# Enumerating the edge-colourings and total colourings of a regular graph 

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## Enumerating the edge-colourings of a regular graph

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## Outline

(1) Introduction

- Colorings
- Algorithmic problems
- Our results
(2) Enumerating the 3-edge colorings of a cubic graph
- The 3-edge colorings of a 3-regular graph
- Turning the proof into algorithm
(3) Extensions: $k$-edge colorings and the total colorings
- $k$-edge colorings
- A more precise bounds for the 3-edge coloring
- Total coloring

4 Conclusion

## Introduction

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4) Conclusion

## Vertex Coloring

## Definition ( $k$-vertex coloring)

A $k$-vertex coloring of a (non oriented) graph $G=(V, E)$ is a function $c: V \rightarrow\{1, \ldots, k\}$ such that $u v \in E$ implies $c(u) \neq c(v)$.

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$\chi(G)$ is the minimum $k$ such that $G$ admits a $k$-vertex coloring.

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## Definition (chromatic index)

$\chi^{\prime}(G)$ is the minimum $k$ such that $G$ admits a $k$-edge coloring.

## Edge Coloring

- The line graph $L(G)$ of $G$ :
- The vertex set of $L(G)$ is the edge set of $G$
- Two edges $e$ and $f$ of $G$ are adjacent in $L(G)$ if there are incident in $G$

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- $\chi^{\prime}(G)=\chi(L(G))$


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## Definition (total chromatic number)

$\chi_{T}(G)$ is the minimum $k$ such that $G$ admits a $k$-total coloring.

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- Vizing's Theorem (1964): $\Delta(G) \leq \chi^{\prime}(G) \leq \Delta(G)+1$
- Total coloring conjecture (Behzad, Vizing, ~1964): $\Delta(G)+1 \leq \chi_{T}(G) \leq \Delta(G)+2$


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- parameterized algorithm: find an algorithm with running time $f(k)$.poly $(n)$ for some parameter $k$.
- exact (exponential) algorithms: find an algorithm with running time $O^{\star}\left(c^{n}\right)\left(=O\left(P(n) c^{n}\right)\right)$


## Examples of exact algorithms

Enum-3-EDGE COLORING:

- Input: a graph $G=(V, E)$ with $\Delta(G) \leq 3$.
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Usual methods to improve: branching algorithms and dynamic programming.

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In the same environment:

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Here: $\frac{\log 3^{\frac{3}{2}}}{\log 2^{\frac{3}{2}}}=1.58$

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Main results in exact algorithms to color a graph:

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- More precise bound on the number of 3-edge coloring of simple graph


## Enumerating the 3-edge colorings

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We denote by $c_{k}(G)$ the number of $k$-edge-colorings of a graph $G$.

Here, we want to compute $c_{3}(G), G$ being a sub-cubic (multiple) graph.

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We can assume that $G$ is 2 -(vertex) connected.


## Number of 3-edge colorings of 3-regular graphs

## Lemma

Let $C_{n}$ be the cycle of length $n$.

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c_{3}\left(C_{n}\right)= \begin{cases}2^{n}+2, & \text { if } n \text { is even }, \\ 2^{n}-2, & \text { if } n \text { is odd. }\end{cases}
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Proof: By induction on $n$. True for $n=2$ and $n=3$.


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## Number of 3-edge colorings of 3-regular graphs

We denote by $n_{i}$ the number of vertices of degree $i$ in $G$.
Theorem
Let $G$ be a 2 -connected subcubic graph. Then $c_{3}(G) \leq 3 \cdot 2^{n-\frac{n_{3}}{2}}$.

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## Theorem

Let $G$ be a 2 -connected subcubic graph. Then $c_{3}(G) \leq 3 \cdot 2^{n-\frac{n_{3}}{2}}$.
Proof: If $G$ is a cycle, it is true.
Otherwise, let $v_{1}$ and $v_{n}$ be two vertices of degree 3 and consider $v_{1}, \ldots, v_{n}$ an st-ordering of $G$ :
for all $1<i<n, \quad d\left(v_{i}\right)_{\left\{v_{1}, \ldots, v_{i-1}\right\}} \geq 1$ and $d\left(v_{i}\right)_{\left\{v_{i+1}, \ldots, v_{n}\right\}} \geq 1$. (always exists: Lempel et al.,1967)


## Number of 3-edge colorings of 3-regular graphs

## Theorem

Let $G$ be a 2 -connected subcubic graph. Then $c_{3}(G) \leq 3 \cdot 2^{n-\frac{n_{3}}{2}}$.

## Proof:



Orient from left to right.
$A^{+}=\left\{v: d^{+}(v)=2\right.$ and $\left.d^{-}(v)=1\right\}$
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We have $\left|A^{+}\right|=\left|A^{-}\right|=\left(n_{3}-2\right) / 2$

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- $c_{1}=6$
- if $v_{i} \in A^{-} \rightarrow c_{i} \leq c_{i-1}$
- if $v_{i} \in A^{+} \rightarrow c_{i} \leq 2 c_{i-1}$
- if $d^{+}\left(v_{i}\right)=d^{-}\left(v_{i}\right)=1 \xrightarrow{v_{i}} c_{i} \leq 2 c_{i-1}$


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$c_{3}(G) \leq 6 \cdot 2^{(n-2)-\left(n_{3}-2\right) / 2}=3 \cdot 2^{n-\frac{n_{3}}{2}}$


## Number of 3-edge colorings of 3-regular graphs

## Theorem

Let $G$ be a 2 -connected subcubic graph. Then $c_{3}(G) \leq 3 \cdot 2^{n-\frac{n_{3}}{2}}$.

## Corollary

Let $G$ be a cubic graph, then $c_{3}(G) \leq 3 \cdot 2^{\frac{n}{2}}$.
This is sharp for multi-graphs:


## Enum-3-EDGE COLORING:

## Corollary (Solving Enum-3-EDGE COLORING:)

There exists a branching algorithm with running time $O^{\star}\left(2^{\frac{n}{2}}\right)=$ $O^{\star}\left(1.4143^{n}\right)$ and polynomial space to solve Enum-3-EDGE COLORING.

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## Proof:

- If $\Delta \geq 4$ answer 0 .
- If the graph is not connected enough, divide the instance.
- Otherwise, run a branching algorithm.


## Enum-3-EDGE COLORING:



## Extensions

(1) Introduction

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(3) Extensions: k-edge colorings and the total colorings
- $k$-edge colorings
- A more precise bounds for the 3-edge coloring
- Total coloring

4) Conclusion

## Number of $k$-edge colorings of $k$-regular graphs

We denote by $c_{k}$ the number of $k$-edge colorings of $G$.
Theorem
Let $G$ be a $k$-regular graph. Then $c_{k}(G) \leq k \cdot(k-1!)^{\frac{n}{2}}$.

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$A_{i}=\left\{v: d^{+}(v)=i\right\}$ and $a_{i}=\left|A_{i}\right|$ for $i=1, \ldots, k-1$
Then, $c_{k}(G) \leq \prod_{i=1}^{n-1} d^{+}\left(v_{i}\right)!\leq k!\prod_{i=1}^{k-1}(i!)^{a_{i}}$

$$
\text { under } \quad \sum_{i=1}^{k-1} a_{i}=n-2 \quad \text { and } \quad \sum_{i=1}^{k-1} i . a_{i}=k(n-2) / 2
$$

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Also sharp for multi-graphs:


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## Corollary (Solving Enum-3-EDGE COLORING:)

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## Proof:



At most $\prod_{i=1}^{n-1} d^{+}\left(v_{i}\right)!=O^{\star}\left((k-1!)^{\frac{n}{2}}\right)$ leaves

## Number of 3-edge colorings for simple graphs

## Corollary

Let $G$ be a cubic graph, then $c_{3}(G) \leq 3 \cdot 2^{\frac{n}{2}}$.
The sharp example for the number of 3-edge coloring of cubic graphs:


## Number of 3-edge colorings for simple graphs

Can we improve a lot?

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Can we improve a lot? Not really: let $H_{n}$ be:


Lemma ( $c_{3}$ of the ladder graph)

$$
c_{3}\left(H_{n}\right)= \begin{cases}2^{n / 2}+8, & \text { if } n / 2 \text { is even, } \\ 2^{n / 2}-2, & \text { if } n / 2 \text { is odd. }\end{cases}
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Can we improve a lot? Not really: let $M_{n}$ be:


Lemma ( $c_{3}$ of the Mœbius ladder graph)

$$
c_{3}\left(M_{n}\right)= \begin{cases}2^{n / 2}+2, & \text { if } n / 2 \text { is even }, \\ 2^{n / 2}+4, & \text { if } n / 2 \text { is odd. }\end{cases}
$$

## Number of 3-edge colorings for simple graphs

So, for simple cubic graphs, we have:
$2^{\frac{n}{2}}<\max \left\{c_{3}(G): G\right.$ cubic graph on $n$ vertices $\} \leq 3.2^{\frac{n}{2}}$

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Let $G$ be a connected simple cubic graph. Then $c_{3}(G) \leq \frac{9}{4} \cdot 2^{n-\frac{n_{3}}{2}}$.

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So, for simple cubic graphs, we have:

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## Proof:



We have $\frac{1}{3} \cdot c_{3}\left(C_{j-i+1}\right) / 2^{j-i} \cdot 3.2^{\frac{n}{2}}$ colorings.

## 4-total colorings

Let $c_{4}^{T}(G)$ be the number of 4-total colorings of $G$.

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- $c_{1}=24$


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Hence:
$c_{4}^{T}(G) \leq 24 \cdot 2^{\frac{(n-2)}{2}} \cdot 4^{\frac{(n-2)}{2}}=3 \cdot 2^{\frac{3 n}{2}}$

## Enum-4-TOTAL COLORING:

## Corollary (Solving Enum-3-edge coloring:)

There exists a branching algorithm with running time $O^{\star}\left(2^{\frac{3 n}{2}}\right)=$ $O^{\star}\left(2.8285^{n}\right)$ and polynomial space to solve Enum-4-Total COLORING.

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## Lemma

Let $T$ be a binary tree (degree $=1$ or 3 ) on $n$ vertices, $c_{4}^{T}(T)=3 \cdot 2^{\frac{3 n}{2}}$.

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Let $T_{n}$ be a binary tree on $n$ vertices + parallel edges between twin leaves, $C_{4}^{T}\left(T_{n}\right)=\frac{3}{\sqrt{2}} \cdot 2^{\frac{5 n}{4}}$.

## Proof:



$$
n=p-2+p+2 p
$$

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$C_{4}^{\top}\left(T^{\prime}\right)=3 \cdot 2 \cdot \frac{3(2(p-2)}{2}$

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## Proof:


$c_{4}^{T}\left(T^{\prime}\right)=3 \cdot 2^{\frac{3(2 p-2)}{2}} \cdot 4^{p}=3 \cdot 2^{5 p-3}=3 \cdot 2^{\frac{5 n}{4}-\frac{1}{2}}$

## Conclusion

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- Minimum vertex coloring in $O^{\star}\left(2^{n}\right)$ (A. Bjöklund, T. Husfeld and M. Koivisto, 2006)


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Is it possible to do better?
Under ETH, there is no $O^{\star}\left(2^{\circ(n)}\right)$ algorithm for 3-vERTEX COLORING (D. Lokshtanov, D. Marx, S. Saurabh, 2012), several scenarios...

## Some (less hard?) open questions

- Enum-3-Edge coloring in $O^{\star}\left(1.4142^{n}\right)$ and a sharp example (multi-graph).


## Some (less hard?) open questions

- Enum-3-EDGE COLORING in $O^{\star}\left(1.4142^{n}\right)$ and a sharp example (multi-graph).
Is it possible to fix $c_{3}(n)=\max \left\{c_{3}(G): G\right.$ simple graph on $n$ vertices $\}$ We know that $2^{\frac{n}{2}}<c_{3}(n) \leq \frac{9}{4} \cdot 2^{\frac{n}{2}}$


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## Thank you for your attention!

