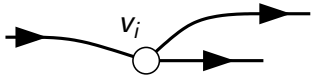
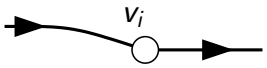
- if $v_i \in A^-$  $c_i \leq c_{i-1}$
- if $v_i \in A^+$  $c_i \leq 2c_{i-1}$
- if $d^+(v_i) = d^-(v_i) = 1$  $c_i \leq 2c_{i-1}$

$c_3(G) \leq 6 \cdot 2^{(n-2)-(n_3-2)/2} = 3 \cdot 2^{n-\frac{n_3}{2}}$ □

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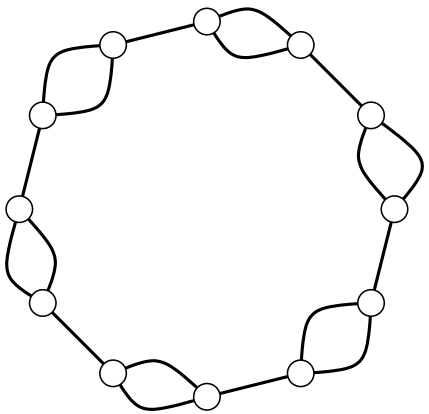
*Let $G$ be a 2-connected subcubic graph. Then $c_3(G) \leq 3 \cdot 2^{n-\frac{n_3}{2}}$.*

**Corollary**

*Let $G$ be a cubic graph, then $c_3(G) \leq 3 \cdot 2^{\frac{n}{2}}$.*

This is sharp for multi-graphs:



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**Corollary (Solving Enum-3-edge coloring:)**

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**Proof:**

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- If the graph is not connected enough, divide the instance.
- Otherwise, run a branching algorithm.

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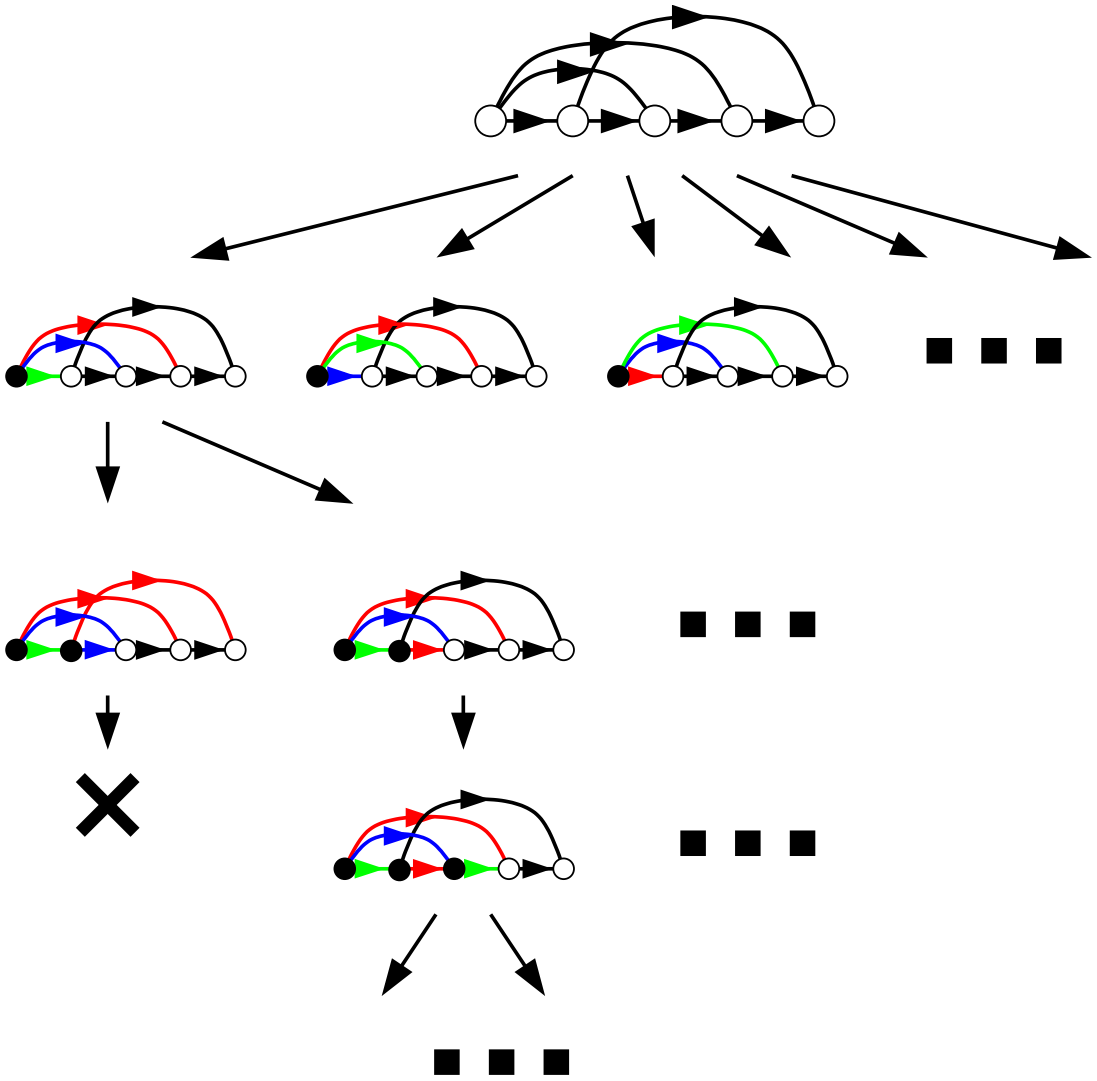

# Extensions


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We denote by $c_k$ the number of $k$-edge colorings of $G$.

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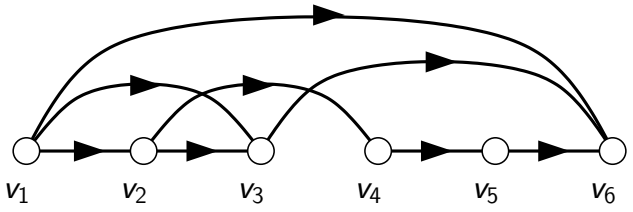

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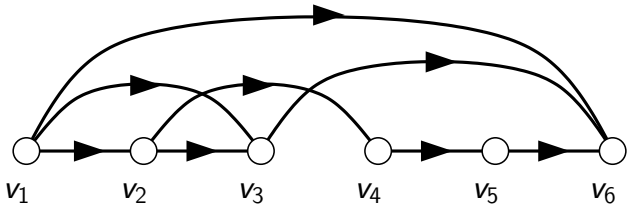
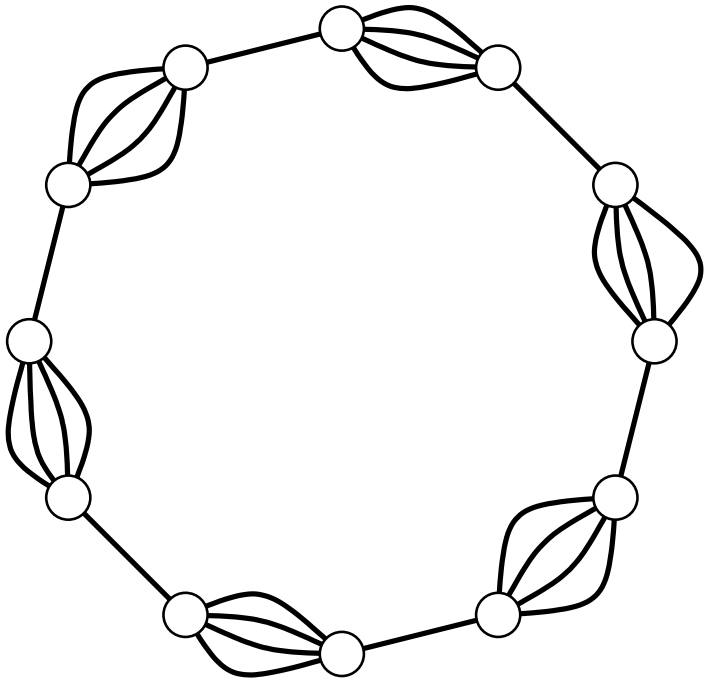



$A_i = \{v \ : \ d^+(v) = i\}$ and $a_i = |A_i|$ for $i = 1, \ldots, k-1$

Then, $c_k(G) \leq \prod_{i=1}^{n-1} d^+(v_i)! \leq k! \prod_{i=1}^{k-1} (i!)^{a_i}$
under $\sum_{i=1}^{k-1} a_i = n-2$ and $\sum_{i=1}^{k-1} i.a_i = k(n-2)/2$ $\square$

# Number of $k$-edge colorings of $k$-regular graphs

Also sharp for multi-graphs:



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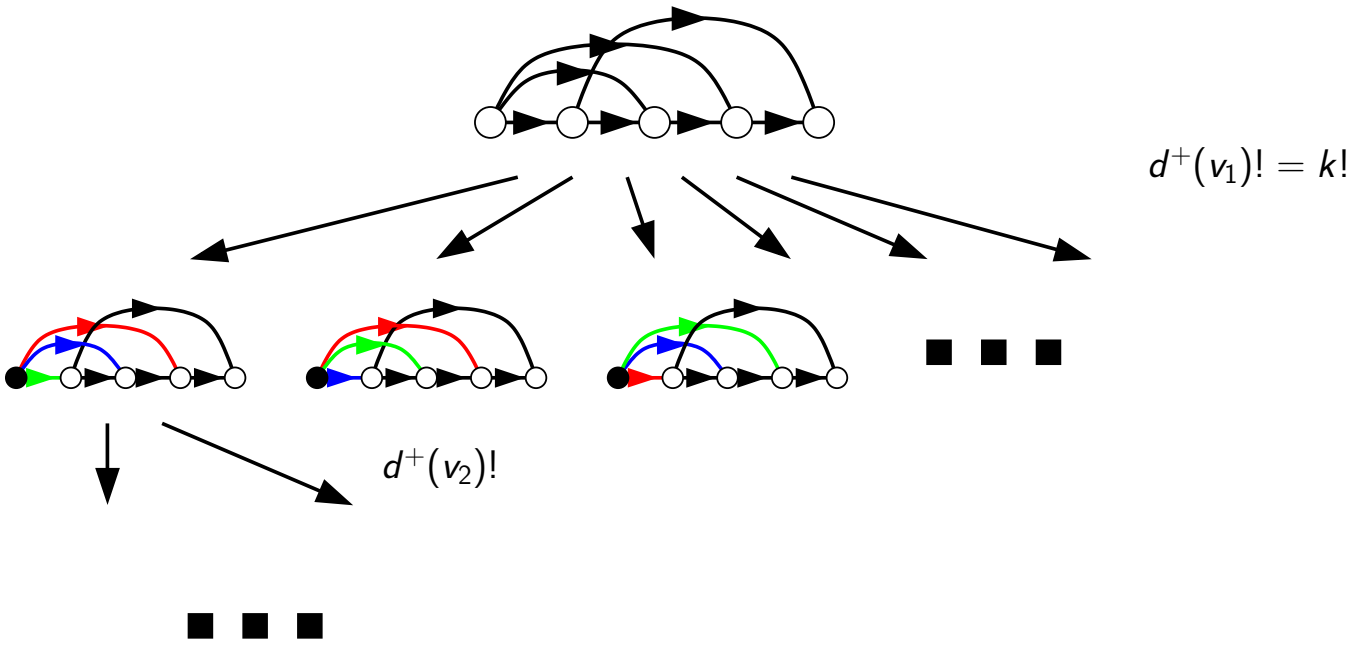


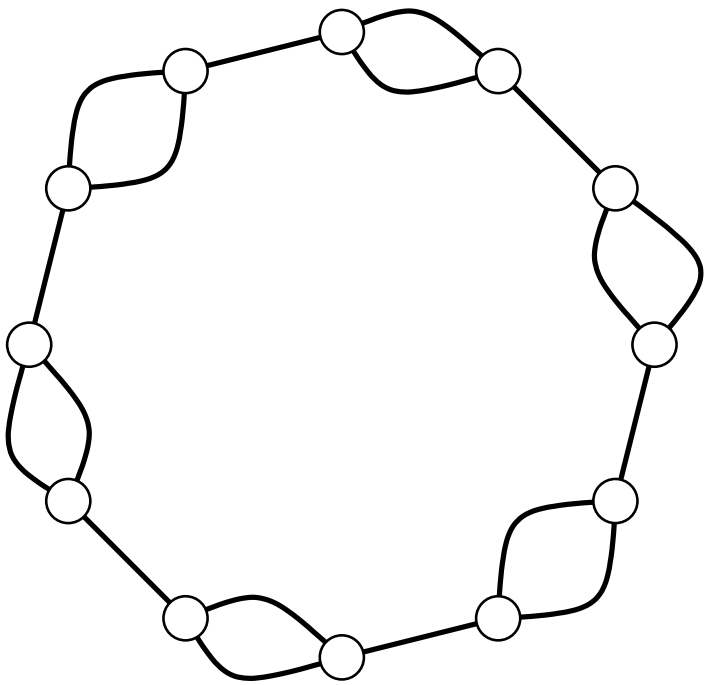

At most $\prod_{i=1}^{n-1} d^+(v_i)! = O^\star((k-1!)^{\frac{n}{2}})$ leaves

# Number of 3-edge colorings for simple graphs

**Corollary**

*Let $G$ be a cubic graph, then $c_3(G) \leq 3 \cdot 2^{\frac{n}{2}}$.*

The sharp example for the number of 3-edge coloring of cubic graphs:

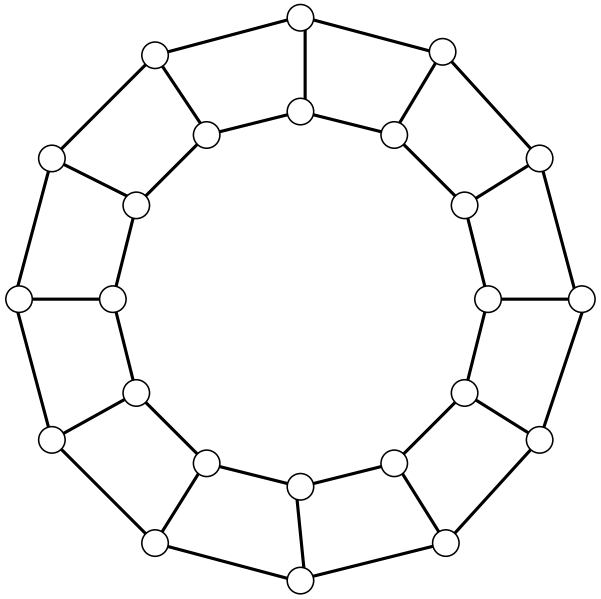


# Number of 3-edge colorings for simple graphs

Can we improve a lot?

# Number of 3-edge colorings for simple graphs

Can we improve a lot? Not really: let $H_n$ be:



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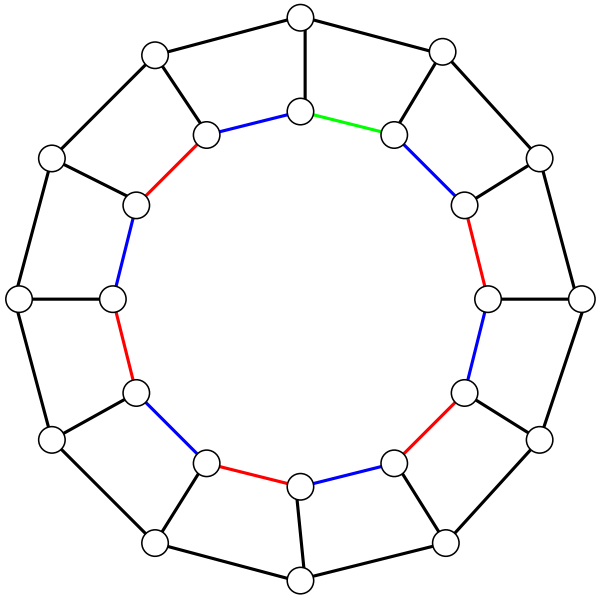

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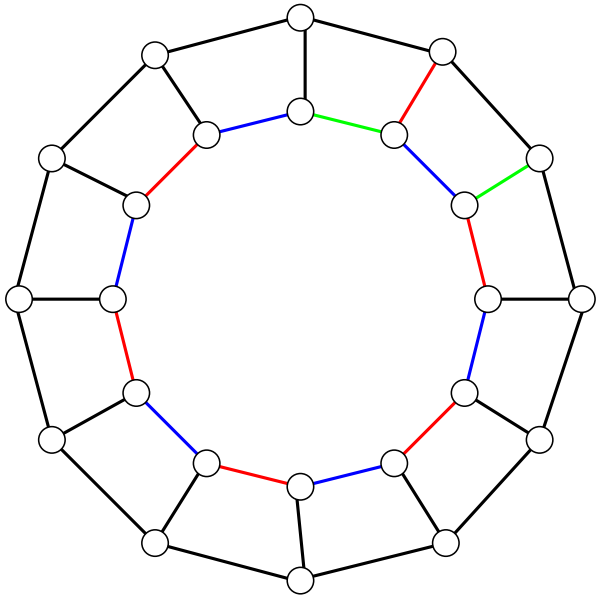

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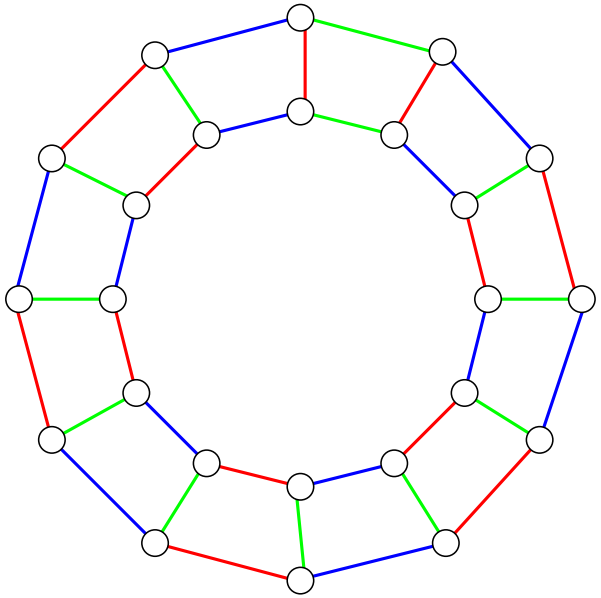

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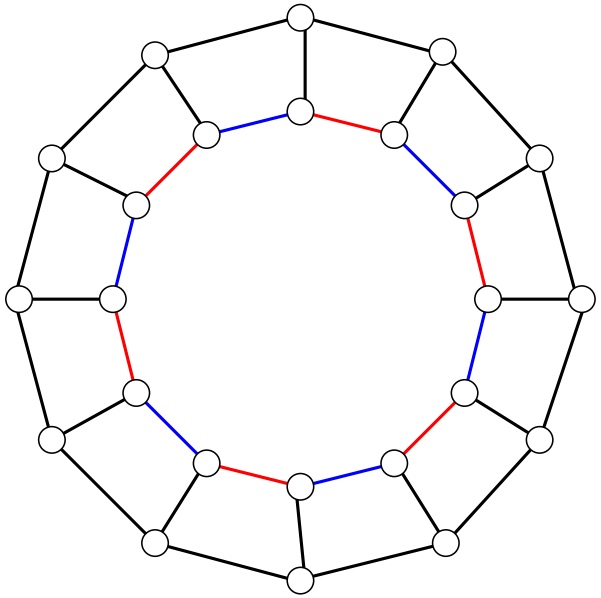

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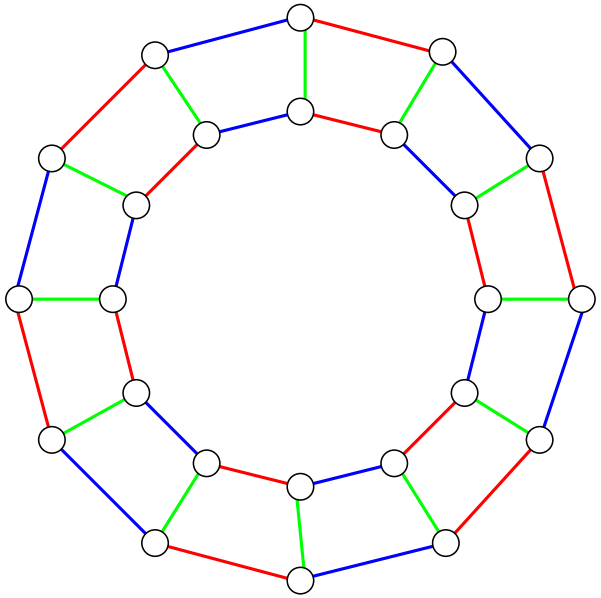

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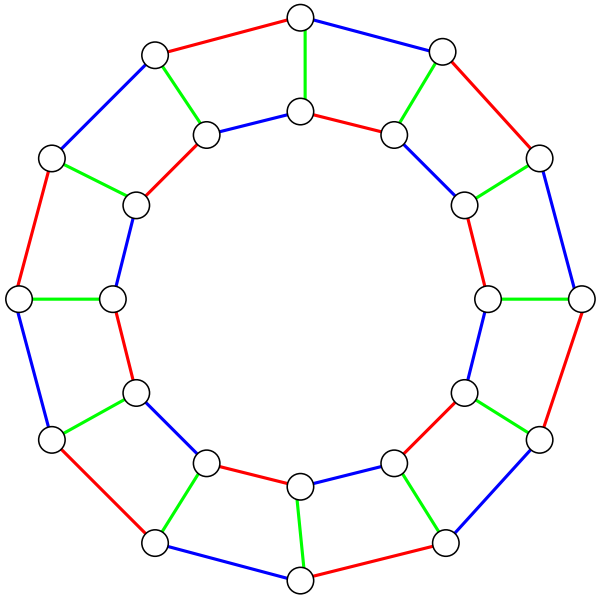

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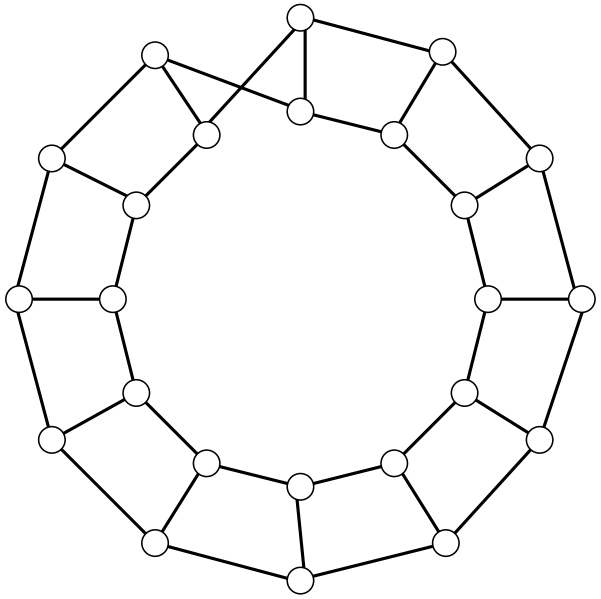

**Lemma ($c_3$ of the Mœbius ladder graph)**

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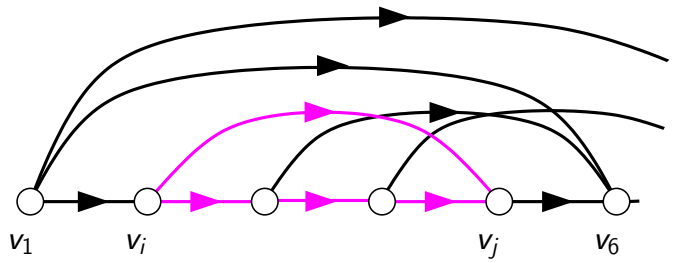


We have $\frac{1}{3} \cdot c_3(C_{j-i+1})/2^{j-i} \cdot 3.2^{\frac{n}{2}}$ colorings.

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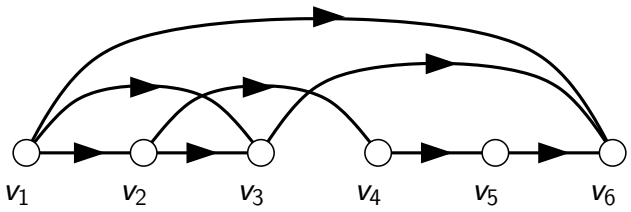


Orient from left to right.
$A^+ = \{v \ : \ d^+(v) = 2 \text{ and } d^-(v) = 1\}$
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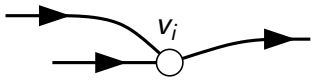
- if $v_i \in A^-$  $c_i \leq 2c_{i-1}$

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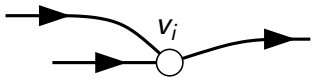
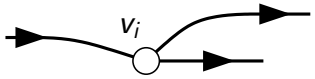
- if $v_i \in A^-$  $c_i \leq 2c_{i-1}$
- if $v_i \in A^+$  $c_i \leq 4c_{i-1}$

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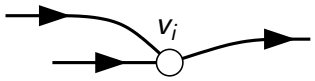
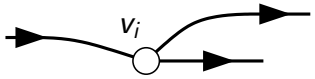
- if $v_i \in A^-$  $c_i \leq 2c_{i-1}$
- if $v_i \in A^+$  $c_i \leq 4c_{i-1}$

Hence:
$c_4^T(G) \leq 24 \cdot 2^{\frac{(n-2)}{2}} \cdot 4^{\frac{(n-2)}{2}} = 3 \cdot 2^{\frac{3n}{2}}$ □

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**Corollary (Solving Enum-3-edge coloring:)**

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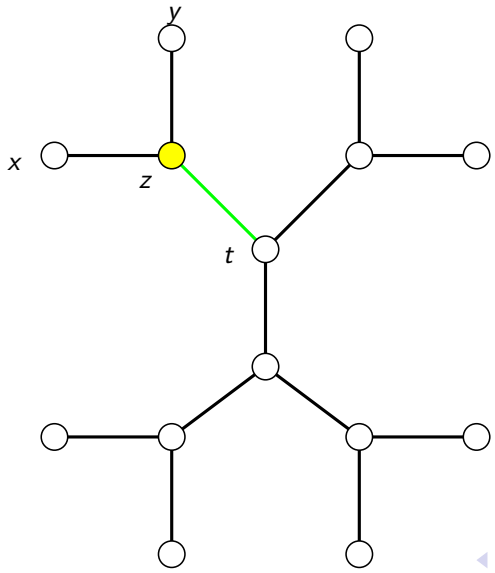

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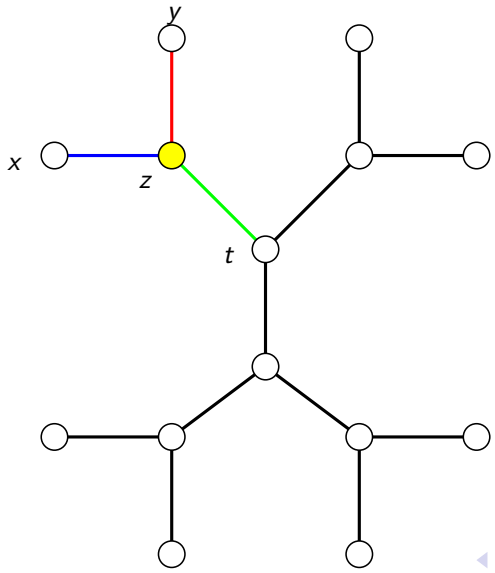

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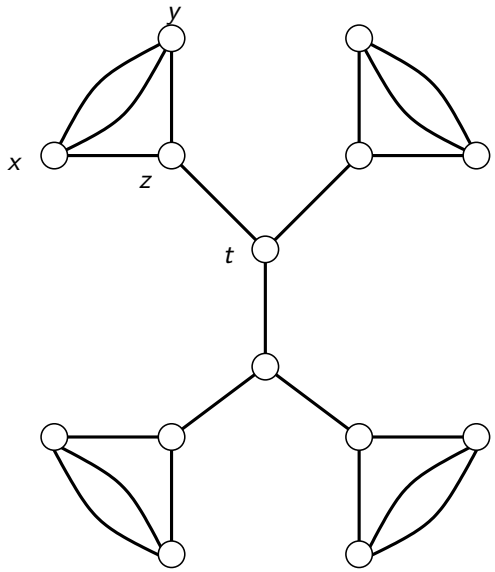


$n = p - 2 + p + 2p$

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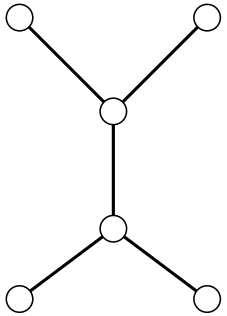

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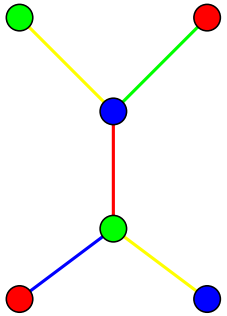

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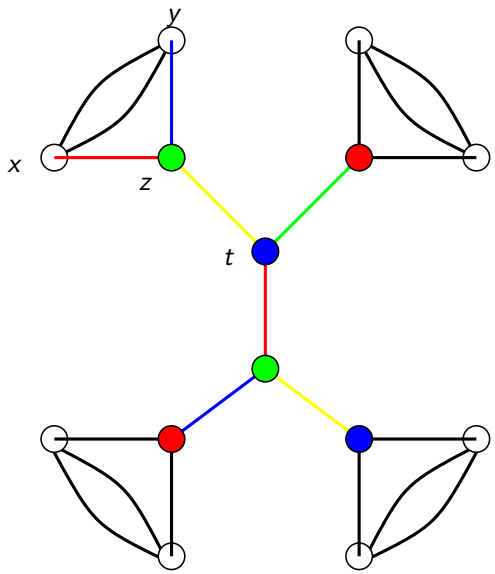


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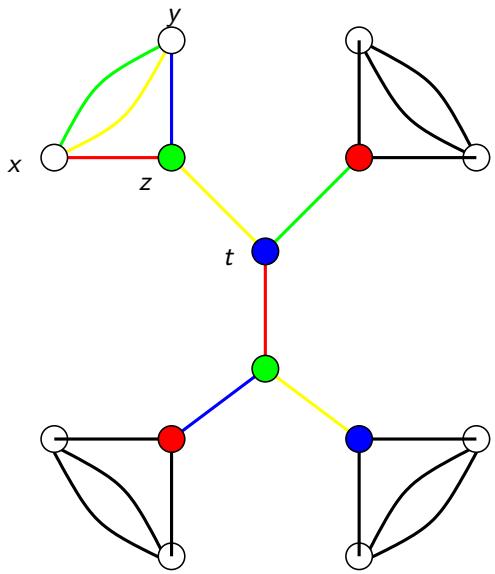


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□

# Conclusion


1. Introduction
   - Colorings
   - Algorithmic problems
   - Our results
2. Enumerating the 3-edge colorings of a cubic graph
   - The 3-edge colorings of a 3-regular graph
   - Turning the proof into algorithm
3. Extensions: $k$-edge colorings and the total colorings
   - $k$-edge colorings
   - A more precise bounds for the 3-edge coloring
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- Enum-4-total coloring in $O^\star(2.8285^n)$. Example in $\Omega(2.3784^n)$.

# Some (less hard?) open questions

- Enum-3-edge coloring in $O^\star(1.4142^n)$ and a sharp example (multi-graph).
  Is it possible to fix
  $c_3(n) = \max\{c_3(G) \ : \ G \text{ simple graph on } n \text{ vertices}\}$
  We know that $2^{\frac{n}{2}} < c_3(n) \leq \frac{9}{4} \cdot 2^{\frac{n}{2}}$

- Enum-4-total coloring in $O^\star(2.8285^n)$. Example in $\Omega(2.3784^n)$.
  Is it possible to find the algorithm with best running time for Enum-4-total coloring.

Thank you for your attention !